Toroidal Grid Minors and Stretch in Embedded Graphs[*]

Markus Chimani[†] Petr Hliněný [‡] Gelasio Salazar[§]

March 3, 2014

Abstract

We investigate the *toroidal expanse* of an embedded graph G, that is, the size of the largest toroidal grid contained in G as a minor. In the course of this work we introduce a new embedding density parameter, the *stretch* of an embedded graph G, and use it to bound the toroidal expanse from above and from below within a constant factor depending only on the genus and the maximum degree. We also show that these parameters are tightly related to the planar *crossing number* of G. As a consequence of our bounds, we derive an efficient constant factor approximation algorithm for the toroidal expanse and for the crossing number of a surface-embedded graph with bounded maximum degree.

Keywords: Graph embeddings, compact surfaces, face-width, edge-width, toroidal grid, crossing number, stretch

AMS 2010 Subject Classification: 05C10, 05C62, 05C83, 05C85, 57M15, 68R10

---

[*]This draws upon and extends partial results presented at ISAAC 2007 [20] and SODA 2010 [19].

[†]Faculty of Mathematics/Computer Science, Osnabrück University. Osnabrück, Germany.

[‡]Faculty of Informatics, Masaryk University. Brno, Czech Republic. Supported by the Czech Science Foundation, Eurocores project GraDR GIG/11/E023, and project 14-03501S (since 2014).

[§]Instituto de Fisica, Universidad Autonoma de San Luis Potosi. San Luis Potosi, Mexico. Supported by CONACYT Grant 106432.

1 Introduction

In their development of the Graph Minors theory towards the proof of Wagner's Conjecture [32], Robertson and Seymour made extensive use of surface embeddings of graphs. Robertson and Seymour introduced parameters that measure the density of an embedding, and established results that are not only central to the Graph Minors theory, but are also of independent interest. We recall that the *face-width* $fw(G)$ of a graph G embedded in a surface Σ is the smallest r such that Σ contains a noncontractible closed curve (a *loop*) that intersects G in r points.

Theorem 1.1 (Robertson and Seymour [31]). *For any graph H embedded on a surface Σ, there exists a constant $c := c(H)$ such that every graph G that embeds in Σ with face-width at least c contains H as a minor.*

This theorem, and other related results, spurred great interest in understanding which structures are forced by imposing density conditions on graph embeddings. For instance, Thomassen [36] and Yu [38] proved the existence of spanning trees with bounded degree for graphs embedded with large enough face-width. In the same paper, Yu showed that under strong enough connectivity conditions, G is Hamiltonian if G is a triangulation.

Large enough density, in the form of edge-width, also guarantees several nice coloring properties. We recall that the *edge-width* $ew(G)$ of an embedded graph G is the length of a shortest noncontractible cycle in G. Fisk and Mohar [15] proved that there is a universal constant c such that every graph G embedded in a surface of Euler genus $g > 0$ with edge-width at least $c \log g$ is 6-colorable. Thomassen [35] proved that larger (namely 2^{14g+6}) edge-width guarantees 5-colorability. More recently, DeVos, Kawarabayashi, and Mohar [11] proved that large enough edge-width actually guarantees 5-choosability.

In a direction closer to our current interest, Fiedler et al. [14] proved that if G is embedded with face-width r, then it has $\lfloor r/2 \rfloor$ pairwise disjoint contractible cycles, all bounding discs containing a particular face. Brunet, Mohar, and Richter [4] showed that such a G contains at least $\lfloor (r-1)/2 \rfloor$ pairwise disjoint, pairwise homotopic, non-separating (in Σ) cycles, and at least $\lfloor (r-1)/8 \rfloor - 1$ pairwise disjoint, pairwise homotopic, separating, noncontractible cycles. We remark that throughout this paper, "homotopic" refers to "freely homotopic" (that is, not to "fixed point homotopic").

For the particular case in which the host surface is the torus, Schrijver [33] unveiled a beautiful connection with the geometry of numbers and proved that G has at least $\lfloor 3r/4 \rfloor$ pairwise disjoint noncontractible cycles, and proved that the factor $3/4$ is best possible.

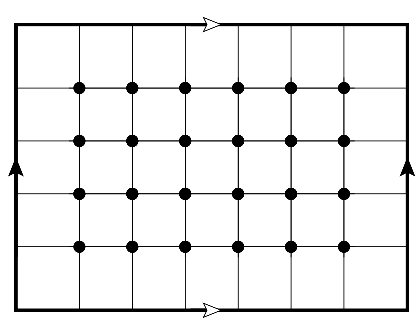


Figure 1: A toroidal embedding of the 4×6 toroidal grid.

The *toroidal* $p \times q$ *-grid* is the Cartesian product $C_p \square C_q$ of the cycles of sizes p and q. See Figure 1. Using results and techniques from [33], de Graaf and Schrijver [10] showed the following:

Theorem 1.2 (de Graaf and Schrijver [10])**.** *Let G be a graph embedded in the torus with face-width $fw(G) = r \geq 5$. Then G contains the toroidal $\lfloor 2r/3 \rfloor \times \lfloor 2r/3 \rfloor$ -grid as a minor.*

De Graaf and Schrijver also proved that $\lfloor 2r/3 \rfloor$ is best possible, by exhibiting (for each $r \geq 3$) a graph that embeds in the torus with face-width r and that does not contain a toroidal $(\lfloor 2r/3 \rfloor + 1) \times (\lfloor 2r/3 \rfloor + 1)$ -grid as a minor. As they observe, their result shows that $c = \lceil 3m/2 \rceil$ is the smallest value that applies in (Robertson-Seymour's) Theorem 1.1 for the case of $H = C_m \square C_m$.

1.1 Our focus: toroidal expanse, stretch, and crossing number

Along the lines of the aforementioned de Graaf-Schrijver result, our aim is to investigate the largest size (meaning the number of vertices) of a toroidal grid minor contained in a graph G embedded in an arbitrary orientable surface of genus greater than zero. We do not restrict ourselves to square proportions of the grid and define this parameter as follows.

Definition 1.3 (Toroidal expanse)**.** *The* toroidal expanse *of a graph G, denoted by* $\mathrm{Tex}(G)$*, is the largest value of $p \cdot q$ over all integers $p, q \geq 3$ such that G contains a toroidal $p \times q$ -grid as a minor. If G does not contain $C_3 \square C_3$ as a minor, then let* $\mathrm{Tex}(G) = 0$.

Our interest is both in the structural and the algorithmic aspects of the toroidal expanse.

The "bound of nontriviality" $p, q \geq 3$ required by Definition 1.3 is natural in the view of toroidal embeddability —the degenerate cases $C_2 \square C_q$ are planar, while $C_p \square C_q$ has orientable genus one for all $p, q \geq 3$. It is not difficult to combine results from [4] and [10] to show that for each positive integer $g > 0$ there is a constant $c := c(g)$ with the following property: if G embeds in the orientable surface Σ_g of genus g with face-width r, then G contains a toroidal $c \cdot r \times c \cdot r$ -grid as a minor; that is, $\mathrm{Tex}(G) = \Omega(r^2)$.

On the other hand, it is very easy to come up with a sequence of graphs G embedded in a fixed surface with face-width r and arbitrarily large $\mathrm{Tex}(G)/r^2$: it is achieved by a natural toroidal embedding of $C_r \square C_q$ for arbitrarily large q. This inadequacy of face-width to estimate the toroidal expanse of an embedded graph is to be expected, due to the one-dimensional character of this parameter.

To this end, we introduce a new density parameter of embedded graphs that captures the truly two-dimensional character of our problem; the *stretch of an embedded graph* in Definition 2.6. Using this tool, we unveil our main result—a tight two-way relationship between the toroidal expanse of a graph G in an orientable surface and its *crossing number* $\mathrm{cr}(G)$ in the plane. We furthermore provide an approximation algorithm for both these numbers under an assumption of a sufficiently dense embedding. A simplified summary of the main results follows:

Theorem 1.4 (Main Theorem)**.** *Let Σ be an orientable surface of fixed genus $g > 0$, and let Δ be an integer. There exist constants $r_0, c_0, c_1, c_2 > 0$, depending only on g and Δ, such that the following holds: If G is a graph of maximum degree Δ embedded in Σ with face-width at least r_0, then*

(a) $c_0 \cdot \mathrm{cr}(G) \leq \mathrm{Tex}(G) \leq c_1 \cdot \mathrm{cr}(G)$*, and*

(b) there is a polynomial time algorithm that outputs a drawing of G in the plane with at most $c_2 \cdot \mathrm{cr}(G)$ crossings.

The density assumption that $fw(G) \geq r_0$ is unavoidable for (a). Indeed, consider a very large planar grid plus an edge. Such a graph clearly admits a toroidal embedding with face-width 1. By suitably placing the additional edge, such a graph would have arbitrarily large crossing number, and yet no $C_3 \square C_3$ minor. However, one could weaken this restriction a bit by considering "nonseparating" face-width instead, as we are going to do in the proof. Furthermore, we shall show later (Section 8.2) how to remove the density assumption $fw(G) \geq r_0$ completely for the algorithm (b), using additional results of [9].

Regarding the constants r_0, c_0, c_1, c_2 we note that, in our proofs,

- r_0 is exponential in g (of order 2^g),
- c_1 is independent of g, Δ, and
- c_2 and $1/c_0$ are quadratic in Δ and exponential in g (of order 8^g).

The rest of this paper is structured as follows. In Section 2 we present some basic terminology and results on graph drawings and embeddings, and introduce the key concept of stretch of an embedded graph. In Section 3 we give a commentated walkthrough on the lemmas and theorems leading to the proof of Theorem 1.4. The exact value of the constants r_0, c_0, c_1, c_2 is given there as well. Some of the presented statements seem to be of independent interest, and their (often long and technical) proofs are deferred to the later sections of the paper. Final Section 8 then outlines some possible extensions of the main theorem and directions for future research.

2 Preliminaries

We follow standard terminology of topological graph theory, see Mohar and Thomassen [28] and Stillwell [34]. We deal with undirected multigraphs by default; so when speaking about a *graph*, we allow multiple edges and loops. The vertex set of a graph G is denoted by $V(G)$, the edge set by $E(G)$, the number of vertices of G (the *size*) by $|G|$, and the maximum degree by $\Delta(G)$.

In this section we lay out several concepts and basic results relevant to this work, and introduce the key new concept of stretch of an embedded graph.

2.1 Graph drawings and embeddings in surfaces

We recall that in a *drawing* of a graph G in a surface Σ, vertices are mapped to points and edges are mapped to simple curves (arcs) such that the endpoints of an arc are the vertices of the corresponding edge; no arc contains a point that represents a non-incident vertex. For simplicity, we often make no distinction between the topological objects of a drawing (points and arcs) and their corresponding graph theoretical objects (vertices and edges). A *crossing* in a drawing is an intersection point of two edges (or a self-intersection of one edge) in a point other than a common endvertex. An *embedding* of a graph in a surface is a drawing with no edge crossings.

If we regard an embedded graph G as a subset of its host surface Σ, then the connected components of $\Sigma \setminus G$ are the *faces* of the embedding. We recall that the vertices of the *topological dual* G^* of G are the faces of G, and its edges are the edge-adjacent pairs of faces of G. There is a natural one-to-one correspondence between the edges of G and the edges of G^*, and so, for an arbitrary $F \subseteq E(G)$, we denote by F^* the corresponding subset of edges of $E(G^*)$. We often use lower case Greek letters (such as α, β, γ) to denote dual cycles. The rationale behind this practice is the convenience to regard a dual cycle as a simple closed curve, often paying no attention to its graph-theoretical properties.

Let G be a graph embedded in a surface Σ of genus $g > 0$, and let C be a two-sided surface-non-separating cycle of G. We denote by $G /\!\!/ C$ the graph obtained by *cutting G through C* as follows. Let F denote the set of edges not in C that are incident with a vertex in C. Orient C arbitrarily, so that F gets naturally partitioned into the set L of edges to the left of C and the set R of edges to the right of C. Now contract (topologically) the whole curve representing C to a point-vertex v, to obtain a pinched surface, and then naturally split v into two vertices, one incident with the edges in L and another incident with the edges in R. The resulting graph $G /\!\!/ C$ is thus embedded on a surface Σ' such that Σ results from Σ' by adding one handle. Clearly $E(G /\!\!/ C) = E(G) \setminus E(C)$, and so for every subgraph $F \subseteq G /\!\!/ C$ there is a unique naturally corresponding subgraph $\hat{F} \subseteq G$ (on the same edge set), which we call the *lift of F into G*.

The "cutting through" operation is a form of a standard surface surgery in topological graph theory, and we shall be using it in the dual form too, as follows. Let G be a graph embedded in a surface Σ and $\gamma \subseteq G^*$ a dual cycle such that γ is two-sided and Σ-nonseparating. Now cut the surface along γ, discarding the set E' of edges of G that are severed in the process. This yields an embedding of $G - E$ in a surface with two holes. Then paste two discs, one along the boundary of each hole, to get back to a compact surface. We denote the resulting embedding by $G /\!\!/ \gamma$, and say that this is obtained by *cutting G along γ*. Note that we may equivalently define $G /\!\!/ \gamma$ as the embedded $(G^* /\!\!/ \gamma)^*$. Note also that $V(G /\!\!/ \gamma) = V(G)$, and that the previous definition of a *lift* applies also to this case.

2.2 Graph crossing number

We further look at drawings of graphs (in the plane) that allow edge crossings. To resolve ambiguity, we only consider drawings where no three edges intersect in a common point other than a vertex. The *crossing number* $cr(G)$ of a graph G is then the minimum number of edge crossings in a drawing of G in the plane.

For the general lower bounds we shall derive on the crossing number of graphs we use the following results on the crossing number of toroidal grids (see [1, 22, 23, 30]).

Theorem 2.1. *For all nonnegative integers p and q, $cr(C_p \square C_q) \geq \frac{1}{2}(p-2)q$. Moreover, $cr(C_p \square C_q) = (p-2)q$ for $p = 3, 4, 5$.*

We note that this result already yields the easy part of Theorem 1.4 (a):

Corollary 2.2. *Let G be a graph embedded on a surface. Then $cr(G) \geq \frac{1}{12}\mathrm{Tex}(G)$.*

Proof. Let $q \geq p \geq 3$ be integers that witness $Tex(G)$ (that is, G contains $C_p \square C_q$ as a minor, and $Tex(G) = pq$). It is known [16] that if G contains H as a minor, and $\Delta(H) = 4$, then $cr(G) \geq \frac{1}{4}\,cr(H)$. We apply this bound with $H = C_p \square C_q$. By Theorem 2.1, we then have for $p \in \{3, 4, 5\}$ that $cr(G) \geq \frac{1}{4}(p-2)q \geq \frac{1}{12}pq$, and for $p \geq 6$ we obtain $cr(G) \geq \frac{1}{4} \cdot \frac{1}{2}(p-2)q \geq \frac{1}{12}pq$. $\square$

2.3 Curves on surfaces and embedded cycles

For the rest of the paper, we shall exclusively focus on orientable surfaces, and for each $g \geq 0$ we let Σ_g denote the orientable surface of genus g. Note that in an embedded graph, paths are simple curves and cycles are simple closed curves in the surface, and hence it makes good sense to speak about their homotopy. In particular, there are no one-sided cycles embedded in Σ_g.

