Decompositions of permutations and book embeddings

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Abstract

In the influential paper in which he proved that every graph with m edges can be embedded in a book with $O(m^{1/2})$ pages, Malitz proved the existence of d-regular n-vertex graphs that require $\Omega(n^{\frac{1}{2}-\frac{1}{d}})$ pages. In view of the $O(m^{1/2})$ bound, this last bound is tight when $d > \log n$, and Malitz asked if it is also tight when $d < \log n$. We answer negatively to this question, by showing that there exist d-regular graphs that require $\Omega(n^{\frac{1}{2}-\frac{1}{2(d-1)}})$ pages. In addition, we show that the bound $O(m^{1/2})$ is not tight either for most d-regular graphs, by proving that for each fixed d, w.h.p. the random d-regular graph can be embedded in $o(m^{1/2})$ pages. We also give a simpler proof of Malitz's $O(m^{1/2})$ bound, and improve the proportionality constant.

As we investigated these questions on book embeddings, we stumbled upon, and shifted our attention to, questions about decompositions of permutations which seem to be of independent interest. For instance, we proved that if A is a $k \times n$-matrix each of whose rows is a random permutation of $[n]$, then w.h.p. there is a column permutation such that in the resulting matrix each row can be decomposed into $o(n^{1/2})$ monotone decreasing subsequences.

Keywords: Book thickness, pagenumber, book embedding, random graph, permutation, decreasing subsequence

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1 Introduction

We recall that the *book with k pages* is the topological space $\mathscr{B}_k$ that consists of a line (the *spine*) plus and k half-planes (the *pages*), such that the boundary of each page is the spine. A *k-page book embedding* (or simply a *k-page embedding*) of a graph G is an embedding of G into $\mathscr{B}_k$ in which the vertices are on the spine, and each edge is contained in one page. If the linear order of the vertices in the spine is π, then the book is a *π-book*.

Book embeddings were introduced by Kainen [15], and later investigated by Bernhart and Kainen [4]. In their seminal paper [7], Chung, Leighton and Rosenberg investigated several theoretical and algorithmical aspects of book embeddings. In [7], several applications of this problem were discussed, such as sorting with parallel stacks, single-row routing, fault-tolerant processor arrays, and Turing machine graphs.

Trivially, any finite graph can be embedded in a book with sufficiently many pages; the natural goal is to use as few pages as possible. Given a graph G, the minimum k such that G can be embedded in a k-page book is the *book thickness* (or *pagenumber*) of G. Determining the pagenumber of an arbitrary graph is NP-complete [7]. Few results are known for particular families of graphs. It is not difficult to show that the pagenumber of the complete graph K_n is $\lceil n/2 \rceil$. On the other hand, with few exceptions, the pagenumbers of the complete bipartite graphs $K_{m,n}$ are unknown (see [8, 13]).

The pagenumbers of graphs embeddable in a given surface have also been investigated. Bernhart and Kainen had conjectured in [4] the existence of graphs with bounded orientable genus and arbitrarily large pagenumber. This was disproved by Heath and Istrail [14], who showed that graphs of (orientable or nonorientable) genus g have pagenumber $O(g)$. Malitz [18] improved this to $O(g^{1/2})$, which is a sharp bound, as witnessed by the complete graphs. Some additional results are known for some low genus surfaces. Yannanakis proved [28] that every planar graph can be embedded in four pages. Endo [12] proved that every toroidal graph can be embedded in a book with at most seven pages, and Nakamoto et al. [20] recently proved that five pages always suffice to embed any toroidal bipartite graph. Shahrokhi et al. investigated the related problem in which the number of pages is fixed, and the goal is to minimize the number of edge crossings [23].

In their quest for general lower and upper bounds, Chung, Leighton, and Rosenberg [7] showed that d-regular graphs on n vertices have pagenumber $O(dn^{1/2})$, and proved the existence of such graphs requiring $\Omega\left(\frac{n^{1/2-1/d}}{\log^2 n}\right)$ pages. Malitz [19] tightened these bounds, establishing a general $O(m^{1/2})$ bound for graphs with m edges (i.e., not only for bounded degree graphs), and showing the existence of d-regular graphs with pagenumber $\Omega(\sqrt{d} \cdot n^{1/2-1/d})$.