If B is a path or a cycle of a graph, then the *length* $\|B\|$ of B is its number of edges. We recall that the *edge-width* $ew(G)$ of an embedded graph G is the length of a shortest noncontractible cycle in G. The *nonseparating edge-width* $ewn(G)$ is the length of a shortest nonseparating (and hence also noncontractible) cycle in G. It is easy to see that the face-width $fw(G)$ of G equals one half of the edge-width of the vertex-face incidence graph of G. It is also an easy exercise to show that $ew(G^*) \geq fw(G) \geq \frac{ew(G^*)}{\lfloor \Delta(G)/2 \rfloor}$. In this paper, we are primarily interested in graphs of bounded degree. We can thus regard $ew(G^*)$ as a suitable (easier to deal with) replacement for $fw(G)$.

For a cycle (or an arbitrary subgraph) C in a graph G, we call a path $P \subset G$ a *C-ear* if the ends r, s of P belong to C, but the rest of P is disjoint from C. We allow $r = s$, i.e., a C-ear can also be a cycle. A C-ear P is a *C-switching ear* (with respect to an orientable embedding of G) if the two edges of P incident with the ends r, s are embedded on opposite sides of C. The following simple technical claim is useful.

Lemma 2.3. *If C is a nonseparating cycle in an embedded graph G of length $\|C\| = ewn(G)$, then all C-switching ears in G have length at least $\frac{1}{2}ewn(G)$.*

Proof. Seeking a contradiction, we suppose that there is a C-switching ear D of length $< \frac{1}{2}ewn(G)$. The ends of D on C determine two subpaths $D_1, D_2 \subseteq C$ (with the same ends as D), labeled so that $||D_1|| \leq ||D_2||$. Then $D \cup D_1$ (respectively, $D \cup D_2$) is a nonseparating cycle, as witnessed by D_2 (respectively, D_1). Since $\|D_1\| \leq \frac{1}{2}\|C\|$, then

$$\|D \cup D_1\| \leq \|D\| + \frac{1}{2}\|C\| < \left(\frac{1}{2} + \frac{1}{2}\right)\|C\| = ewn(G)\,,$$

a contradiction. □

Even though surface surgery can drastically decrease (and also increase, of course) the edge-width of an embedded graph in general, we now prove that this is not the case if we cut through a short cycle (in Lemma 6.3 we shall establish a surprisingly powerful extension of this simple claim).

Lemma 2.4. *Let G be a graph embedded in the orientable surface Σ_g of genus $g \geq 2$, and let C be a nonseparating cycle in G of length $\|C\| = ewn(G)$. Then $ewn(G/\!\!/C) \geq \frac{1}{2}ewn(G)$.*

Proof. Let c_1, c_2 be the two vertices of $G/\!\!/C$ that result from cutting through C, i.e., $\{c_1, c_2\} = V(G/\!\!/C) \setminus V(G)$. Let $D \subseteq G/\!\!/C$ be a nonseparating cycle of length $ewn(G/\!\!/C)$. If D avoids both c_1, c_2, then its lift $\hat{D}$ in G is a nonseparating cycle again, and so $ewn(G) \leq \|D\| = ewn(G/\!\!/C)$. If D hits both c_1, c_2 and $P \subseteq D$ is (any) one of the two subpaths with the ends c_1, c_2, then the lift $\hat{P}$ is a C-switching ear in G. Thus, by Lemma 2.3,

$$ewn(G/\!\!/C) = \|D\| \geq \|\hat{P}\| \geq \frac{1}{2}ewn(G)\,.$$

In the remaining case D, up to symmetry, hits c_1 and avoids c_2. Then its lift $\hat{D}$ is a C-ear in G. If $\hat{D}$ itself is a cycle, then we are done as above. Otherwise, $\hat{D} \cup C \subseteq G$ is the union of three nontrivial internally disjoint paths with common ends, forming exactly three cycles $A_1, A_2, A_3 \subseteq \hat{D} \cup C$. Since D is nonseparating in $G /\!\!/ C$, each of A_1, A_2, A_3 is nonseparating in G, and hence $\|A_i\| \geq ewn(G)$ for $i = 1, 2, 3$. Since every edge of $\hat{D} \cup C$ is in exactly two of A_1, A_2, A_3, we have $\|A_1\| + \|A_2\| + \|A_3\| = 2\|C\| + 2\|\hat{D}\| = 2ewn(G) + 2\|\hat{D}\|$ and $\|A_1\| + \|A_2\| + \|A_3\| \geq 3ewn(G)$, from which we get

$$ewn(G /\!\!/ C) = \|D\| = \|\hat{D}\| \geq \frac{1}{2} ewn(G)\,. \qquad \square$$

Many arguments in our paper exploit the mutual position of two graph cycles in a surface. In topology, the *geometric intersection number*[1] $i(\alpha, \beta)$ of two (simple) closed curves α, β in a surface is defined as $\min\{\alpha' \cap \beta'\}$, where the minimum is taken over all pairs (α', β') such that α' (respectively, β') is homotopic to α (respectively, β). For our purposes, however, we prefer the following slightly adjusted discrete view of this concept.

Let $A \neq B$ be cycles of a graph embedded in a surface Σ. Let $P \subseteq A \cap B$ be a connected component of the graph intersection $A \cap B$ (a path or a single vertex), and let $f_A, f'_A \in E(A)$ (respectively, $f_B, f'_B \in E(B)$) be the edges immediately preceding and succeeding P in A (respectively, B). See Figure 2. Then P is called a *leap of A,B* if there is a sufficiently small open neighborhood Ω of P in Σ such that the mentioned edges meet the boundary of Ω in this cyclic order; f_A, f_B, f'_A, f'_B (i.e., A and B *meet transversely in* P). Note that $A \cap B$ may contain other components besides P that are not leaps.

Definition 2.5 (k-leaping)**.** *Two cycles A, B of an embedded graph are in a k-*leap position *(or simply k-*leaping*), if their intersection $A \cap B$ has exactly k connected components that are leaps of A, B. If k is odd, then we say that A, B are in an* odd-leap position*.*

We now observe some basic properties of the k-leap concept:

- If A, B are in an odd-leap position, then necessarily each of A, B is noncontractible and nonseparating.
- It is not always true that A, B in a k-leap position have geometric intersection number exactly k, but the parity of the two numbers is preserved. Particularly, A, B are in an odd-leap position if and only if their geometric intersection number is odd. (We will not directly use this fact herein, though.)
- We will later prove (Lemma 6.1) that the set of embedded cycles that are odd-leaping a given cycle A satisfies the useful 3-*path condition* (cf. [28, Section 4.3]).

2.4 Stretch of an embedded graph

In the quest for another embedding density parameter suitable for capturing the two-dimensional character of the toroidal expanse and crossing number problems, we put forward the following concept improving upon the original "orthogonal width" of [20].

Definition 2.6 (Stretch)**.** *Let G be a graph embedded in an orientable surface Σ. The* stretch *$Str(G)$ of G is the minimum value of $\|A\| \cdot \|B\|$ over all pairs of cycles $A, B \subseteq G$ that are in a one-leap position in Σ.*

[1] Note that this quantity is also called the "crossing number" of the curves, and a pair of curves may be said to be "k-crossing". Such a terminology would, however, conflict with the graph crossing number, and we have to avoid it. Following [19], we thus use the term "k-leaping", instead.

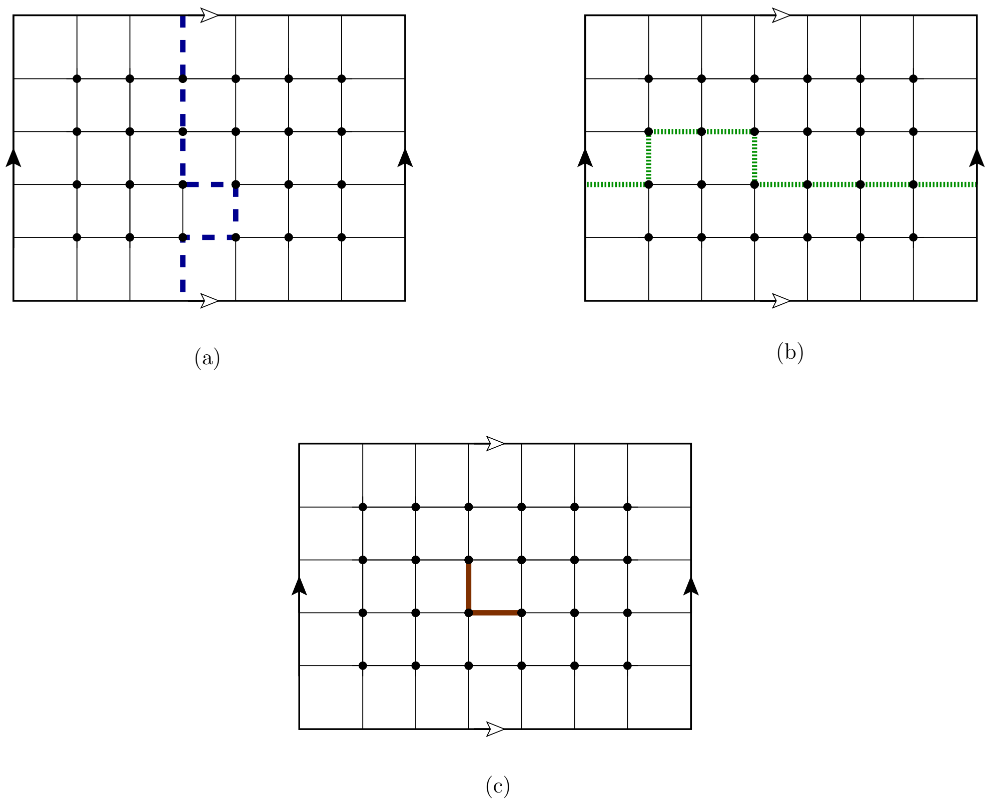


Figure 2: A toroidal embedding of $C_4 \Box C_6$. In (a) and (b) we indicate two cycles A and B (one with dashed edges and one with stripy edges). The intersection of A and B is the 2-edge path indicated in (c) with thick edges. This path is a leap of A and B.

As we noted above, if A, B are in an odd-leap position, then both A and B are noncontractible and nonseparating. Thus it follows that $Str(G) \geq ewn(G)^2$. We postulate that stretch is a natural two-dimensional analogue of edge-width, a well-known and often used embedding density parameter. Actually, one may argue that the dual edge-width is a more suitable parameter to measure the density of an embedding, and so we shall mostly deal with *dual stretch*—the stretch of the topological dual G^*—later in this paper (starting since Lemma 2.8 and Section 3). Analogously to face-width, we can also define the *face stretch* of G as one quarter of the stretch of the vertex-face incidence graph of G, and this is to be discussed later in Section 8.1.

We note in passing that although our paper does not use nor provide an algorithm to compute the stretch of an embedding, this can be done efficiently on any surface [6].

We now prove several basic facts about the stretch of an embedded graph. We start with an easy observation.

Lemma 2.7. *If C is a nonseparating cycle in an embedded graph G, and P is a C-switching ear in G, then $Str(G) \leq \|C\| \cdot \left(\|P\| + \frac{1}{2}\|C\|\right) \leq 2\|C\| \cdot \|P\|$.*

Proof. The ends of P partition C into two paths $C_1, C_2 \subseteq C$, which we label so that $||C_1|| \leq ||C_2||$. (In a degenerate case, C_1 can be a single vertex). Thus $\|C_1\| \leq \frac{1}{2}\|C\|$. Since C and $P \cup C_1$ are in a one-leap position, we have $Str(G) \le \|C\| \cdot (\|P\| + \|C_1\|) \le \|C\| \cdot 2\|P\|$. ☐

A tight relation of stretch to the topic of our paper is illustrated in the following claims.

Lemma 2.8. *If G is a graph embedded in the torus, then $cr(G) \le Str(G^*)$.*

Proof. Let $\alpha, \beta \subseteq G^*$ be a pair of dual cycles witnessing $Str(G^*)$, and let $K := E(\alpha)^*$, $L := E(\beta)^* \setminus K$, and $M := E(\alpha \cap \beta)^*$. Note that K, L, and M are edge sets in G. Then, by cutting G along α, we obtain a plane (cylindrical) embedding G_0 of $G - K$. It is easily possible to draw the edges of K into G_0 in one parallel "bunch" along the fragment of β such that they cross only with edges of L and $M \subseteq K$ (indeed, crossings between edges of K are necessary when $M \neq \emptyset$), thus getting a drawing of G in the plane. See Figure 3. The total number of crossings in this particular drawing, and thus the crossing number of G, is at most $|K| \cdot |L| + |K| \cdot |M| = |K| \cdot (|L| + |M|) = \|\alpha\| \cdot \|\beta\| = Str(G^*)$. ☐

Corollary 2.9. *If G is a graph embedded in the torus, then $Tex(G) \le 12Str(G^*)$.*

Proof. This follows immediately using Corollary 2.2. ☐

We finish this section by proving an analogue of Lemma 2.4 for the stretch of an embedded graph, showing that this parameter cannot decrease too much if we cut the embedding through a short cycle. This will be important to us since cutting through handles of embedded graphs will be our main inductive tool in the proofs of lower bounds on $cr(G)$ and $Tex(G)$.

Lemma 2.10. *Let G be a graph embedded in the orientable surface Σ_g of genus $g \ge 2$, and let C be a nonseparating cycle in G of length $\|C\| = ewn(G)$. Then $Str(G/\!\!/C) \ge \frac{1}{4}Str(G)$.*

Proof. Let c_1, c_2 be the two vertices of $G/\!\!/C$ that result from cutting through C, i.e., $\{c_1, c_2\} = V(G/\!\!/C) \setminus V(G)$. Suppose that $Str(G/\!\!/C) = ab$ is attained by a pair of one-leaping cycles A, B in $G/\!\!/C$, with $a = \|A\|$ and $b = \|B\|$. Our goal is to show that $Str(G) \le 4ab$. Using Lemma 2.4 and the fact that both A, B are nonseparating, we get

$$a, b \ge ewn(G/\!\!/C) \ge \frac{1}{2} ewn(G) = \frac{1}{2}\|C\|. \tag{1}$$

Suppose first that both $c_1, c_2 \in V(A \cup B)$. Then there exists a path $P \subseteq A \cup B$ connecting c_1 to c_2 such that $\|P\| \le \frac{1}{2}(a+b)$. Clearly, its lift $\hat{P}$ is a C-switching ear in G, and so by Lemma 2.7 and (1),

$$\begin{aligned} Str(G) &\le \|C\| \cdot \big(\|\hat{P}\| + \frac{1}{2}\|C\|\big) \le \|C\| \cdot \frac{1}{2}(a + b + \|C\|) \\ &\le \frac{1}{2}(2ba + 2ab + 4ab) = 4ab = 4\,Str(G/\!\!/C). \end{aligned}$$

Finally suppose that, up to symmetry, $c_2 \notin V(A \cup B)$ but possibly $c_1 \in V(A \cup B)$. Consider the lift $\hat{A}$ in G (which is a C-ear in the case $c_1 \in V(A)$). We define $\bar{A}$ to be $\hat{A}$ if $\hat{A}$ is a cycle, and otherwise $\bar{A} = \hat{A} \cup C_0$ where $C_0 \subseteq C$ is a shortest subpath with the same ends in C as $\hat{A}$. We define $\bar{B}$ analogously. With the help of a simple case-analysis, it is straightforward to verify that the cycles $\bar{A}, \bar{B}$ form a one-leaping pair in G, and so again using Lemma 2.7 we obtain

$$\begin{aligned} Str(G) &\le \|\bar{A}\| \cdot \|\bar{B}\| \le (a + \frac{1}{2}\|C\|) \cdot (b + \frac{1}{2}\|C\|) \\ &\le (a + a) \cdot (b + b) = 4ab = 4\,Str(G/\!\!/C). \end{aligned}$$

☐

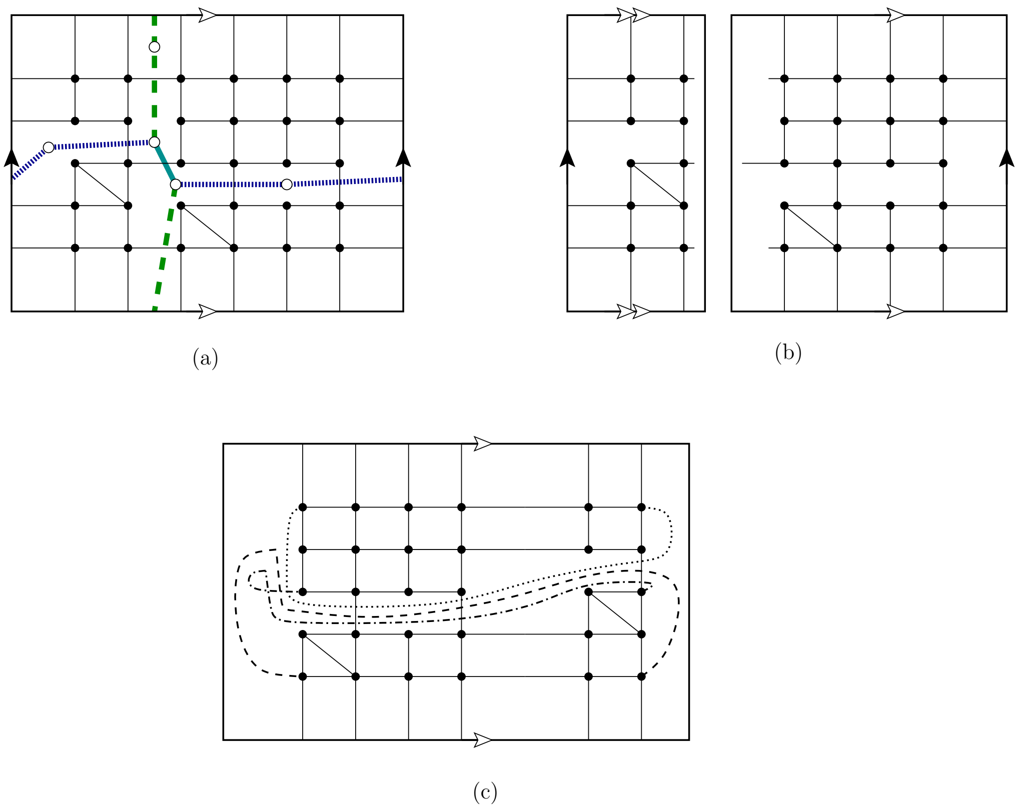


Figure 3: In (a) we show a graph G embedded in the torus (black vertices and solid thin edges), together with dual cycles α, β witnessing the dual stretch (white vertices and dashed/stripy edges). The thick dual edge is common to α and β. We let K denote the set of three edges in G that correspond to the edges of α. In (b) we have cut the torus along the curve defined by α, to obtain a cylindrical embedding of $G_0 := G - K$. In (c) we start with the same embedding of G_0 as in (b) (we have simply identified the black arrows); the three severed edges of K can be drawn along the remaining fragment of β, to get a cylindrical drawing of G. Notice that a bunch of edges of K follows the whole fragment of β, including the section common to α and β—this is to maintain the "right order" of edges in K (although not being optimal, this is very simple).

3 Breakdown of the proof of Theorem 1.4

In this section we shall state the results leading to the proof of Theorem 1.4, which is given in Section 3.4. The proofs of (most of) these statements are long and technical, and so they are deferred to the later sections of the paper.

To reach our main goal, i.e., to provide a proof for Theorem 1.4, we aim to:

(I) extend the upper estimate of Lemma 2.8 to surfaces of higher genus than the torus; and

(II) provide asymptotically matching lower bounds on $cr(G)$ and $Tex(G)$ in terms of the dual stretch of G.

While the upper bounds are given (cf. Lemma 2.8) for the crossing number, the lower bounds here will be investigated for the toroidal expanse. At first glance, goal (I) would appear to be much easier than (II), but it is not really so straightforward due to some complications in expressing the upper bound (cf. Theorem 3.6 below). Such difficulties are to be expected: for instance, a graph embedded in the double torus could have a huge toroidal grid living in one of the handles, and yet very small dual stretch due to a very small dual edge width in the other handle.

Since we will frequently deal with dual graphs in our arguments, we introduce several conventions in order to help comprehension. When we add an adjective *dual* to a graph term, we mean this term in the topological dual of the (currently considered) graph. We will denote the faces of an embedded graph G using lowercase letters, treating them as vertices of its dual G^*. As we already mentioned in Section 2.1, we use lowercase Greek letters to refer to subgraphs (cycles or paths) of G^*, and when there is no danger of confusion, we do not formally distinguish between a graph and its embedding. In particular, if $\alpha \subseteq G^*$ is a dual cycle, then α also refers to the loop on the surface determined by the embedding G. Finally, we will denote by $ewn^*(G) := ewn(G^*)$ the nonseparating edge-width of the dual G^* of G, and by $Str^*(G) := Str(G^*)$ the dual stretch of G.

3.1 Estimating the toroidal expanse

We first give some basic lower bound estimates for the toroidal expanse, aimed at goal (II) above. These estimates ultimately rely on the following basic result, which appears to be of independent interest. Loosely speaking, it states that if a graph has two collections of cycles that mimic the topological properties of the cycles that build up a $p \times q$-toroidal grid, then the graph does contain such a grid as a minor. We say that a pair (C, D) of curves in the torus is a *basis* (for the fundamental group) if there are no integers m, n such that C^m is homotopic to D^n.

Theorem 3.1. *Let G be a graph embedded in the torus. Suppose that G contains a collection $\{C_1, \dots, C_p\}$ of $p \geq 3$ pairwise disjoint, pairwise homotopic cycles, and a collection $\{D_1, \dots, D_q\}$ of $q \geq 3$ pairwise disjoint, pairwise homotopic cycles. Further suppose that the pair (C_1, D_1) is a basis. Then G contains a $p \times q$-toroidal grid as a minor.*

The proof of this statement is in Section 4.

Now recall that in the torus $ewn(G) = ew(G)$, and so $fw(G) \geq \frac{ewn^*(G)}{\lfloor \Delta(G)/2 \rfloor}$. Hence, for instance, one can formulate Theorem 1.2 in terms of nonseparating dual edge-width. Along these lines we shall derive the following as a consequence of Theorem 3.1 (the proof is also in Section 4):

Theorem 3.2. *Let G be a graph embedded in the torus and $k := ewn^*(G)$. Let ℓ be the largest integer such that, in the dual graph G^*, there exists a dual cycle α of length k and the shortest α-switching dual ear has length ℓ (recall from Lemma 2.3 that $\ell \geq k/2$). If $k \geq 5\lfloor \Delta(G)/2 \rfloor$, then G contains as a minor the toroidal grid of size*

$$\left\lceil \frac{\ell}{\lfloor \Delta(G)/2 \rfloor} \right\rceil \times \left\lfloor \frac{2}{3} \left\lceil \frac{k}{\lfloor \Delta(G)/2 \rfloor} \right\rceil \right\rfloor .$$

Hence the toroidal expanse of G is at least $\lceil \frac{\ell}{\lfloor \Delta(G)/2 \rfloor} \rceil \cdot \lfloor \frac{2}{3} \lceil \frac{k}{\lfloor \Delta(G)/2 \rfloor} \rceil \rfloor$. On the other hand, since $fw(G) \geq \frac{k}{\lfloor \Delta(G)/2 \rfloor}$, by Theorem 1.2 it follows that the toroidal expanse of G is at least $\lfloor \frac{2}{3} \lceil \frac{k}{\lfloor \Delta(G)/2 \rfloor} \rceil \rfloor^2$. Therefore our estimate becomes useful roughly whenever $\ell > \frac{2}{3}k$. Now by Lemma 2.7 (applied to G^*), we have $Str^*(G) \leq k \cdot (\ell + k/2)$, and so $\ell > \frac{2}{3}k$ whenever $Str^*(G) > \frac{7}{6}k^2$.

Moreover, Theorem 3.2 can be reformulated in terms of $Str^*(G)$ (instead of "$\ell \cdot k$"). This reformulation is important for the general estimate on the toroidal expanse of G:

Corollary 3.3. *Let G be a graph embedded in the torus with* $\mathrm{ewn}^*(G) \geq 5\lfloor\Delta(G)/2\rfloor$*. Then*

$$Tex(G) \ \geq \ \frac{2}{7}\left\lfloor\Delta(G)/2\right\rfloor^{-2} \cdot Str^*(G) \ \geq \ \frac{8}{7}\Delta(G)^{-2} \cdot Str^*(G)\,.$$

Furthermore, for any $\varepsilon > 0$ there is a $k_0 := k_0(\Delta, \varepsilon)$ such that if $\mathrm{ewn}^*(G) > k_0$, then $\mathrm{Tex}(G) \geq (\frac{8}{21} - \varepsilon) \cdot \lfloor\Delta(G)/2\rfloor^{-2} \cdot Str^*(G)$*.*

For the proof of this statement, we again refer to Section 4.

Stepping up to orientable surfaces of genus $g > 1$, we use Lemmas 2.4 and 2.10 and Corollary 3.3 iteratively ($g - 1$ times), cutting through shortest nonseparating dual cycles. This easily leads by induction to the following lower estimate:

Corollary 3.4. *Let G be a graph embedded in the orientable surface Σ_g, $g \geq 1$, such that* $\mathrm{ewn}^*(G) \geq 5 \cdot 2^{g-1}\lfloor\Delta(G)/2\rfloor$*. Then*

$$Tex(G) \ \geq \ \frac{2}{7}\,4^{1-g}\left\lfloor\Delta(G)/2\right\rfloor^{-2} \cdot Str^*(G) \ \geq \ \frac{1}{7}\,2^{5-2g}\Delta(G)^{-2} \cdot Str^*(G)\,. \qquad \square$$

This bound is, unfortunately, not strong enough to give the desired conclusion for $g \geq 2$, but it is nevertheless useful in the course of deriving a stronger estimate later on (cf. Lemma 3.7).

3.2 Algorithmic upper estimate for higher surfaces

We now tackle task (I): to give an algorithmically efficient upper bound on the crossing number of a graph embedded in Σ_g.

Peter Brass conjectured the existence of a constant c such that the crossing number of a toroidal graph on n vertices is at most $c\Delta n$. This conjecture was proved by Pach and Tóth [29]. Moreover, Pach and Tóth showed that for every orientable surface Σ there is a constant c_Σ such that the crossing number of an n-vertex graph embeddable on Σ is at most $c_\Sigma \Delta n$; this result was extended to any surface by Böröczky, Pach, and Tóth [3]. The constant c_Σ proved in these papers is exponential in the genus of Σ. This was later refined by Djidjev and Vrt'o [12], who decreased the bound to $\mathcal{O}(g\Delta n)$, and proved that this is tight within a constant factor.

At the heart of these results lies the technique of (perhaps recursively) cutting along a suitable *planarizing* subgraph (most naturally, a set of short cycles), and then redrawing the missing edges without introducing too many crossings. Our techniques and aims are of a similar spirit, although our cutting process is more delicate, due to our need to (eventually) find a matching lower bound for the number of crossings in the resulting drawing. Our cutting paradigm is formalized in the following definition.

Definition 3.5 (Good planarizing sequence)**.** *Let G be a graph embedded in the orientable surface Σ_g. A sequence $(G_1, C_1), (G_2, C_2), \ldots, (G_g, C_g)$ is called a* good planarizing sequence *for G if the following holds for $i = 1, \ldots, g$, letting $G_0 = G$:*

- *G_i is a graph embedded in Σ_{g-i},*
- *C_i is a nonseparating cycle in G_{i-1} of length* $\mathrm{ewn}(G_{i-1})$*, and*
- *G_i results by cutting the embedding G_{i-1} through C_i.*

We implicitly associate such a planarizing sequence with the values $\{k_i, \ell_i\}_{i=1,\dots,g}$*, where* $k_i = \|C_i\|$ *and* ℓ_i *is the length of a shortest* C_i*-switching ear in* G_{i-1}*, for* $i = 1, \dots, g$*.*

In order to upper bound the crossing number of an embedded graph, we make use of good planarizing sequences in the dual graph, as stated in the following result.

Theorem 3.6. *Let* G *be a graph embedded in* Σ_g*. Let* $(G_1^*, \gamma_1), \dots, (G_g^*, \gamma_g)$ *be a good planarizing sequence for the topological dual* G^**, with associated lengths* $k_1, \ell_1, \dots, k_g, \ell_g$*. Then*

$$cr(G) \;\le\; 3 \cdot \left(2^{g+1} - 2 - g\right) \cdot \max\{k_i \ell_i\}_{i=1,2,\dots,g}\,. \tag{2}$$

There is an algorithm that produces a drawing of G *in the plane with at most* (2) *crossings in time* $\mathcal{O}(n \log n)$ *for fixed* g*.*

The proof of this theorem is given in Section 5.

3.3 Bridging the approximation gap

Let us briefly revise where we stand in our way towards proving Theorem 1.4. The right hand side of part (a) already follows from Corollary 2.2, and so to finish this part we need an estimate of the form $Tex(G) = \Omega(cr(G))$. We currently have a lower bound for $Tex(G)$ in terms of $Str^*(G)$ (Corollary 3.4) and an upper bound for $cr(G)$ in terms of $\max\{k_i \ell_i\}$. It may thus appear that our next task is to bridge the gap by proving that $Str^*(G) = \Omega(\max\{k_i \ell_i\})$. As it happens, no such statement is true in general, and so we need to find a way around this difficulty.

The following is a key technical claim that allows us to bridge the aforementioned gap.

Lemma 3.7. *Let* H *be a graph embedded in the orientable surface* Σ_g*. Let* $k := \mathrm{ewn}^*(H)$*, and let* ℓ *be the largest integer such that there is a cycle* γ *of length* k *in* H^* *whose shortest* γ*-switching ear has length* ℓ*. Assume* $k \ge 2^g$*. Then there exists an integer* g'*,* $0 < g' \le g$*, and a subgraph* H' *of* H *embedded in* $\Sigma_{g'}$ *such that*

$$ewn^*(H') \ge 2^{g'-g} k \qquad \text{and} \qquad Str^*(H') \ge 2^{2g'-2g} \cdot k\ell\,.$$

In a nutshell, the main idea behind the proof of this statement is to cut along handles that (may) cause small stretch, until we arrive to the desired toroidal $\Omega(k \times \ell)$ grid.

The arguments required to prove Lemma 3.7 span two sections. In Section 6 we establish several basic results on the stretch of an embedded graph. As we believe this new parameter may be of independent interest, it makes sense to gather these results in a standalone section for possible further reference. The proof of Lemma 3.7 is then presented in Section 7.

The importance of Lemma 3.7 is its crucial role in establishing the following result, the final step in bridging the approximation gap.