Malitz observed that (in view of the $O(m^{1/2})$ result) the bound $\Omega(\sqrt{d} \cdot n^{1/2-1/d})$ is tight for $d > \log n$, and he asked if it is tight also for $d < \log n$. In this paper we answer negatively to this question:

Theorem 1. *The pagenumber of the random d-regular graph on n vertices is w.h.p. at least*

$$c_d \cdot \left(\frac{n}{\log n}\right)^{\frac{1}{2}-\frac{1}{2(d-1)}},$$

where c_d is a constant that depends only on d.

Moreover, we show that the answer is negative even in the bipartite case:

Theorem 2. *The pagenumber of the random bipartite d-regular graph on n vertices is w.h.p. at least*

$$c_d \cdot \left(\frac{n}{\log n}\right)^{\frac{1}{2}-\frac{1}{2(d-1)}},$$

where c_d is a constant that depends only on d.

Regarding upper bounds for d-regular graphs, the Chung-Leighton-Rosenberg bound and the Malitz bound are essentially the same for each fixed d, namely $O(n^{1/2})$. In this direction, we prove that the pagenumber of most d-regular graphs is actually smaller:

Theorem 3. *The pagenumber of the random d-regular graph on n vertices is w.h.p. at most*

$$C_d \cdot n^{\frac{1}{2}-\frac{1}{2+8\cdot 3^{d-2}}},$$

where C_d is a constant that depends only on d.

We have a corresponding statement for the bipartite case:

Theorem 4. *The pagenumber of the random d-regular bipartite graph on n vertices is w.h.p. at most*

$$C_d \cdot n^{\frac{1}{2}-\frac{1}{2+8\cdot 3^{d-2}}},$$

where C_d is a constant that depends only on d.

It remains an open question whether or not for each fixed d, the pagenumber of all d-regular graphs is $o(n^{1/2})$.

Malitz [19] gave a Las Vegas algorithm to embed a graph with m edges in $31m^{1/2}$ pages. Shahrokhi and Shi [22] improved this bound to $(tm)^{1/2}$ for t-partite graphs, and also gave a deterministic polynomial time algorithm for these graphs.

For general graphs, Malitz's $31m^{1/2}$ bound is still the best known. Using the techniques we developed to prove the statements given above, we improve on this result and provide a somewhat simpler proof.

Theorem 5. *Let G be a graph with n vertices and m edges. Let π be a random linear ordering of the vertices of G. If we place the vertices on the spine in the order given by π, then w.h.p. the edges of G can be embedded into at most $11m^{1/2}$ pages.*

With the original motivation of investigating these problems, we stumbled upon (and shifted our attention to) questions about decompositions of permutations which are of independent interest. The quest for subsequences of permutations with special properties is of great interest in combinatorics. Notable examples include the longest increasing subsequence [2, 3], the longest common subsequences of two permutations [16, 17], the longest alternating subsequences of permutations [24], and the longest subsequences avoiding a given pattern [1]. Let us now present one such result, which we find particularly interesting.

Let $A = \{a_{i,j}\}_{i\in[k],j\in[n]}$ be a $k \times n$ matrix, where each of the k rows is a permutation of $[n]$. Let $\mu = \mu(A)$ be the minimum number over all column permutation of A, such that each row of A can be decomposed into at most μ monotone decreasing subsequences. For n sufficiently large compared to k, it is not difficult to show that a random column permutation yields $\mu \leq 3\sqrt{n}$; moreover, it is not hard to see that this bound is tight within a constant factor (see Section 6 for more details). The problem is much more interesting when each row is a random permutation. In this case, we can prove a bound of $o(n^{1/2})$:

Theorem 6. *Let k be a fixed integer. Let A be a $k \times n$ matrix, each of whose rows is a random permutation of $[n]$, chosen independently of each other. Then w.h.p. $\mu(A) \leq 3n^{\frac{1}{2}-a_k}$, where $a_k := 1/(2^{k+1} - 2)$.*

The rest of this paper is structured as follows.

In Section 2 we establish some basic results on decompositions of permutations into monotone subsequences, which are a major tool to tackle book embedding problems. The proofs of Theorems 1 and 2 are given on Section 3; the proofs of Theorems 3 and 4 are given in Section 4; and the proof of Theorem 5 is in Section 5. The proof of Theorem 6, as well as further discussions and results on decompositions of permutations, are given in Section 6.

Throughout this paper, $\log x$ means the natural logarithm of x. For simplicity, we often omit explicitly taking the integer part of a quantity; this practice has no effect in the (asymptotic) results we are interested on in this work.