Corollary 3.8. *Let* G *be a graph embedded in* Σ_g*. Let* $(G_1^*, \gamma_1), \dots, (G_g^*, \gamma_g)$ *be a good planarizing sequence of* G^**, with associated lengths* $k_1, \ell_1, \dots, k_g, \ell_g$*. Suppose that* $\mathrm{ewn}^*(G) \ge 5 \cdot 2^{g-1} \lfloor \Delta(G)/2 \rfloor$*. Then*

$$Tex(G) \;\ge\; \frac{1}{7}\, 2^{3-2g} \left\lfloor \Delta(G)/2 \right\rfloor^{-2} \cdot \max\{k_i \ell_i\}_{i=1,2,\dots,g}\,.$$

Consequently,

$$cr(G) \;\ge\; \frac{1}{21}\, 2^{1-2g} \left\lfloor \Delta(G)/2 \right\rfloor^{-2} \cdot \max\{k_i \ell_i\}_{i=1,2,\dots,g}\,.$$

Proof. Let j be the smallest integer such that $k_j\ell_j = \max\{k_i\ell_i\}_{i=1,2,\dots,g}$, and let $H := G_{j-1}$ (in case $j = 1$, recall that we set $G_0 := G$). Thus H is embedded in a surface of genus $g_1 = g - j + 1$. An iterative application of Lemma 2.4 yields that $ewn^*(H)/\lfloor\Delta(H)/2\rfloor \ge 5 \cdot 2^{g-1} \cdot 2^{g_1-g} = 5 \cdot 2^{g_1-1}$.

We now apply Lemma 3.7 to H. Thus the resulting graph H' of genus $g' \ge 1$ satisfies $ewn^*(H')/\lfloor\Delta(H')/2\rfloor \ge 5 \cdot 2^{g'-1}$ and $Str^*(H') \ge 2^{2g'-2g_1} \cdot k_j\ell_j \ge 2^{2g'-2g} \cdot k_j\ell_j$. Note that, even though $H^* = G^*_{j-1}$ may not be a subgraph of G^*, we have that H (and thus H') is a subgraph of G, and so $Tex(G) \ge Tex(H')$. Using Corollary 3.4 we finally get

$$\begin{aligned} Tex(G) \ge Tex(H') \;&\ge\; \frac{2}{7}\, 4^{1-g'} \big\lfloor \Delta(H')/2 \big\rfloor^{-2} \cdot Str^*(H') \\ &\ge \frac{1}{7}\, 2^{3-2g'} \big\lfloor \Delta(G)/2 \big\rfloor^{-2} \cdot 2^{2g'-2g} k_j\ell_j \;=\; \frac{1}{7}\, 2^{3-2g} \big\lfloor \Delta(G)/2 \big\rfloor^{-2} \cdot k_j\ell_j \,. \end{aligned}$$

The second inequality then follows immediately by Corollary 2.2. ☐

3.4 Proof of Theorem 1.4

Having deferred the long and technical proofs of the previous subsections for the later sections of the paper, all the ingredients are now in place to prove Theorem 1.4.

The right hand side inequality in (a) follows from Corollary 2.2 (with $c_1 = 12$), and the left hand side follows at once by combining Theorem 3.6 and Corollary 3.8. Finally we note that part (b) follows from Theorem 3.6 and (the crossing number inequality in) Corollary 3.8. ☐

4 Finding grids in the torus

In this section we prove Theorems 3.1 and 3.2 and Corollary 3.3.

Proof of Theorem 3.1. Let α, β be oriented simple closed curves such that (α, β) is a basis, and such that α and β intersect (cross) each other exactly once. Using a standard surface homeomorphism argument (cf. [34]), we may assume without loss of generality that each C_i has the same homotopy type as α (we assign an orientation to the cycles C_i to ensure this). Thus it follows that the cycles D_j may be oriented in such a way that there exist integers $r \ge 0, s \ge 1$ such that the homotopy type of each D_j is $\alpha^r\beta^s$.

We assume without loss of generality that $p \ge q \ge 3$. We let $C_+ := C_1 \cup C_2 \cup \cdots \cup C_p$ and $D_+ := D_1 \cup D_2 \cup \cdots \cup D_q$. We shall assume that among all possible choices of the collections $\{C_1, \dots, C_p\}$ and $\{D_1, \dots, D_q\}$ that satisfy the conditions in the theorem (for the given values of p and q), our collection $\mathcal{C} := \{C_1, \dots, C_p\}$ minimizes $|E(C_+) \setminus E(D_+)|$.

The indices of the C_i-cycles (respectively, the D_j-cycles) are read modulo p (respectively, modulo q). We may assume that the cycles $C_1, C_2, \dots, C_p$ appear in this cyclic order around the torus; that is, for each $i = 1, 2, \dots, p$, one of the cylinders bounded by C_i and C_{i+1} does not intersect any other curve in $\mathcal{C}$. Moreover, we may choose this labeling so that β intersects $C_1, C_2, \dots, C_p$ in this cyclic order.

At first glance it may appear that it is easy to get the desired grid as a minor of $C_+ \cup D_+$, since every D_j has to intersect each C_i in some vertex of G (this follows since each pair (C_i, D_j) is a basis). There are, however, two possible complications. First, two cycles C_i, D_j could have many "zigzag" intersections, with D_j intersecting C_i, then C_{i+1}, then C_i again, etc. Second, D_j

may "wind" many times in the direction orthogonal to C_i. These are the problems to overcome in the upcoming proof.

We start by showing that, even though we may intersect some C_i several times when traversing some D_j, it follows from the choice of $\mathcal{C}$ that, after D_j intersects C_i, it must hit either C_{i-1} or C_{i+1} before coming back to C_i.

Claim 4.1. *No C_+-ear contained in D_+ has both ends on the same cycle C_i.*

Proof. Suppose that there is a C_+-ear $P \subset D_+$ with both ends on the same C_i. Modify C_i by following P in the appropriate section, and let C_i' be the resulting cycle. The families $\{C_1, \ldots, C_{i-1}, C_i', C_{i+1}, \ldots, C_p\}$ and $\{D_1, \ldots, D_q\}$ satisfy the conditions in the theorem. The fact that $|E(C_1 \cup \cdots \cup C_{i-1} \cup C_i' \cup C_{i+1} \cdots \cup C_p) \setminus E(D_+)| < |E(C_+ \setminus D_+)|$ contradicts the choice of $\{C_1, \ldots, C_p\}$. □

For any cycle C, a *quasicycle* is a graph-homomorphic image of C without degree-1 vertices, implicitly retaining its cyclic ordering of vertices.

Let D_j' be a quasicycle in G homotopic to D_1, with its same orientation. We say that D_j' is C_+-*ear good* if (cf. Claim 4.1) no C_+-ear contained in D_j' has both ends on the same C_i. The *rank* s_j of D_j' is the number of connected components of $C_+ \cap D_j'$. By traversing D_j' once and registering each time it intersects a curve in $\mathcal{C}$, starting with (some intersection with) C_1, we obtain an *intersection sequence* $a_j(i)$, $i = 1, \ldots, s_j$, where each $a_j(i)$ is in $\{1, \ldots, p\}$. Since we chose the starting point of the traversal of D_j' so that the first curve of $\mathcal{C}$ it intersects is C_1, it follows that $a_j(1) = 1$. We read the indices of this subsequence modulo s_j. We denote by $Q_{j,t}$, $t = 1, 2, \ldots, s_j$, the path of D_j' (possibly a single vertex) forming the corresponding intersection with the cycle $C_{a_j(t)}$, and by $T_{j,t}$ the path of D_j' between $Q_{j,t}$ and $Q_{j,t+1}$. If D_j' is C_+-ear good then $a_j(t+1) \neq a_j(t)$, and hence in this case $|a_j(t+1) - a_j(t)| \in \{1, p-1\}$ for $t = 1, 2, \ldots, s_j$.

A collection of C_+-ear good quasicycles $D_1', D_2', \ldots, D_q'$ in G is *quasigood* if it satisfies the property that whenever D_n' intersects D_m' in a path P (counting also the case of a self-intersection with $m = n$), the following hold up to symmetry between n and m: (i) $P \subseteq Q_{n,x}$ for an appropriate index x of the intersection sequence of D_n' for which $a_n(x-1) = a_n(x+1)$ and $a_n(x) - a_n(x-1) \in \{1, 1-p\}$; and (ii) the path $T_{n,x-1} \cup Q_{n,x} \cup T_{n,x}$ of D_n' stays locally on one side of the (embedded) quasicycle D_m'. Informally, this means that if D_n' intersects D_m' in P, then D_n' makes a $C_{a_n(x-1)}$-ear with P "touching" D_m' from the left side. For further reference we say that D_n' is locally on the *left side* of the intersection P.

Since D_j is clearly a C_+-ear good quasicycle for each $j = 1, 2, \ldots, q$, it follows that $D_1, D_2, \ldots, D_q$ is a quasigood collection. Now among all choices of a quasigood collection $D_1', D_2', \ldots, D_q'$ in G, we select one minimizing the sum of the ranks of its quasicycles. For each D_j', as above we let s_j denote its rank.

Claim 4.2. *For all $1 \leq j \leq q$ the intersection sequence of D_j' satisfies $a_j(t-1) \neq a_j(t+1)$ for any $1 < t \leq s_j$. Consequently, $D_1', D_2', \ldots, D_q'$ is a collection of pairwise disjoint cycles in G.*

Proof. The conclusion that $D_1', D_2', \ldots, D_q'$ is a collection of pairwise disjoint cycles directly follows from the first statement in the claim, since it is a quasigood collection. We hence focus on the first statement in the following.

The main idea in the proof is quite simple: if $a_j(t-1) = a_j(t+1)$, then we could modify D_j' rerouting it through $C_{a_j(t-1)}$ instead of $T_{j,t-1} \cup Q_{j,t} \cup T_{j,t}$, thus decreasing s_j (and hence the total sum of the ranks) by 2, and consequently contradicting the choice of $\mathcal{D} := \{D'_1, D'_2, \ldots, D'_q\}$. We now formalize this idea.

Let Π_i denote the cylinder bounded by C_i and C_{i+1}. Note that if for some j, t we have $a_j(t-1) = a_j(t+1)$ and $a_j(t) - a_j(t-1) \in \{-1, p-1\}$, then necessarily for some t' we must have $a_j(t'-1) = a_j(t'+1)$ and $a_j(t') - a_j(t'-1) \in \{1, 1-p\}$. So, seeking a contradiction, we may suppose that there exist j, t such that $a_j(t-1) = a_j(t+1) = i$ and $a_j(t) = i+1$. Then the path $P = T_{j,t-1} \cup Q_{j,t} \cup T_{j,t}$ is drawn in Π_i with both ends on C_i and "touching" (i.e., not intersecting transversally) C_{i+1}. We denote by $R_0 \subset \Pi_i$ the open region bounded by P and C_i, and by P' the section of the boundary of R_0 not belonging to D'_j.

Assuming that R_0 is minimal over all choices of j for which $a_j(t-1) = a_j(t+1)$, we show that no D'_m, $m \in \{1, \ldots, q\}$, intersects R_0. Indeed, if some D'_m intersected R_0, then D'_m could not enter R_0 across P by the "stay on one side" property of a quasigood collection. Hence D'_m should enter and leave R_0 across $P' \subseteq C_i$, but not touch $Q_{j,t} \subseteq C_{i+1}$, by the minimality of R_0. But then, D'_m would make a C_+-ear with both ends on C_i, contradicting the assumption that D'_m is C_+-ear good.

Now we form D^o_j as the symmetric difference of D'_j with the boundary of R_0 (so that D^o_j follows P'). To argue that $D'_1, \ldots, D^o_j, \ldots, D'_q$ is a quasigood collection again, it suffices to verify all possible new intersections of D^o_j along P'. Suppose there is an D'_n such that its intersection $Q_{n,x}$ with C_i contains some internal vertex of P'. Since D'_n is disjoint from (the open region) R_0, it will "stay on one side" of D^o_j. If $Q_{n,x}$ intersects D'_j, then D'_n must be locally on the left side of this intersection, and so it is also on the left side of the intersection with D^o_j. If, on the other hand, $Q_{n,x}$ is disjoint from D'_j, then the adjacent paths $T_{n,x-1}$ and $T_{n,x}$ have to connect to C_{i-1} by Claim 4.1, and so we have $a_n(x) = i$ and $a_n(x-1) = a_n(x+1) = i-1$ as required by the definition for D'_n on the left side. Let $\mathcal{D}^o$ be the collection derived from $\mathcal{D}$ by substituting D'_j with D^o_j. In every case, $\mathcal{D}^o$ is quasigood as well, but the sum of the ranks of its elements is strictly smaller (by 2) than it is for $\mathcal{D}$. This contradicts the choice of $\mathcal{D}$. □

Claim 4.3. *There is a collection of q pairwise disjoint, pairwise homotopic noncontractible cycles in G, each of which has a connected nonempty intersection with each cycle in $\mathcal{C}$.*

Proof. It follows from Claim 4.2 that the intersection sequence of each D'_j is a t-fold repetition of the subsequence $\langle 1, 2, \ldots, p\rangle$, for some nonnegative integer t. If $t = 1$, we are obviously done, so assume $t \geq 2$. Our task is to "shortcut" each D'_j such that it "winds only once" in the direction orthogonal to α (more formally, to modify each D'_j so that its homotopy type is $\alpha^r\beta$ for some integer r).

Note that, for all $i = 1, \ldots, p$, every C_i-ear contained in any D'_j is C_i-switching by Claim 4.2. Each such ear naturally inherits an orientation from D'_j, so that after leaving C_i it intersects $C_{i+1}, C_{i+2}, \ldots, C_{i-1}$ in this order, and then intersects C_i again. Let $T_1 \subset D'_1$ be any C_1-ear, and let x_1, y_1 be their start and end points, respectively. Then let $W_1 \subset C_1$ be (any) one of the two paths contained in C_1 with endpoints x_1, y_1. It is clear that the cycle $D''_1 = T_1 \cup W_1$ is a simple closed curve that has a connected nonempty intersection with each C_i. Our final task is to find, for each $j = 2, \ldots, q$, a homotopic, similarly constructed cycle D''_j, so that the cycles $D''_1, D''_2, \ldots, D''_j$ are pairwise disjoint.

Since D''_1 is not homotopic to D'_1, every D'_j has to intersect D''_1 in W_1; this intersection is a path P_j (possibly a single vertex). Since the curves D'_j are pairwise disjoint, it follows that the paths P_j are also pairwise disjoint. For $j = 2, \ldots, q$, let x_j be the point in P_j closest to x_1, and let T'_j be the unique C_1-ear starting at x_j. Now let T_j be the unique C_j-ear starting on a vertex in T'_j, and let $W_j \subset C_j$ be the path joining the ends of T_j that is disjoint from T_1. Finally, set $D''_j = T_j \cup W_j$, for $j = 2, \ldots, q$. It is straightforward to check that the curves $D''_1, D''_2, \ldots, D''_q$ satisfy the required properties. □

To conclude the proof of Theorem 3.1, we let $\{D''_1, D''_2, \ldots, D''_q\}$ be the collection guaranteed by this last claim. For each $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$, we contract the path $C_i \cap D''_j$ to a single vertex (unless it already is a single vertex). Since the curves $D''_1, D''_2, \ldots, D''_q$ are pairwise disjoint and pairwise homotopic, it directly follows that the resulting graph is isomorphic to a subdivision of the $p \times q$-toroidal grid. □

Proof of Theorem 3.2. First we show the following.

Claim 4.4. *G has a set of at least $\frac{\ell}{\lfloor \Delta/2 \rfloor}$ pairwise disjoint cycles, all homotopic to α.*

Proof. Let F be the set of those edges of G intersected by α. Let α_1, α_2 be loops very close to and homotopic to α, one to each side of α, so that the cylinder bounded by α_1 and α_2 that contains α intersects G only in the edges of F. Now we cut the torus by removing the (open) cylinder bounded by α_1 and α_2, thus leaving an embedded graph $H := G - F$ on a cylinder Π with boundary curves ("rims") α_1 and α_2. Let δ be a curve on Π connecting a point of α_1 to a point of α_2, such that δ has the fewest possible points in common with the embedding H. We note that we may clearly assume that the p points in which δ intersects H are vertices.

We claim that $p \geq \frac{\ell}{\lfloor \Delta/2 \rfloor}$. Indeed, if $p < \frac{\ell}{\lfloor \Delta/2 \rfloor}$, then the union of all faces incident with the p vertices intersected by δ would contain a dual path β of length at most $p \cdot \lfloor \Delta/2 \rfloor < \frac{\ell}{\lfloor \Delta/2 \rfloor} \cdot \lfloor \Delta/2 \rfloor = \ell$. Such β would be an α-switching dual ear in G^* of length less than ℓ, a contradiction.

We now cut open the cylinder Π along δ, duplicating each vertex intersected by δ. As a result we obtain a graph H' embedded in the rectangle with sides $\alpha_1, \delta_1, \alpha_2, \delta_2$ in this cyclic order, so that δ_1 (respectively, δ_2) contains p vertices $w_i^1, i = 1, 2, \ldots, p$ (respectively, $w_i^2, i = 1, 2, \ldots, p$).

We note that there is no vertex cut of size at most $p - 1$ in H' separating $\{w_1^1, \ldots, w_p^1\}$ from $\{w_1^2, \ldots, w_p^2\}$, for such a vertex cut would imply the existence of a curve ε from α_1 to α_2 on Π intersecting H in fewer than p points, contradicting our choice of δ. Thus applying Menger's Theorem we obtain p pairwise disjoint paths from $\{w_1^1, \ldots, w_p^1\}$ to $\{w_1^2, \ldots, w_p^2\}$ in H'. Moreover, it follows by planarity of H' that each of these paths connects w_i^1 to the corresponding w_i^2 for $i = 1, \ldots, p$. By identifying back w_i^1 and w_i^2 for $i = 1, \ldots, p$, we get a collection of p pairwise disjoint cycles in H, each of them homotopic to α. □

We have thus proved the existence of a collection $\mathcal{C}$ of $\ell / \lfloor \Delta(G)/2 \rfloor$ pairwise disjoint, pairwise homotopic noncontractible cycles. By Theorem 1.2, since $fw(G) \geq ewn^*(G)/\lfloor \Delta(G)/2 \rfloor$, it follows that G also contains two collections $\mathcal{D}, \mathcal{E}$ of cycles such that: (i) the cycles in $\mathcal{D}$ are noncontractible, pairwise disjoint, and pairwise homotopic; (ii) the cycles in $\mathcal{E}$ are noncontractible, pairwise disjoint, and pairwise homotopic; (iii) for any $D \in \mathcal{D}$ and $E \in \mathcal{E}$, the pair (D, E) is a basis; and (iv) each of $|\mathcal{D}|$ and $|\mathcal{E}|$ is at least $\lfloor \frac{2}{3} \lceil \frac{k}{\lfloor \Delta(G)/2 \rfloor} \rceil \rfloor$.

Let $C \in \mathcal{C}$, $D \in \mathcal{D}$, and $E \in \mathcal{E}$. From properties (i)–(iii) it follows that either (C, D) or (C, E) is a basis. Therefore the result follows from Theorem 3.1. □

Proof of Corollary 3.3. Let $k := ewn^*(G)$, and let ℓ and α be as in Theorem 3.2. By Lemma 2.7, $Str^*(G) \leq 2k\ell$. Let $r = \lceil \frac{k}{\lfloor \Delta(G)/2 \rfloor} \rceil$. Since $r \geq 5$, it follows that $\lfloor 2r/3 \rfloor \geq \frac{6}{7}(2r/3) = \frac{4}{7}r$ (with equality at $r = 7$). Letting $s = \lceil \frac{\ell}{\lfloor \Delta(G)/2 \rfloor} \rceil$ we then get, by Theorem 3.2,

$$Tex(G) \geq s \cdot \left\lfloor \frac{2}{3} r \right\rfloor \geq \frac{4}{7} rs \geq \frac{4}{7} k\ell \cdot \lfloor \Delta(G)/2 \rfloor^{-2} \geq \frac{2}{7} Str^*(G) \cdot \lfloor \Delta(G)/2 \rfloor^{-2} .$$

In order to get the better asymptotic estimate $Tex(G) \geq (\frac{8}{21} - \varepsilon) \cdot \lfloor \Delta(G)/2 \rfloor^{-2} \cdot Str^*(G)$, we directly apply Theorem 1.2 in the case $s \leq 2r/3$; otherwise, we use the stronger bound $Str^*(G) \leq k\ell + k \cdot k/2 \leq k(\ell + 3\ell/4) = \frac{7}{4} k\ell$. □

5 Drawing embedded graphs into the plane

In this section, we prove Theorem 3.6. That is, we provide an efficient algorithm that, given a graph G embedded in some orientable surface, yields a drawing of G (with a controlled number of crossings) in the plane. Although our algorithm takes an embedded graph as its input, we might as well take the non-embedded graph as input without any loss of efficiency; indeed, Mohar [26] showed that, for any fixed genus g, there is a linear time algorithm that takes as input any graph G embeddable in Σ_g and outputs an embedding of G in Σ_g.

We start with an informal outline of the proof.

We proceed in g steps, working at the i-th step with the pair (G_i^*, γ_i). For convenience, let $G_0 = G$, and define $F_i = E(G_{i-1}) \setminus E(G_i) = E(\gamma_i)$. The idea at the i-th step is to cut from G_{i-1} the edges intersected by γ_i (that is, the set F_i). We could then to draw these edges into the embedded graph G_i along the route determined by a γ_i-switching ear of length ℓ_i in G_{i-1}. This would result in at most $k_i(\ell_i + k_i)$ new crossings in G_i (similarly as in Figure 3). For technical reasons, we consider routing the edges of each F_i in one bunch (i.e., along the same route), even though routing every edge separately could perhaps save a small number of crossings.

In reality, the situation is not as simple as in the previous sketch. The main complication comes from the fact that subsequent cutting (step $j > i$) could "destroy" the chosen route for F_i (or at least part of it). Then it would be necessary to perform a further re-routing for the edges of F_i in step j. This could essentially happen in each subsequent step until the end of the process (when obtaining planar G_g).

We handle this complication in two ways: Proof-wise, we track a possible insertion route (and its necessary modifications) for F_i through the full cutting process. In particular, we show that the final insertion route is never longer than $\ell_i + \ell_{i+1} + \cdots + \ell_g$, for each index i. Another detail one has to take care of, is to ensure that such a detour for F_i would not produce significantly more additional crossings than $k_j \ell_j$, over all $j = i+1, \ldots g$; this holds as long as k_j is never much smaller than k_i (cf. Lemma 2.4).

Algorithmically, we will reinsert all edges $\bigcup_{i=1}^{g} F_i$ only at the very end, into G_g. The previously tracked routes are then upper bounds for the so-achieved solution.

In the proof, we briefly use the concept of an *angle* of a pair of edges in an embedded graph. For this, we recall that the *rotation* of a vertex v in an embedded graph is the (say, counterclockwise, by convention) cyclic order in which the edges incident with v leave this vertex. Suppose now that the rotation of a degree-d vertex is $e_0, e_1, \ldots, e_{d-1}$, and let (e_i, e_j) be an ordered pair. Then the *angle* of (e_i, e_j) is the set of edges $\{e_i, e_{i+1}, \ldots, e_{j-1}, e_j\}$ (with indices read modulo d).

Proof of Theorem 3.6. As outlined in the sketch above, we proceed in g steps. At the i-th step, for $i = 1, 2, \ldots, g$, we take the embedded graph G_{i-1} and cut the surface open along γ_i, thus severing the edges in the set $F_i := E(G_{i-1}) \setminus E(G_i) = E(\gamma_i)$. This decreases the genus by one, and creates two holes, which we repair by pasting a closed disc on each hole. Thus we get the graph G_i embedded in a compact surface with no holes.

Claim 5.1. *Let $i = 1, \ldots, g$, and let f be an edge in F_i. Then, f can be drawn into the plane graph G_g with at most $\sum_{j=i}^{g} \ell_j$ crossings.*

Proof. Let $i \in \{1, \ldots, g\}$ be fixed. In the graph G_i, we let a, b denote the two new faces created by cutting G_{i-1} along γ_i (thus each of these faces contains one of the pasted closed discs). Let f be an edge in F_i, with endpoints f_a (incident with a in G_i) and f_b (incident with b in G_i).

For each $j = i, i+1, \ldots, g$, we associate two faces $a_j(f), b_j(f)$ of G_j with f. Loosely speaking, these faces are the natural heirs in G_j of the faces a and b, if we stand in G_j on the vertices f_a and f_b (we note that a, b are faces in G_i, but by the further cutting process, they may not be faces in G_j for some $j > i$). The faces $a_j(f), b_j(f)$ are recursively defined as follows. First, let $a_i(f) = a$ and $b_i(f) = b$. Now suppose $a_{j-1}(f), b_{j-1}(f)$ have been defined for some j, $i < j \leq g$. We then let $a_j(f)$ be the unique face h of G_j that satisfies the following: if (e, e') is the pair of edges of h incident with f_a, ordered so that the angle of (e, e') in G_j consists only of e and e', then the angle of (e, e') in G_{j-1} includes the edges of the face $a_{j-1}(f)$ that are incident with f_a. The face $b_j(f)$ is defined analogously.

The vertex f_a (respectively, f_b) is incident to the face $a_g(f)$ (respectively, $b_g(f)$) in the plane embedding G_g. To finish the proof, it suffices to show that the dual distance between $a_g(f)$ and $b_g(f)$ in G_g is at most $\sum_{j=i}^{g} \ell_j$. We prove this via induction over $j = i, i+1, \ldots, g$, i.e., we show that the dual distance between $a_j(f)$ and $b_j(f)$ in G_j is at most $\ell_i + \ell_{i+1} + \cdots + \ell_j$.

This holds (with equality) for $j = i$ by the definition of ℓ_i. For $j > i$, take a shortest dual path π in G_{j-1} connecting $a_{j-1}(f)$ to $b_{j-1}(f)$. Unless π intersects γ_j, its length also bounds the dual distance in G_j. Assuming $\pi \cap \gamma_j \neq \emptyset$ in G_{j-1}, we can replace (in G_j) the section of π between the first and the last intersection with γ_j by a γ_j-switching ear of length ℓ_j. It follows that the dual distance between $a_j(f)$ and $b_j(f)$ is at most $\|\pi\| + \ell_j \leq \ell_i + \cdots + \ell_{j-1} + \ell_j$, as claimed. □

Now recall that $|F_i| = k_i$, for $i = 1, \ldots, g$. From Claim 5.1 it follows that the edges in F_i can be added to the plane embedding G_g by introducing at most $k_i \cdot \sum_{j=i}^{g} \ell_j$ crossings with the edges of G_g. This measure disregards any additionally crossings arising between edges of F_i. We add to G_g the edges of F_g, then the edges of F_{g-1}, and so on. As we add the edges of F_i, in the worst case scenario each edge we add crosses every edge already or currently inserted; thus the total cost of adding the edges of F_i is at most $k_i \cdot \sum_{j=i}^{g} \ell_j + k_i \cdot \sum_{j=i}^{g} k_j$. Overall, the edges $F_1 \cup F_2 \cup \cdots \cup F_g$ can be added to the plane embedding by introducing at most $\sum_{i=1}^{g} \Big(k_i \cdot \sum_{j=i}^{g} (k_j + \ell_j) \Big)$ crossings. Using that $2\ell_i \geq k_i$ (cf. Lemma 2.3), this process yields a drawing of G in the plane with at most

$$\begin{aligned} \sum_{i=1}^{g} \left(k_i \cdot \sum_{j=i}^{g} (k_j + \ell_j) \right) &\leq \sum_{i=1}^{g} \left(k_i \cdot \sum_{j=i}^{g} 3\ell_j \right) \\ &= 3 \sum_{j=1}^{g} \left(\ell_j \cdot \sum_{i=1}^{j} k_i \right) \end{aligned}$$

crossings. The inductive application of Lemma 2.4 yields $k_i \le 2^{j-i}k_j$ for all $1 \le i < j \le g$. Therefore

$$\begin{aligned} 3\sum_{j=1}^{g}\left(\ell_j \cdot \sum_{i=1}^{j} k_i\right) &\le 3\sum_{j=1}^{g} \ell_j k_j (2^{j-1} + \cdots + 2^1 + 2^0) \\ &= 3\sum_{j=1}^{g} k_j \ell_j (2^j - 1) \\ &\le 3 \max_{1\le i\le g}\{k_i\ell_i\} \cdot (2^1 + 2^2 + \cdots + 2^g - g) \\ &= 3 \cdot (2^{g+1} - 2 - g) \cdot \max_{1\le i\le g}\{k_i\ell_i\}. \end{aligned} \tag{3}$$

We have thus shown how to produce a drawing of G with at most $3\cdot(2^{g+1}-2-g)\cdot\max_{1\le i\le g}\{k_i\ell_i\}$ crossings. It remains to show how such a drawing can be computed efficiently from an embedding of G in Σ_g. The algorithm runs two phases:

1. A good planarizing sequence $(G_1^*, \gamma_1), \dots, (G_g^*, \gamma_g)$ for G^* is computed using g calls to the $\mathcal{O}(n \log n)$ algorithm of Kutz [24], which finds a cycle witnessing nonseparating edge-width in orientable surfaces. During the computation, we represent G^* by its rotation scheme which allows fast implementation of the cutting operation as well.

2. In the planar graph G_g, optimal insertion routes are found for all the missing edges $F = E(G) \setminus E(G_g)$ using linear-time breadth-first search in G_g^*. A key observation is that we are looking for these insertion routes only between predefined pairs of faces $a_g(f)$ and $b_g(f)$ for each $f \in F$. Since each of $\{a_g(f)\ : f \in F_i\ \}$ and $\{b_g(f)\ : f \in F_i\ \}$ has at most 2^{g-i} elements for each $i = 1, 2, \dots, g$, it follows that we need to perform at most $2^{g-1} + \cdots + 2^1 + 2^0 < 2^g$ searches in total (independently of $|F|$), a process that takes an overall linear time for fixed g. From the practical point of view, it may be worthwhile to mention that $|G_g|$ also serves as a natural upper bound for the considered faces.

In view of this, the overall runtime of the algorithm is $\mathcal{O}(n \log n)$ for each fixed g. □

6 More properties of stretch

In this section, we establish several basic properties on the stretch of an embedded graph. Even though we could have alternatively included these in the next section, as we only require them in the proof of Lemma 3.7, we prefer to present them in a separate section, for an easier further reference of the basic properties of this new parameter which may be of independent interest.