2 Decomposing permutations into decreasing sequences

The motivation to investigate decompositions of a permutation (of a set or multiset) into monotone decreasing subsequences is given by the following lemma. Given a permutation π of a set S, and $i, j \in S$, we write $i \leq_\pi j$ if i appears before j in π, and define $\geq_\pi$ similarly.

Lemma 7. *Let $M = \{a_1b_1, a_2b_2, \ldots, a_sb_s\}$ be a matching, and let π be a permutation of $a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_s$ such that $a_1 \geq_\pi a_2 \geq_\pi \cdots \geq_\pi a_s$ and $b_s \geq_\pi b_{s-1} \geq_\pi b_{s-2} \geq_\pi \cdots \geq_\pi b_1$. Then M can be embedded into a π-book with 2 pages.*

Proof. Let k be the smallest integer such that $a_k \geq_\pi b_k$ (if no such integer exists, then let $k = s + 1$). Since $a_1 \geq_\pi a_2 \geq_\pi \ldots \geq_\pi a_{k-1} \geq_\pi b_{k-1} \geq_\pi b_{k-2} \geq_\pi \ldots \geq_\pi b_1$, it follows that all the edges $a_1b_1, a_2b_2, \ldots a_{k-1}b_{k-1}$ can be embedded in a single page. If $k = s + 1$ then we are done; suppose then that $k \leq s$. Then, since $b_s \geq_\pi b_{s-1} \geq_\pi \ldots \geq_\pi b_k \geq_\pi a_k \geq_\pi a_{k+1} \geq_\pi \ldots \geq_\pi a_s$, it follows that all the edges $a_kb_k, a_{k+1}b_{k+1}, \ldots a_sb_s$ can be embedded in a single page. □

The main tool to decompose a sequence into (few) decreasing sequences is to invoke the close relationship between such a decomposition and the length of the longest increasing subsequence.

In his alternative proof of the Erdős-Szekeres theorem [10], Blackwell [5] describes a canonical (i.e., *leftmost maximal*) decomposition of a sequence of integers into monotone decreasing sequences. He shows that if a sequence S gets partitioned into t monotone decreasing sequences, then S has a monotone increasing subsequence of length t. This implies the following:

Proposition 8. *Let S be a sequence of distinct integers. If the length of the longest increasing subsequence of S is ℓ, then S can be decomposed into ℓ decreasing subsequences.* □

In the particular case of a random permutation of integers, we have the following well-known fact:

Lemma 9. *Let π be a random permutation of a set of n distinct integers. Then w.h.p. π can be decomposed into at most $3\sqrt{n}$ decreasing subsequences.* □

Combining this last result with Lemma 7, we obtain the following:

Corollary 10. *Let $M = \{a_1b_1, a_2b_2, \ldots, a_sb_s\}$ be a matching. Let π be a permutation of the vertices obtained by the concatenation of a random permutation of $\{a_1, a_2, \ldots, a_s\}$ followed by a random permutation of $\{b_1, b_2, \ldots, b_s\}$. Then w.h.p. M can be embedded in a π-book with at most $6\sqrt{s}$ pages.* □

3 Proof of Theorems 1 and 2

The strategy of the proofs is as follows. Let d be fixed. For each positive integer p, let $\mathcal{G}_p(n)$ (respectively, $\mathscr{B}_p(n)$) denote the set of d-regular (respectively, bipartite d-regular) labelled graphs on n vertices that can be embedded in p pages. Let $\mathcal{G}^d(n)$ (respectively, $\mathscr{B}^d(n)$) denote the set of d-regular (respectively, bipartite d-regular) labelled graphs. Thus the goals are to show that $|\mathcal{G}_p(n)|/|\mathcal{G}^d(n)|$ is $o(1)$ (Theorem 1) and that $|\mathscr{B}_p(n)|/|\mathscr{B}^d(n)|$ is also $o(1)$ (Theorem 2). Note that since $\mathscr{B}_p(n) \subseteq \mathcal{G}_p(n)$ and $\mathscr{B}^d(n) \subseteq \mathcal{G}^d(n)$, both quotients are less than or equal to $|\mathcal{G}_p(n)|/|\mathscr{B}^d(n)|$, and so it suffices to show that this last quotient is $o(1)$. We achieve this by establishing an upper bound for $|\mathcal{G}_p(n)|$ (Lemma 11), and then invoking a lower bound for $|\mathscr{B}^d(n)|$.