We recall that a graph property $\mathcal{P}$ satisfies the 3-*path condition* (cf. [28, Section 4.3]) if the following holds: Let T be a *theta graph* (a union of three internally disjoint paths with common endpoints) such that two of the three cycles of T do not possess $\mathcal{P}$; then neither does the third cycle. In the proof of the following lemma we make use of halfedges. A *halfedge* is a pair $\langle e, v\rangle$ ("e at v"), where e is an edge and v is one of the two ends of e.

Lemma 6.1. *Let G be embedded on an orientable surface, and let C be a cycle of G. The set of cycles of G satisfies the 3-path condition for the property of odd-leaping C. Furthermore, not all three cycles in any theta subgraph of G can be odd-leaping C.*

Proof. Let a theta graph $T \subseteq G$ be formed by three paths $T = T_1 \cup T_2 \cup T_3$ connecting the vertices s, t in G. We consider a connected component M of $C \cap T$. If $M = \emptyset$ or $M = C$, then the 3-path condition trivially holds. Otherwise, M is a path with ends m_1, m_2 in G. We denote by f_1, f_2 the edges in $E(C) \setminus E(M)$ incident with m_1, m_2, respectively, and by M^+ the union of M and the halfedges $\langle f_1, m_1\rangle$ and $\langle f_2, m_2\rangle$. We show that the number q of leaps of M^+ summed over all three cycles in T is always even.

If $m_i \notin \{s, t\}$ for $i \in \{1, 2\}$, then contracting the edge of M incident to m_i clearly does not change the number q. Iteratively applying this argument, we can assume that finally either (i) $m_1 = m_2$ (and possibly $m_1 \in \{s, t\}$), or (ii) $m_1 = s$, $m_2 = t$, and $M = T_1$. In case (i), M^+ leaps either none or two of the cycles of T in the single vertex m_1, and so $q \in \{0, 2\}$. Thus we assume for the rest of the proof that (ii) holds.

For $i = 1, 2, 3$, let e_i (respectively, e'_i) be the edge of T_i incident with s (respectively, t). By relabeling e_1, e_2, e_3 if needed, we may assume that the rotation around s is one of the cyclic permutations (e_1, f_1, e_2, e_3) or (e_1, e_2, f_1, e_3). The rotation around t could be any of the six cyclic permutations of e'_1, e'_2, e'_3, f_2. This yields a total of twelve possibilities to explore. A routine analysis shows that in every case we get $q \in \{0, 2\}$, except for the case in which the rotation around s is (e_1, e_2, f_1, e_3) and the rotation around t is (e'_1, e'_2, f_2, e'_3); in this case, M^+ leaps twice the cycle $T_2 \cup T_3$, and $q = 4$.

Altogether, the number of leaps of C summed over all three cycles in T is even. Hence the number of cycles of T which are odd-leaping with C is also even, and the 3-path condition follows. □

The next claim shows that stretch (Definition 2.6) could have been equivalently defined as an *odd-stretch*, using pairs of odd-leaping cycles instead of one-leaping cycles.

Lemma 6.2 (Odd-stretch equals stretch)**.** *Let G be a graph embedded in an orientable surface. If C, D is an odd-leaping pair of cycles in G, then $Str(G) \le \|C\| \cdot \|D\|$.*

Proof. We choose an odd-leaping pair C, D that minimizes $\|C\| \cdot \|D\|$. Up to symmetry, $\|C\| \le \|D\|$. Since $C \cap D \ne \emptyset$, there is a set $\mathcal{D} = \{D_1, \dots, D_k\}$ of pairwise edge-disjoint C-ears in D, such that $E(D_1) \cup \dots \cup E(D_k) = E(D) \setminus E(C)$. By a simple parity argument, there exists a C-switching ear in $\mathcal{D}$. Hence if $|\mathcal{D}| = 1$, then C, D are one-leaping, and the lemma immediately follows.

If more than one C-ear in $\mathcal{D}$ is switching, then we pick, say, D_1 as the shorter of these. By the choice of D we have $\|D_1\| \le \frac{1}{2}\|D\|$, and so by Lemma 2.7 we have

$$Str(G) \le \|C\| \cdot \left(\|D_1\| + \frac{1}{2}\|C\|\right) \le \|C\| \cdot \left(\frac{1}{2}\|D\| + \frac{1}{2}\|D\|\right) = \|C\| \cdot \|D\|,$$

as required.

In the remaining case, we have that $|\mathcal{D}| > 1$ and exactly one C-ear in $\mathcal{D}$ (say D_1) is switching. We pick any $D_j \in \mathcal{D}$, $j > 1$, let u, v be the ends of D_j on C, and compare the distance d between u and v on C with $\|D_j\|$. If $d > \|D_j\|$, then both cycles of $C \cup D_j$ containing D_j are shorter than $\|C\|$, and one of them is odd-leaping with D by Lemma 6.1. This contradicts the choice of C (for the pair C, D, that is). Hence $\|D_j\| \ge d$, and summing these inequalities over all $j = 1, \dots, k$ we get $\|D_1\| \le \|D\| - s$, where s is the distance between the ends of D_1 on C. Similarly as in Lemma 2.7, we then get

$$Str(G) \le \|C\| \cdot (\|D_1\| + s) \le \|C\| \cdot (\|D\| - s + s) = \|C\| \cdot \|D\|.$$ □

Lemma 6.3. *Let H be a graph embedded in an orientable surface of genus $g \geq 2$, and let $A, B \subseteq H$ be a one-leaping pair of cycles witnessing the stretch of H, such that $\|A\| \leq \|B\|$. Then $ewn(H /\!\!/ A) \geq \frac{1}{2} ewn(H)$.*

Proof. Let C be a nonseparating cycle in $H /\!\!/ A$ of length $ewn(H /\!\!/ A)$. If its lift $\hat{C}$ is a cycle again, then (since $\hat{C}$ is nonseparating in H) $ewn(H) \leq \|\hat{C}\| = ewn(H /\!\!/ A)$, and we are done. Thus we may assume that $\hat{C}$ contains an A-ear $P \subseteq \hat{C}$ such that $A \cup P$ is a theta graph. Let $A_1, A_2 \subseteq A$ be the subpaths into which the ends of P divide A. By Lemma 6.1, exactly two of the three cycles of $A \cup P$ are odd-leaping with B. One of these cycles is A; let the other one, without loss of generality, be $A_1 \cup P$. Then $\|A_1 \cup P\| \geq \|A\|$ using Lemma 6.2, and so $\|P\| \geq \|A_2\|$. Furthermore, $A_2 \cup P$ is nonseparating in H, and we conclude that

$$ewn(H) \leq \|A_2 \cup P\| \leq 2\|P\| \leq 2\|\hat{C}\| = 2ewn(H /\!\!/ A)\,. \qquad \square$$

At this point, an attentive reader may wonder why we do not use the cutting paradigm as in Lemma 6.3 in a good planarizing sequence for Theorem 3.6 (Section 5). Indeed, it would seem that the same proof as in Section 5 works in this new setting, and the added benefit would be an immediately matching lower bound in the form provided by Corollary 3.4. The caveat is that the proof of Theorem 3.6 strongly uses the fact that subsequent cuts in a planarizing sequence do not involve *much fewer* edges (recall "$k_i \leq 2^{j-i} k_j$ for all $1 \leq i < j \leq g$" from the proof). If one cuts along the shortest cycle of a pair that witnesses the dual stretch, then the number of cut edges may jump up or down arbitrarily. Thus an attempted proof along the lines of the proof we gave in Section 5 would (inevitably?) fail at this point.

7 Finding a subgraph of large stretch

In this section we prove Lemma 3.7. Therefore, we need to generalize the concepts of switching and leaping. Given an embedded graph H and an embedded subgraph $D \subset G$, we want to talk about D-switching ears, and walks that are k-leaping D, also in cases when D is a not necessarily a cycle. The essential property of a cycle used in these definitions is that it has two clearly defined sides. We generalize this feature (to subgraphs that are not necessarily cycles) to a property we call *polarity.*

7.1 Polarity

Let H be a graph cellularly embedded in a surface Σ, and let D be a (not necessarily connected) subgraph of H. The H*-induced* embedding $\tilde{D}$ of the graph D is determined by the system of H-rotations around vertices of D restricted to $E(D)$. Intuitively, $\tilde{D}$ is obtained from the usual subembedding of D in Σ via replacing all non-cellular faces with discs. Notice that $\tilde{D}$ has a separate surface for each connected component of D. If $\tilde{D}$ can be face-bicolored, then we say that D is *bipolar in* H, and we associate one chosen facial bicoloring of $\tilde{D}$ with D (notice that this bicoloring is not unique when D is not connected). We will refer to the facial colors of $\tilde{D}$ (white and black) as the D*-polarities* in H (positive and negative, respectively).

More formally, for $v \in V(D)$ and $e \notin E(D)$, the halfedge $\langle e, v \rangle$ has a *positive* (*negative*) D-polarity if the position of e in the H-rotation around v is between consecutive edges of a white (black) $\tilde{D}$-face. Clearly, a cycle in any orientable embedding is always bipolar. Also, if D is bipolar, then it is Eulerian.

A D-ear P is *D-polarity switching* if the halfedges of P incident with the ends of P are of distinct D-polarities. If D is a cycle, then being "D-polarity switching" is equivalent to being "D-switching". We now consider a (possibly closed) walk $W \subseteq H$. A proper subwalk M of W is called a *polarity leap (of W and D)* if

- $M \subseteq D \cap W$ and neither the edge f_0 preceding M in W nor the edge f_1 succeeding M in W belong to D (in particular, M is neither a prefix nor a suffix of W), and
- the halfedges of f_0, f_1 incident with M are of distinct D-polarities.

We say that W is *odd-leaping* bipolar D if the number of all proper subwalks of W which are polarity leaps is odd; otherwise W is *even-leaping* D. Notice that being "one-leaping" (Definition 2.5) implies "odd-leaping" in this new sense.

7.2 The workhorse

Informally speaking, the intuition behind our proof of Lemma 3.7 is to suitably cut down the embedding G to a smaller surface (destroying handles causing small stretch; remember our aim is to find a subgraph with large stretch), while approximately preserving γ and its switching distance.

The main tool behind the proof of Lemma 3.7 is the following lemma. To make sense of this statement, and to grasp how this easily leads to the proof of Lemma 3.7, we refer the reader to the informal discussion provided immediately after the statement.

Lemma 7.1. *Let H be a graph embedded in an orientable surface. Suppose that:*

a) there is a bipolar dual subgraph δ in H^*;

b) there exists a closed walk in H^* that is odd-leaping δ; and

c) the shortest δ-polarity switching ear in H^* has length h.

Let α, β be a one-leaping pair (any one) of dual cycles in H^* such that $\|\alpha\| \leq \|\beta\|$ and $Str^*(H) = \|\alpha\| \cdot \|\beta\|$. Then, unless (d) $\|\beta\| \geq h$, the following hold:

a') there is a bipolar dual subgraph δ_1 ("induced" by δ) in $(H/\!\!/\alpha)^*$;

b') there exists a closed walk in $(H/\!\!/\alpha)^*$ that is odd-leaping δ_1; and

c') the shortest δ_1-polarity switching ear in $(H/\!\!/\alpha)^*$ has length $h_1 \geq h - \frac{1}{2}\|\alpha\|$.

Conditions (a) and (a') address the "preservation of γ" requisite from Lemma 3.7, and (c),(c') address the necessarily long "switching distance". Conditions (b) and (b') have a purely technical purpose. Notice, for instance, that if (b) is true, then the embedding H is not planar (and so the stretch of H is well defined). Indeed, a closed walk odd-leaping a bipolar plane δ cannot exist since such a δ would equal its H^*-induced embedding $\tilde{\delta}$, which means that δ is face-bicolored, too; a simple parity argument then gives a contradiction. For a similar parity reason, (b) implies that a δ-polarity switching ear in H^* (implicitly required in (c)) must exist. Moreover, as we proceed in the cutting process, the non-planarity implied by (b') guarantees that we will eventually arrive at the desired exceptional conclusion (d) $\|\beta\| \geq h$, which is the ultimately desired outcome for Lemma 7.1.

Proof of Lemma 7.1. Recall the definition of cutting an embedding H along a dual cycle α. The dual graph $H^*/\!\!/\alpha = (H/\!\!/\alpha)^*$ is obtained from H^* by successive contractions of all the dual edges in $E(\alpha)$ into one dual vertex a, and then "splitting" a into two a_1, a_2 (giving the two α-cut faces of $H/\!\!/\alpha$). This "stepwise contraction" perspective of cutting turns out to be very useful in our proof.

Proof of (a'). Let ε denote the subgraph of H_1^* induced by the edges in $E(\delta) \setminus E(\alpha)$. If $\alpha = \delta$, then clearly (d) $\|\beta\| \geq h$, and so we may assume that ε is nonempty. We show that we can choose $\delta_1 = \varepsilon$, under the assumption that α contains a δ-polarity switching ear (the validity of this assumption follows since, if no such switching ear existed, then by (c) it would follow that $\|\beta\| \geq \|\alpha\| \geq h$, thus implying (d)).

The following is immediate from the definition of bipolarity:

Fact 7.2. *If $f \in E(H^*)$ is not a loop-edge and not a δ-polarity switching ear, then the dual graph H^*/f (obtained by* contraction *of f) is embedded in the same surface as H^*, and the dual subgraph δ' induced by $E(\delta) \setminus \{f\}$ in H^*/f is bipolar again, where the δ'-polarities are naturally inherited from the δ-polarities.*

Since we assume that α contains no δ-polarity switching ear, we can iteratively apply Fact 7.2 to all edges of α except some (the last one) $f_1 \in E(\alpha) \setminus E(\beta)$. In this way we get an "intermediate" embedding $H_1^* = H^*/\big(E(\alpha) \setminus \{f_1\}\big)$ such that f_1 is a nonseparating dual loop-edge in H_1^*, and bipolar $\varepsilon_1 \subseteq H_1^*$ is naturally derived from δ. Let a be the face of H_1 that is the double end of f_1, and let the H_1^*-rotation of edges around a be $e_1, \ldots, e_i, f_1, e'_1, \ldots, e'_j, f_1$. The last step in the construction of H_1^* (and of ε) is to remove f_1 and split a into a_1, a_2 such that the H_1^*-rotation around a_1 (respectively, a_2) is $e_1, \ldots, e_i$ (respectively, $e'_1, \ldots, e'_j$).

Clearly, $\varepsilon_1 = \varepsilon$ stays bipolar in H_1^* if $a \notin V(\varepsilon_1)$, and so we assume $a \in V(\varepsilon_1)$. Let $\tilde{\varepsilon}$ denote the H_1^*-induced embedding of ε. Let e_a and e_b be the first and last element of the list $e_1, \ldots, e_i$, respectively, that are also edges of ε. Note that both ends of f_1 in the H_1^*-rotation around a are between e_b and e_a. Then, e_b, e_a appear consecutively on a unique face x of $\tilde{\varepsilon}$. Analogously, we find a face x' at a_2 in $\tilde{\varepsilon}$. Loosely speaking, x, x' are the dual $\tilde{\varepsilon}$-faces "inheriting" the two H_1^*-faces incident with f_1. If $f_1 \notin E(\varepsilon_1)$, then both halfedges of f_1 are of the same ε_1-polarity (by our assumption on α), say positive. Hence both $\tilde{\varepsilon}$-faces x and x' will get (consistently) positive polarity, and so ε is bipolar in H_1^*.

If, on the other hand, $f_1 \in E(\varepsilon_1)$, then one of the two faces incident with f_1 in the H_1^*-induced embedding $\tilde{\varepsilon}_1$ of ε_1 is positive, say the one containing edge(s) from $e_1, \ldots, e_i$, and the other one is negative. Then the $\tilde{\varepsilon}$-face x will be (consistently) positive and x' negative. Thus also in this case $\varepsilon = \delta_1$ is bipolar in H_1^*.

Proof of (b'). As in (a'), we may assume that α contains no δ-polarity switching ear. We can make a similar assumption with β: if there is a δ-polarity switching ear contained in β, then $\|\beta\| \geq h$ (that is, (d) holds).

The following counterpart of Fact 7.2, formulated for any closed dual walk ψ in H^*, is easily derived from our definition of a leap.

Fact 7.3. *Suppose $f \in E(H^*)$ is not a loop-edge and not a δ-polarity switching ear, and denote by δ', ψ' the dual subgraphs induced by $E(\delta) \setminus \{f\}$ and $E(\psi) \setminus \{f\}$ in H^*/f (i.e., after contraction of f). Then the number of leaps of δ' and ψ' in H^*/f is the same as the number of leaps of δ and ψ in H^*, with an exception when $f \in E(\psi) \setminus E(\delta)$ and both ends of f are incident with leaps of δ and ψ in H^* (in which case the two leaps vanish in H^*/f).*

We now proceed in the same way as in (a'), and use the same notation H_1^*, f_1, a, ε_1, etc. Let ω be a dual closed walk in H^* odd-leaping δ, and ω_1, β_1 denote the dual closed walks in H_1^* induced by $E(\omega)\cap E(H_1^*)$ and $E(\beta)\cap E(H_1^*)$. By an iterative application of Fact 7.3 to all edges in $E(\alpha)\setminus\{f_1\}$, we get that the parity of leaping between δ and ω (respectively, δ and β) in H^* is the same as that between ε_1 and ω_1 (respectively, ε_1 and β_1) in H_1^*. Hence ω_1 is odd-leaping ε_1, and β_1 is even-leaping ε_1, since β contains no δ-polarity switching ear in H^* and so β is not odd-leaping δ.

We note that $a \in V(\beta_1)$ since α intersects β, and recall $f_1 \notin E(\beta)$. If $f_1 \in E(\omega)$, then we moreover remove f_1 from ω_1; this does not change the parity of leaping between ε_1 and ω_1. We say that the dual walk ω_1 *passes through* a in H_1^* if one edge of ω_1 is from $e_1, \dots, e_i$ and the next edge of ω_1 is among $e'_1, \dots, e'_j$, or vice versa. Every time ω_1 passes through a, we replace this pass by one iteration of the cycle β_1. The resulting closed dual walk ω_2 in H_1^* (which does not pass through a) is again odd-leaping ε_1, since β_1 is even-leaping ε_1. Then, the subgraph ω_0 induced by $E(\omega_2)$ in the graph H_1^* is a closed dual walk odd-leaping $\varepsilon = \delta_1$.

Proof of (c'). Let σ be a δ_1-polarity switching ear in H_1^* of length h_1. If $V(\sigma)$ contains both α-cut faces a_1, a_2, then the lift $\hat{\nu}$ of a subpath $\nu \subseteq \sigma$ between a_1 and a_2 is a δ-polarity switching ear, and hence $h \le \|\hat{\nu}\| \le h_1$, thus implying (c'). Otherwise, the lift $\hat{\sigma}$ in H^* is an $(\alpha \cup \delta)$-ear which means that, for some subpath $\pi \subseteq \alpha$ of length at most $\frac{1}{2}\|\alpha\|$ (possibly empty), $\hat{\sigma} \cup \pi$ is a δ-ear. Since σ is δ_1-polarity switching in H_1^*, and the δ_1-polarities are inherited from those of δ in H^* by (a') and Fact 7.2, we conclude that $\hat{\sigma} \cup \pi$ is a δ-polarity switching ear. Therefore, $h \le \|\hat{\sigma} \cup \pi\| \le h_1 + \frac{1}{2}\|\alpha\|$ as claimed. ☐

7.3 Proof of Lemma 3.7

We proceed by induction, using Lemma 7.1. Notice that all the conditions (a),(b),(c) of Lemma 7.1 are satisfied by the graph H, its bipolar dual cycle $\delta := \gamma$, and by $h := \ell$. Let $H_0 = H$, $\gamma_0 = \gamma$, and $\ell_0 = \ell$. Until we reach the condition (d) $\|\beta\| \ge h$, we repeatedly apply Lemma 7.1 for $i = 1, 2, \dots$ to $H := H_{i-1}$ and $\delta := \gamma_{i-1}$, $h := \ell_{i-1}$, obtaining $H_i := H /\!\!/ \alpha$ and $\gamma_i := \delta_1$, $\ell_i := h_1$. After the maximum possible number i of iterations in which (d) does not hold:

- the graph H_i has genus $g - i$, and it is $i \le g - 1$ since (b') implies nonplanarity of H_i;
- the nonseparating dual edge-width is $ewn^*(H_i) \ge 2^{-i} \cdot ewn^*(H) > 1$ (this follows by iterating Lemma 6.3 i times); and
- the shortest γ_i-polarity switching ear in H_i^* has length at least $\ell_i \ge 2^{-i} \cdot \ell$, since one can iterate $h_1 \ge h - \frac{1}{2}\|\alpha\| \ge h - \frac{1}{2}\|\beta\| \ge \frac{1}{2}h$ at each of the i steps.

Hence (as no further iteration is possible), we have gotten an $i \le g-1$ such that (cf. Lemma 7.1) there exists a pair of odd-leaping dual cycles α_i, β_i in H_i^* such that $Str^*(H_i) = \|\alpha_i\| \cdot \|\beta_i\|$, and (d) $\|\beta_i\| \ge \ell_i$ holds. Consequently,

$$Str^*(H_i) = \|\alpha_i\| \cdot \|\beta_i\| \ge ewn^*(H_i) \cdot \ell_i \ge 2^{-i} ewn^*(H) \cdot 2^{-i}\ell = 2^{-2i} \cdot k\ell \,.$$

By setting $H' = H_i$ for $g' = g - i$, Lemma 3.7 follows. ☐

8 Concluding remarks

There are several natural questions that arise.

Extension to nonorientable surfaces. One can wonder whether our results, namely about approximating planar crossing number of an embedded graph, can also be extended to nonorientable surfaces of higher genus. Indeed, the upper-bound result of [3] holds for any surface, and there is an algorithm to approximate the crossing number for graphs embeddable in the projective plane [17]. We currently do not see any reason why such an extension would be impossible.

However, the individual steps become much more difficult to analyze and tie together, since the "cheapest" cut through the embedding can cut (a) a handle along a two-sided loop, (b) an antihandle along a two-sided loop, or (c) a crosscap along a one-sided loop. Hence it then does not suffice to consider toroidal grids as the sole base case (and a usable definition of "nonorientable stretch" should reflect this), but the lower bound may also arise from a projective or Klein-bottle grid minor. Already for the latter, there are currently no non-trivial results known. We thus leave this direction for future investigation.

Dependency of the constants in Theorem 1.4 on Δ and g. Taking a toroidal grid with sufficiently multiplied parallel edges (possibly subdividing them to obtain a simple graph) easily shows that a relation between the toroidal expanse and the crossing number must involve a factor of Δ^2. Regarding an efficient approximation algorithm for the crossing number, general dependency on the maximum degree seems unavoidable as well, as is suggested by comparison with related algorithmic results. However, considering the so-called minor crossing number (see Section 8.1 below), one can avoid this dependency at least in a special case.

The exponential dependency of the constants and the approximation ratio on g, on the other hand, is very interesting. It pops up independently in multiple places within the proofs, and these occurrences seem unavoidable on a local scale, when considering each inductive step independently. However, it seems very hard to construct any example where such an exponential jump or decrease can actually be observed. It might be that a different approach with a global view can reduce the dependency in Theorem 1.4 to some $poly(g)$ factor, cf. also [12].

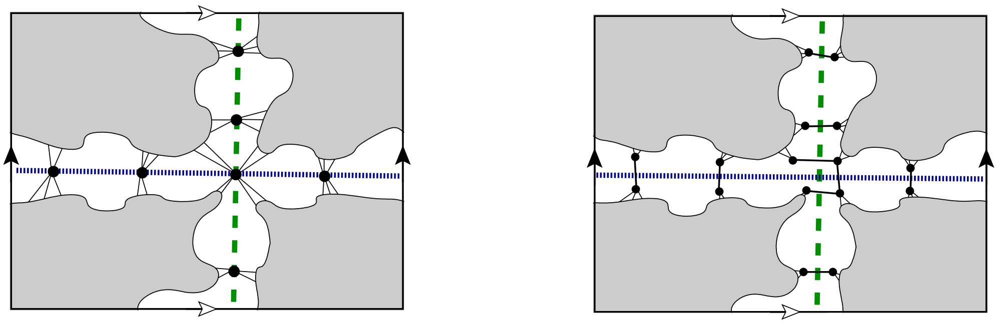
8.1 Toroidal grids and minor crossing number

The *minor crossing number* $mcr(G)$ [2] is the smallest crossing number over all graphs H that have G as their minor. Hence it is, by definition and in contrast to the traditional crossing number, a well-behaved minor-monotone parameter. In general, however, minor crossing number is not any easier to compute [18] than ordinary crossing number. We note the following intuitive observation related to our topic: if G is embedded in Σ with face-width r, then G is a surface minor of a graph H (in particular, H is embedded in Σ as well) such that $ewn(H) = r$. Indeed, consider a loop λ in Σ attaining $fw(G)$ and split every vertex intersected by λ into an edge "perpendicular" to λ. This results in desired H (for formal details, see the proof of Lemma 8.1).

For an embedded graph G, let G_f denote the vertex-face incidence (bipartite) graph of G. It is well-known that $fw(G) = \frac{1}{2}ew(G_f)$. We can analogously define the *face stretch* of an embedded graph G as $FStr(G) = \frac{1}{4}Str(G_f)$, and claim:

Lemma 8.1. *Let G be a graph embedded in an orientable surface Σ. Then there is a graph H also embedded in Σ, such that G is a minor of H and*

$$Str^*(H) \leq FStr(G) + \sqrt{FStr(G)}.$$



(a) (b)

Figure 4: (a) A toroidal embedding of a sample graph G, with the two loops defining $FStr(G)$ in thick dashed and stripy lines. (b) A toroidal embedding of a graph H such that G is a minor of H where the two loops from (a) now represent a pair of one-leaping dual cycles in H.

Proof. Let A, B be one-leaping cycles of G_f witnessing $FStr(G)$. When viewing A and B as simple loops α and β, respectively, on the surface Σ, they intersect the embedding of G only in $a = \|A\|/2$ and $b = \|B\|/2$ vertex points. Consider a vertex v of G intersected by α. We replace v in the embedding with two new vertices v_l, v_r, where v_l is incident with those edges of v on the left-hand side of α and v_r with the edges of v on the right-hand side of α. We join v_l to v_r with a new edge; it is "perpendicular" to α in the embedding in Σ (Figure 4). Let H_0 be the new graph having G as its minor. If v belongs also to β, and there is an edge (or two) of $E(B) \setminus E(A)$ in G_f incident to v, then we position the corresponding one (or two) of v_l, v_r right on this section of β close to original v. So, β intersects the embedded graph H_0 only in vertex points, as well. We apply the same construction to the vertices of H_0 intersected by β, resulting in the desired embedded graph H having G as its minor.

In H, the loop α now intersects exactly a edges (and no vertex), while the loop β intersects b or $b+1$ edges. The latter case happens when α, β intersect each other in exactly one vertex point v of G, and hence both v_l, v_r belong to β in H'. (Generally, this odd case is unavoidable in the situation illustrated in Figure 4.) Therefore, up to symmetry between α, β, H witnesses that $Str^*(H) \leq \min\{a(b+1), b(a+1)\} = ab + \min(a,b) \leq ab + \sqrt{ab}$, where $FStr(G) = ab$. □

From Lemma 2.8 we then immediately obtain:

Corollary 8.2. *If G is a graph embedded in the torus, then $\mathrm{mcr}(G) \leq FStr(G) + \sqrt{FStr(G)}$. Assuming $\mathrm{fw}(G) \geq 5$, we have $\mathrm{mcr}(G) \leq \frac{6}{5} FStr(G)$.* □

The next logical step is to translate the findings from Section 3.1 to the face stretch notion. In the special case of the torus, this translation in fact makes some things simpler. Consider a graph embedded in the torus Σ_1. Let α be a loop in Σ_1 intersecting G only in vertex points. When cutting along α we obtain a cylindrical surface Γ with two borders, corresponding to the former left and right-hand sides of α. We naturally obtain the graph G' embedded on Γ from G by duplicating the vertices v cut by α along the two borders. As in the previous proof, each copy of v in G' retains the edges formerly incident to v on the respective side of α on Σ_1. We say that G' embedded in Γ is obtained by *cutting G along α*.

Theorem 8.3. *Let G be a graph embedded in the torus Σ_1 with $k := \mathrm{fw}(G)$. Let α be a loop in Σ_1 witnessing the face-width of G, and let G' be a graph embedded in the cylinder Γ, obtained by cutting G along α. Among all pairs of points x, y on the opposite bounderies of Γ, let ℓ be the least number of points in which a simple arc from x to y in Γ intersects G', not counting x, y themselves. If $k \geq 5$, then G contains a toroidal $\lfloor 2k/3 \rfloor \times \ell$ -grid as a minor.*

Proof. Analogously to Claim 4.4 we prove that G has a set of at least ℓ pairwise disjoint cycles, all homotopic to α in Σ_1. Then we finish as in the proof of Theorem 3.2, using Theorems 1.2 and 3.1. □

Lemma 8.4. *Let G, $k \geq 5$, and ℓ be as in Theorem 8.3. Then $FStr(G) \leq 3k\ell$.*

Proof. The proof is analogous to that of Lemma 2.7, but slightly more complicated. Let γ' be the curve in Γ defining ℓ as above, and let γ denote the corresponding curve back in G in Σ_1. We can consider α and γ as a cycle and a path, respectively, in the vertex-face incidence graph G_f. Let $\alpha \cap \gamma = \{a, b\}$ (where possibly $a = b$), and let α' denote the component of $\alpha \setminus \{a, b\}$ having not more intersecting points with the drawing G than the other component. Then $\alpha' \cup \gamma$ is a noncontractible loop intersecting G in $\ell' \leq \ell + k/2 + 1$ points, as a simple case analysis shows (observe that, indeed, ℓ' may be larger than $\ell + k/2$ when some of a, b are vertices of G). In particular, $\ell' \geq k \geq 5$ and so $k/2 \leq \ell + 1$ and $\ell \geq 2$. Therefore, α and $\alpha' \cup \gamma$ define a pair of one-leaping cycles in G_f witnessing $FStr(G) \leq k\ell' \leq 3k\ell$. □

We may now conclude, in the toroidal case:

Theorem 8.5 (cf. Theorem 1.4)**.** *Let G be a graph embedded in the torus. If $\mathrm{fw}(G) \geq 5$, then*

(a) $\frac{10}{63} \cdot mcr(G) \leq Tex(G) \leq 12 \cdot mcr(G)$, and

(b) there is a polynomial time algorithm that computes a graph H having G as its minor and outputs a drawing of H in the plane with at most $76 \cdot mcr(G)$ crossings.