Lemma 11. *Let ϵ be any (small enough) positive number. Let $\mathcal{G}_p(n)$ denote the set of d-regular labelled graphs on n vertices that can be embedded in $p := \frac{1}{2\epsilon^{1/(1-d)}} \cdot \left(\frac{n}{\log n}\right)^{\frac{1}{2}-\frac{1}{2(d-1)}}$ pages. Then, for all sufficiently large n,*

$$|\mathcal{G}_p(n)| \le \left(\epsilon \left(\frac{2e}{d}\right)^{d/2} \cdot e^{\frac{1}{d-1}}\right)^n \cdot n^{dn/2}.$$

Proof. Let $s := (\frac{n}{\epsilon^2 \cdot \log n})^{\frac{1}{1-d}}$ and $t := \epsilon \cdot s^{-\frac{d}{2}}$, so that $p = st/2$. Note that $st^2 = n/\log n$.

Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of all graphs in $\mathcal{G}_p(n)$. Let $G \in \mathcal{G}_p(n)$, and consider a fixed embedding of G into p pages. We associate to G a *block graph* B_G, also embedded into p pages, with vertices $b_1, b_2, \ldots, b_t$ placed on the spine in this order, defined as follows. Suppose that in the p-page embedding of G the vertices appear on the spine in the order $v_{i_1}, v_{i_2}, \ldots, v_{i_n}$. For simplicity, let us assume that t divides n. For $j = 1, 2, \ldots, n/t$, let B_j be the set (or *block*) of vertices $\{v_{i_{(j-1)t+1}}, v_{i_{(j-1)t+2}}, \ldots, v_{i_{jt}}\}$. For $k, \ell \in \{1, 2, \ldots, n/t\}$, let vertices b_k, b_ℓ be adjacent in B_G if and only if G has a vertex in B_k adjacent to a vertex in B_ℓ. We ask that B_G has no parallel edges, but allow the possibility of loops (at most one loop per vertex). Thus B_G gets unambiguously defined.

The given p-page embedding of G naturally induces a p-page embedding of B_G. Now in any p-page embedding of such a graph on t vertices (without parallel edges and at most one loop per vertex), each page contains at most $t-2$ edges joining non-neighboring vertices, there are at most $t-1$ edges joining neighboring vertices, and at most t loops. Thus B_G has at most $p(t-2) + (t-1) + t = t(p+2) - 2p - 1 < 2pt = st^2$ edges (for the strict inequality we use that $p \geq 2$).

Each edge of a block graph joins an unordered pair of vertices in $\{b_1, b_2, \ldots, b_t\}$, and there are $\binom{t}{2} + t = (t+1)^2/2$ such unordered pairs (recall that one loop per vertex is allowed). Since each block graph has at most st^2 edges, it follows that the total number of distinct possible block graphs is at most

$$\sum_{i=1}^{st^2} \binom{\frac{(t+1)^2}{2}}{i} \leq st^2 \cdot \binom{\frac{(t+1)^2}{2}}{st^2} \leq st^2 \cdot \left(\frac{e \cdot (t+1)^2}{2st^2}\right)^{st^2} < \left(\frac{e}{s}\right)^{st^2}. \qquad (1)$$

Next we estimate (upper bound) how many graphs in $\mathcal{G}_p$ can possibly get mapped to a given block graph H with vertices $b_1, b_2, \ldots, b_t$ (and respective blocks $B_1, B_2, \ldots, B_t$) and edge set F.

First we note that there are fewer than t^n ways in which the vertices $v_1, v_2, \ldots, v_n$ can be assigned to the blocks $B_1, B_2, \ldots, B_t$. Now fix any such assignment of vertices to blocks. Then for each edge in F, say joining b_i to b_j, there are $(n/t)(n/t)$ pairs (that is, potential edges) with one element in b_i and another element in b_j. Since a graph in $\mathcal{G}_p$ has exactly $dn/2$ edges, it follows that for any assignment of vertices to blocks, there are at most $\binom{|F|(n/t)^2}{dn/2}$ possible graphs in $\mathcal{G}_p$ having H as its block graph. Since there are fewer than t^n possible assignments of vertices to blocks, we have that there are fewer than

$$t^n \cdot \binom{|F|(n/t)^2}{dn/2} \leq t^n \cdot \binom{st^2 \cdot (n/t)^2}{dn/2} \leq t^n \cdot \left(\frac{2esn}{d}\right)^{dn/2}$$

graphs in $\mathcal{G}_p$ associated to each block graph.