Proof. Let G, $k \geq 5$, and ℓ be as in Theorem 8.3. Combining Corollary 8.2 with Lemma 8.4 we get $mcr(G) \leq \frac{18}{5} k\ell$. Then, Theorem 8.3 gives $Tex(G) \geq \lfloor 2k/3 \rfloor \cdot \ell \geq \frac{4}{7} k\ell$ and the left-hand side of (a) follows. For the right-hand side, we simply use the fact that $Tex(G)$ is minor monotone and apply Corollary 2.2 to the graph witnessing $mcr(G)$.

For (b) we compute the graph H from Lemma 8.1 and apply the algorithm of Theorem 1.4. The resulting drawing of H has at most $\frac{18}{5} k\ell$ crossings by the previous, and $mcr(G) \geq \frac{1}{12} \cdot \frac{4}{7} k\ell = \frac{1}{21} k\ell$. Hence the number of crossings in H is at most $21 \cdot \frac{18}{5}\, mcr(G) \leq 76\, mcr(G)$. □

Obviously, the approximation constants in Theorem 8.5 are very rough and can likely be improved a lot. However, the important point is that these constants are independent of the maximum degree. It is interesting to ask whether Theorem 8.5 can be extended to all orientable surfaces analogously to Theorem 1.4. Although this seems quite plausible, there are complications similar to those seen already in the proofs of Lemmas 8.1 and 8.4. Consequently, the nice technical properties of stretch presented in Section 6 cannot be straighforwardly extended to face stretch, and the whole question is left for future research.

8.2 Removing the density requirement

Our algorithmic technique in Section 5 starts with a graph on a higher surface, and brings the graph to the plane without introducing too many crossings. As mentioned before, focusing only on surface-operations will inevitably require a certain lower bound on the density of the original embedding. However, we can naturally combine this algorithm with some other algorithmic results on inserting a *small* number of edges into a planar graph, to obtain a polynomial algorithm with essentially the same approximation ratio but without the density requirement. This combination of algorithms can be sketched as follows:

1. As long as the embedding density requirement of Theorem 1.4 is violated, we cut the surface along the violating loops. Let $K \subseteq E(G)$ be the set of edges affected by this; we know that $|K|$ is small, bounded by a function of g and Δ. Let $G_K := G - K$.

2. By Theorem 3.6, applied to G_K, we obtain a suitable set $F \subseteq E(G_K)$ such that $G_{KF} := G_K - F$ is plane. (F is the union of the edge sets corresponding to dual cycles in the considered dual planarizing sequence of G_K.)

3. We would like to apply independently [9] to insert the edges of K back to G_{KF} with not many crossings, and Theorem 3.6 to insert F back to G_{KF}. The number of possible mutual crossing $|F| \cdot |K|$ is neglectable, but the real trouble is that [9] is allowed to change the planar embedding of G_{KF} and hence the insertion routes assumed by Theorem 3.6 may no longer exist. Fortunately, the number of the insertion routes for F is bounded in the genus (unlike $|F|$), and so the algorithm from [9] can be adapted to respect these routes without a big impact on its approximation ratio.

Unfortunately, turning this simple sketch into a formal proof would not be short, due to the necessity to bring up many fine algorithmic details from [9]. That is why we consider another option, allowing short self-contained proof at the expense of giving a weaker approximation guarantee. We use the following simplified formulation of the main result of [9]. For a graph H and a set of edges K with ends in $V(H)$, but $K \cap E(H) = \emptyset$, let $H + K$ denote the graph obtained by adding the edges K into H.

Theorem 8.6 (Chimani and Hliněný [9])**.** *Let H be a connected planar graph with maximum degree Δ, K an edge set with ends in $V(H)$ but $K \cap E(G) = \emptyset$, and $k = |K|$. There is a polynomial-time algorithm that finds a drawing of $H + K$ in the plane with at most $d \cdot \mathrm{cr}(H + K)$ crossings, where d is a constant depending only on Δ and k. In this drawing, subgraph H is drawn planarly, i.e., all crossings involve at least one edge of K.*

An algorithmic strengthening of our Theorem 1.4 now reads:

Theorem 8.7. *Let Σ be an orientable surface of fixed genus $g > 0$, and let Δ be an integer constant. Assume G is a graph of maximum degree Δ embedded in Σ. There is a polynomial time algorithm that outputs a drawing of G in the plane with at most $c_3 \cdot \mathrm{cr}(G)$ crossings, where c_3 is a constant depending on g and Δ.*

Proof. Let r_0, c_2 be the constants from Theorem 1.4, depending on g and Δ. Recall that r_0 is nondecreasing in g, and so we may just fix it for the rest of the proof. If $ewn^*(G) < r_0\lfloor\Delta/2\rfloor$, let γ be the witnessing dual cycle of G. We cut G along γ, and repeat this operation until we arrive at an embedded graph $G_K \subseteq G$ of genus $g_K < g$ such that $ewn^*(G_K) \geq r_0\lfloor\Delta/2\rfloor$ (and hence $fw(G_1) \geq r_0$). Let $K = E(G) \setminus E(G_K)$ be the affected edges, where $|K| \leq g\, r_0\lfloor\Delta/2\rfloor$ is bounded by a constant.

If $g_K = 0$, then we simply finish by applying Theorem 8.6. Otherwise, we apply the algorithm of Theorem 3.6 to G_K, which results in a planar graph $G_{KF} \subseteq G_K$ and the edge set $F = E(G_K) \setminus E(G_{KF})$, such that F can be drawn into G_{KF} using at most $c_2 \cdot cr(G_K)$ crossings by Theorem 1.4. In this resulting drawing of G_K we replace each crossing by a new subdividing vertex. This gives a planarly embedded graph G'_K that contains a planarly embedded subdivision G'_{KF} of G_{KF}. Let $F_2 = E(G'_K) \setminus E(G'_{KF})$. Since we clearly may assume that every edge of F required at least one crossing in G_{KF}, we have $|F_2| \leq 2c_2 \cdot cr(G_K)$. Now we apply Theorem 8.6 to $H = G'_{KF}$ and K (from the previous paragraph). This gives a drawing G_F of $G'_{KF} + K$ with at most $d \cdot cr(G_{KF} + K)$ crossings in the plane. The final task is to put back the edges of F_2 into G_F; note, however, that the planar subdrawing of G'_{KF} within G_F is generally different from the original embedding of G'_{KF}.

For the latter task use the following technical claim:

Claim 8.8 (Hliněný and Salazar [21, Lemma 2.4])**.** *Suppose H is a connected graph embedded in the plane, and $e, f \notin E(H)$ are two edges joining vertices of H such that $H + f$ is a planar graph. If e can be drawn in H with ℓ crossings, then there is a planar embedding of $H + f$ in which e can be drawn with at most $\ell + 2 \cdot \lfloor\Delta(H)/2\rfloor$ crossings.*

Although [21] does not explicitly handle the algorithmic aspect of Claim 8.8, it is easily seen there that the claimed drawing of $H + f + e$ can be found in polynomial time from the assumed drawing of $H + e$ (for the algorithm of [9], for example, this is a simple special case).

Let $F_2 = \{f_1, f_2, \dots, f_a\}$. By induction on $i = 1, 2, \dots, a$, we apply Claim 8.8 to $f := f_i$ and $H := G'_{KF} + f_1 + \cdots + f_{i-1}$, and simultaneously to each e from K. As the final result we obtain a planar embedding of $G'_{KF} + F_2 = G'_K$. Into this G'_K, we can draw K with at most $|K| \cdot 2\lfloor\Delta/2\rfloor \cdot |F_2| + |K|^2/2$ additional crossings (compared to the number of crossings achieved by Theorem 8.6 to draw K into G_K). By turning the vertices of $V(G'_K) \setminus V(G_K)$ back into edge crossings of G_K this leads to a drawing of $G_K + K = G$ with at most

$$\begin{aligned} & c_2 \cdot cr(G_K) + d \cdot cr(G_{KF} + K) + |K| \cdot 2\lfloor\Delta/2\rfloor|F_2| + |K|^2/2 \\ \leq\ & c_2 \cdot cr(G_K) + d \cdot cr(G_{KF} + K) + g\, r_0\Delta^2 c_2 \cdot cr(G_K) + (g\, r_0\Delta)^2/8 \\ & \leq (c_2 + d + g\, r_0\Delta^2 c_2) \cdot cr(G) + (g\, r_0\Delta)^2/8 \end{aligned}$$

crossings where all the remaining terms are constants depending only on g and Δ. □

References

[1] Lowell W. Beineke and Richard D. Ringeisen, *On the crossing numbers of products of cycles and graphs of order four*, J. Graph Theory **4** (1980), no. 2, 145–155.

[2] Drago Bokal, Gašper Fijavž, and Bojan Mohar, *The minor crossing number*, SIAM J. Discrete Math. **20** (2006), no. 2, 344–356.

[3] Károly J. Böröczky, János Pach, and Géza Tóth, *Planar crossing numbers of graphs embeddable in another surface*, Internat. J. Found. Comput. Sci. **17** (2006), no. 5, 1005–1015.

[4] Richard Brunet, Bojan Mohar, and R. Bruce Richter, *Separating and nonseparating disjoint homotopic cycles in graph embeddings*, J. Combin. Theory Ser. B **66** (1996), no. 2, 201–231.

[5] Sergio Cabello, *Hardness of Approximation for Crossing Number*, Discrete Comp. Geom. **49** (2013), 348–358.

[6] Sergio Cabello, Markus Chimani, and Petr Hliněný, *Computing the stretch of an embedded graph*, Submitted (2013).

[7] Sergio Cabello and Bojan Mohar, *Crossing number and weighted crossing number of near-planar graphs*, Algorithmica **60** (2011), no. 3, 484–504.

[8] Markus Chimani, Petr Hliněný, and Petra Mutzel, *Vertex insertion approximates the crossing number of apex graphs*, European J. Combin. **33** (2012), no. 3, 326–335.

[9] Markus Chimani and Petr Hliněný, *A tighter insertion-based approximation of the crossing number*, Automata, languages and programming. Part I, Lecture Notes in Comput. Sci., vol. 6755, Springer, Heidelberg, 2011, pp. 122–134.

[10] Maurits de Graaf and Alexander Schrijver, *Grid minors of graphs on the torus*, J. Combin. Theory Ser. B **61** (1994), no. 1, 57–62.

[11] Matt DeVos, Ken-ichi Kawarabayashi, and Bojan Mohar, *Locally planar graphs are 5-choosable*, J. Combin. Theory Ser. B **98** (2008), no. 6, 1215–1232.

[12] Hristo N. Djidjev and Imrich Vrt'o, *Planar Crossing Numbers of Graphs of Bounded Genus*, Discrete Comput. Geom. **48** (2012), no. 2, 393–415.

[13] Vida Dujmović, Ken-ichi Kawarabayashi, Bojan Mohar, and David R. Wood, *Improved upper bounds on the crossing number*, Computational geometry (SCG'08), ACM, New York, 2008, pp. 375–384.

[14] J. R. Fiedler, J. P. Huneke, R. B. Richter, and N. Robertson, *Computing the orientable genus of projective graphs*, J. Graph Theory **20** (1995), no. 3, 297–308.

[15] Steve Fisk and Bojan Mohar, *Coloring graphs without short nonbounding cycles*, J. Combin. Theory Ser. B **60** (1994), no. 2, 268–276.

[16] Enrique Garcia-Moreno and Gelasio Salazar, *Bounding the crossing number of a graph in terms of the crossing number of a minor with small maximum degree*, J. Graph Theory **36** (2001), no. 3, 168–173.

[17] I. Gitler, P. Hliněný, J. Leaños, and G. Salazar, *The crossing number of a projective graph is quadratic in the face-width*, Electron. J. Combin. **15** (2008), no. 1, Research paper 46, 8.

[18] Petr Hliněný, *Crossing number is hard for cubic graphs*, J. Combin. Theory Ser. B **96** (2006), no. 4, 455–471.

[19] Petr Hliněný and Markus Chimani, *Approximating the crossing number of graphs embeddable in any orientable surface*, Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms, SIAM, Philadelphia, PA, 2010, pp. 918–927.

[20] Petr Hliněný and Gelasio Salazar, *Approximating the crossing number of toroidal graphs*, Algorithms and computation, Lecture Notes in Comput. Sci., vol. 4835, Springer, Berlin, 2007, pp. 148–159.

[21] ———, *On the Crossing Number of Almost Planar Graphs*, Graph Drawing 2006, Lecture Notes in Comput. Sci., vol. 4372, Springer, Berlin, 2007, pp. 162–173.

[22] Hector A. Juarez and Gelasio Salazar, *Drawings of $C_m \times C_n$ with one disjoint family. II*, J. Combin. Theory Ser. B **82** (2001), no. 1, 161–165.

[23] Marián Klešč, R. Bruce Richter, and Ian Stobert, *The crossing number of $C_5 \times C_n$*, J. Graph Theory **22** (1996), no. 3, 239–243.

[24] Martin Kutz, *Computing shortest non-trivial cycles on orientable surfaces of bounded genus in almost linear time*, Computational geometry (SCG'06), ACM, New York, 2006, pp. 430–438.

[25] Bojan Mohar, *Uniqueness and minimality of large face-width embeddings of graphs*, Combinatorica **15** (1995), no. 4, 541–556.

[26] ———, *A linear time algorithm for embedding graphs in an arbitrary surface*, SIAM J. Discrete Math. **12** (1999), no. 1, 6–26.

[27] Bojan Mohar and Neil Robertson, *Disjoint essential cycles*, J. Combin. Theory Ser. B **68** (1996), no. 2, 324–349.

[28] Bojan Mohar and Carsten Thomassen, *Graphs on surfaces*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 2001.

[29] János Pach and Géza Tóth, *Crossing number of toroidal graphs*, Topics in discrete mathematics, Algorithms Combin., vol. 26, Springer, Berlin, 2006, pp. 581–590.

[30] Richard D. Ringeisen and Lowell W. Beineke, *The crossing number of $C_3 \times C_n$*, J. Combin. Theory Ser. B **24** (1978), no. 2, 134–136.

[31] Neil Robertson and P. D. Seymour, *Graph minors. VII. Disjoint paths on a surface*, J. Combin. Theory Ser. B **45** (1988), no. 2, 212–254.

[32] ———, *Graph minors. XX. Wagner's conjecture*, J. Combin. Theory Ser. B **92** (2004), no. 2, 325–357.

[33] Alexander Schrijver, *Graphs on the torus and geometry of numbers*, J. Combin. Theory Ser. B **58** (1993), no. 1, 147–158.

[34] John Stillwell, *Classical topology and combinatorial group theory*, 2nd ed., Graduate Texts in Mathematics, vol. 72, Springer-Verlag, New York, 1993.

[35] Carsten Thomassen, *Five-coloring maps on surfaces*, J. Combin. Theory Ser. B **59** (1993), no. 1, 89–105.

[36] ———, *Trees in triangulations*, J. Combin. Theory Ser. B **60** (1994), no. 1, 56–62.

[37] David R. Wood and Jan Arne Telle, *Planar decompositions and the crossing number of graphs with an excluded minor*, New York J. Math. **13** (2007), 117–146.

[38] Xingxing Yu, *Disjoint paths, planarizing cycles, and spanning walks*, Trans. Amer. Math. Soc. **349** (1997), no. 4, 1333–1358.