Using this last expression and (1), it follows that

$$|\mathcal{G}_p| \leq t^n \left(\frac{2esn}{d}\right)^{dn/2} \left(\frac{e}{s}\right)^{st^2} \leq \left(t \left(\frac{2es}{d}\right)^{d/2} \left(\frac{e}{s}\right)^{\frac{1}{\log n}} \right)^n \cdot n^{dn/2} < \left(\epsilon \left(\frac{2e}{d}\right)^{d/2} \cdot e^{\frac{1}{d-1}} \right)^n \cdot n^{dn/2},$$

where in this last step we used the equality $ts^{d/2} = \epsilon$ (which follows from the definition of t) and the inequality $(e/s)^{1/\log n} < e^{1/(d-1)}$, which follows easily from the definition of s. $\square$

We now derive an lower bound for $\mathscr{B}^d(n)$. We know from [21] that asymptotically

$$
\begin{aligned}
|\mathscr{B}^d(n)| &\approx e^{-(d-1)^2/2} \cdot \left(\frac{dn}{2}\right)! \cdot \left(d!\right)^{-n} \\
&\approx e^{-(d-1)^2/2} \cdot \sqrt{\pi d n}\left(\frac{dn}{2e}\right)^{dn/2} \left(\sqrt{2\pi d}\left(\frac{d}{e}\right)^{d}\right)^{-n} \\
&\approx e^{-(d-1)^2/2} \cdot \sqrt{\pi d n}\left(\frac{dn}{2e}\right)^{dn/2} \left(\sqrt{2\pi d}\left(\frac{d}{e}\right)^{d}\right)^{-n} \\
&= e^{-(d-1)^2/2} \cdot \sqrt{\pi d n} \cdot \left(\frac{e^{d/2}}{\sqrt{2\pi} 2^{d/2} d^{(d+1)/2}}\right)^{n} \cdot n^{dn/2} \\
&> e^{-(d-1)^2/2} \cdot \left(\frac{e^{d/2}}{\sqrt{2\pi} 2^{d/2} d^{(d+1)/2}}\right)^{n} \cdot n^{dn/2}. \qquad (2)
\end{aligned}
$$

Proofs of Theorems 1 and 2. Let $\epsilon := (6 \cdot 2^d \cdot d^{1/2})^{-1}$, and $p := \frac{1}{2\epsilon^{1/(1-d)}} \cdot \left(\frac{n}{\log n}\right)^{\frac{1}{2} - \frac{1}{2(d-1)}}$.

Now:

(a) The probability that a randomly chosen d-regular n-vertex graph can be embedded into p pages equals $|\mathcal{G}_p(n)|/|\mathcal{G}^d(n)|$.

(b) The probability that a randomly chosen bipartite d-regular n-vertex graph can be embedded into p pages equals $|\mathscr{B}_p(n)|/|\mathscr{B}^d(n)|$.

Since $\mathscr{B}^d(n) \subseteq \mathcal{G}^d(n)$ and $\mathscr{B}_p(n) \subseteq \mathcal{G}_p(n)$, we have the obvious inequalities

$$
\frac{|\mathcal{G}_p(n)|}{|\mathcal{G}^d(n)|} \le \frac{|\mathcal{G}_p(n)|}{|\mathscr{B}^d(n)|} \quad \text{and} \quad \frac{|\mathscr{B}_p(n)|}{|\mathscr{B}^d(n)|} \le \frac{|\mathcal{G}_p(n)|}{|\mathscr{B}^d(n)|}. \qquad (3)
$$

Using Lemma 11 and (2), we have

$$
\frac{|\mathcal{G}_p(n)|}{|\mathscr{B}^d(n)|} \le \frac{\left(\epsilon(2e/d)^{d/2} \cdot e^{\frac{1}{d-1}}\right)^{n} \cdot n^{dn/2}}{e^{-(d-1)^2/2} \cdot \left(\frac{e^{d/2}}{\sqrt{2\pi} 2^{d/2} d^{(d+1)/2}}\right)^{n} \cdot n^{dn/2}} = \frac{\left(\epsilon\sqrt{2\pi}\ 2^d d^{1/2} \cdot e^{\frac{1}{d-1}}\right)^{n}}{e^{-(d-1)^2/2}}.
$$

Recalling that $\epsilon := (6 \cdot 2^d \cdot d^{1/2})^{-1}$, we get that this quotient goes to 0 as n goes to infinity, as it is easy to check that $\epsilon\sqrt{2\pi}\ 2^d d^{1/2} \cdot e^{\frac{1}{d-1}} < 1$. Therefore $|\mathcal{G}_p(n)|/|\mathscr{B}^d(n)|$ is $o(1)$.

Thus it follows from (a) and the first inequality in (3) that w.h.p. the pagenumber of a randomly chosen d-regular n-vertex graph is at least p. Similarly, it follows from (b) and the second inequality in (3) that w.h.p. the pagenumber of a randomly chosen bipartite d-regular n-vertex graph is at least p. Thus Theorems 1 and 2 follow, with $c_d := 1/(2\epsilon^{1/(1-d)}) = (1/2)(6 \cdot 2^d \cdot d^{1/2})^{1/(1-d)}$. $\square$

4 Proof of Theorems 3 and 4

For most of this section we work on random d-regular graphs (Theorem 3). The adjustments needed for random bipartite d-regular graphs (Theorem 4) will be described at the end of the section.

We use the following model for the d-regular random graph. Let $M^1, \ldots, M^d$ be d matchings on n labelled vertices, chosen independently and uniformly at random, and let $G(n, d)$ be their union. This is sufficiently close to the uniform model [27], as long as d is a constant and n is sufficiently large.

Thus in order to establish Theorem 3 it suffices to prove that w.h.p. $M^1 \cup M^2 \cup \cdots \cup M^d$ can be embedded in at most $C_d \cdot n^{\frac{1}{2} - \frac{1}{2+8\cdot 3^{d-2}}}$ pages, where C_d depends only on d.

Setup and strategy

For each edge we randomly assign one endpoint as a *head*, and the other as a *tail*. We let H^i (respectively, T^i) denote the set of heads (respectively, tails) in M^i. Now for each vertex u and each $i \in \{1, 2, \ldots, d\}$, we let $M^i(u)$ denote the vertex matched to u under M^i.

We use a randomized algorithm to order the vertices along the spine, using d steps. At the beginning of Step $t+1$, for $0 \le t \le d-1$, we have a linear ordering of the vertices which is a concatenation of blocks. Throughout this section, a *block* is simply an ordered set of vertices. Roughly speaking, in Step $t+1$ we (i) deterministically refine and rearrange the block partition, so that M^{t+1} can be embedded in relatively few pages; then we (ii) refine again the partition, subdividing each block; and finally (iii) randomly reorder the vertices within each (smaller) block. The blocks themselves do not get rearranged in the process, in the sense that in each step of the iteration, only the order of the vertices inside a block is changing. That is, if u is in block A and v is in block B, and A is to the left of B, then u will always remain to the left of v. This last property is essential: after accomodating the vertices in Step $t+1$ so that M^{t+1} can be embedded into relatively few pages, we want in the subsequent steps to destroy as little as possible what has been achieved for M^{t+1}.

The algorithm

Define the sequence of integers $k_0, k_1, k_2, \ldots, k_t$ as follows: $k_0 := 1$, $k_1 := n^{1/(1+4\cdot 3^{d-2})}$, and $k_i := k_{i-1}^3$ for $1 < i \le d$. For simplicity we assume that k_1 (and hence every k_i) is an integer that divides n.

Step 0. Place the vertices along the spine, in any order, defining the initial block $A^0 = A^0_1$.

Step $t+1$, for $0 \le t \le d-1$. When we enter this step the vertices are placed in the spine as a block $\mathcal{A}^t$ (attained in Step t), which is the concatenation of blocks $A^t_1, \ldots, A^t_{k_t}$. At the end of the step, the vertices will have been reordered into a block $\mathcal{A}^{t+1}$, which will be the concatenation of blocks $A^{t+1}_1, \ldots, A^{t+1}_{k_{t+1}}$. This is done by following these substeps:

(a) In this substep we partition each A^t_i. The idea is first to identify, for each vertex u in A^t_i, whether it is a head or a tail in M^{t+1}, and then to identify in which block A^t_j its matching vertex $M^{t+1}(u)$ lies. Formally, for each $i, j \in [k_t]$, let

$$H_i^{t+1}(j) := \{u \in A_i^t \cap H^{t+1} \ : \ M^{t+1}(u) \in A_j^t\}, \ \text{and}$$
$$T_i^{t+1}(j) := \{u \in A_i^t \cap T^{t+1} \ : \ M^{t+1}(u) \in A_j^t\}.$$

Thus, for each fixed i, A_i^t is the disjoint union $H_i^{t+1}(1) \cup H_i^{t+1}(2) \cup \cdots \cup H_i^{t+1}(k_t) \cup T_i^{t+1}(1) \cup T_i^{t+1}(2) \cup \cdots \cup T_i^{t+1}(k_t)$.

Note that for each edge e of M^{t+1} there exist $i, j \in [k_t]$ such that e matches a vertex in $H_i^{t+1}(j)$ to a vertex in $T_j^{t+1}(i)$.

(b) For each $i, j \in [k_t]$, $H_i^{t+1}(j)$ and $T_i^{t+1}(j)$ are sets, and in this substep we turn them into blocks (recall that a block is an ordered set) as follows. First we let each $H_i^{t+1}(j)$ become a block by simply letting its elements inherit the order from A_i^t. Now suppose that for a particular pair i, j the block $H_i^{t+1}(j)$ reads $u_1 u_2 \cdots u_r$. Then the elements of $T_j^{t+1}(i)$ are $M^{t+1}(u_1), M^{t+1}(u_2), \ldots, M^{t+1}(u_r)$. We turn $T_j^{t+1}(i)$ into a block by letting its elements be ordered as $M^{t+1}(u_r)\, M^{t+1}(u_{r-1}) \cdots M^{t+1}(u_1)$.

(c) Let B_i^{t+1} be the block defined by the concatenation

$$H_i^{t+1}(i-1), \ldots, H_i^{t+1}(1), H_i^{t+1}(k_t), H_i^{t+1}(k_t-1), \cdots, H_i^{t+1}(i), T_i^{t+1}(1), T_i^{t+1}(2), \ldots, T_i^{t+1}(k_t).$$

Thus A_i^t and B_i^{t+1} have the same elements, only differently ordered.

(d) Let $\mathcal{B}^{t+1}$ be the block defined by the concatenation

$$\mathcal{B}^{t+1} := B_1^{t+1}, B_2^{t+1}, \ldots, B_{k_t}^{t+1}.$$

Thus $\mathcal{B}^{t+1}$ is an ordering along the spine of all the vertices of G. The key property of this ordering is the following immediate consequence of how the blocks B_i^{t+1} are constructed:

Remark 12. *In the ordering $\mathcal{B}^{t+1}$, for each $i, j \in [k_t]$ all the edges of M^{t+1} that have their heads in $H_i^{t+1}(j)$ can be simultaneously embedded in one page. Thus all the edges of M^{t+1} with its head in B_i^{t+1} can be simultaneously embedded in k_t pages.*

If we were to stop the process at this point, it follows from this remark that all the M^{t+1}-edges could be embedded in a book with k_t^2 pages. However, unless we are already in Step d (the last step), there are still iterations to be performed. (Actually, if we are already in Step d, the next last substep is unnecessary, and thus we omit it.) The crucial idea is to preserve as much as possible of what we have achieved for M^{t+1} in the subsequent reorderings. This is done by further refining the basic elements of the partition $\mathcal{B}^{t+1}$ (the blocks $H_i^{t+1}(j)$ and $T_i^{t+1}(j)$) and then reshuffling the vertices inside these refined subblocks, but without changing the relative order of these subblocks. This feature of not changing the relative order of the subblocks, allows us to do in the next step a reordering suitable for the edges of M^{t+2}, without totally destroying what we have already achieved for M^{t+1}.

Formally, this last substep of further refining and randomly shuffling is the following.

Note: *If we are already on Step d, we let $\mathcal{A}^d := \mathcal{B}^d$, and stop, omitting the next substep.*

(e) Working with the ordering $\mathcal{B}^{t+1}$, partition each of the blocks $H_i^{t+1}(j)$ and $T_i^{t+1}(j)$ (there are $k_t \cdot 2k_t = 2k_t^2$ such blocks in total) into $k_t/2$ blocks of sizes as equal as possible (in the particular case $t = 0$, partition each of these $2k_0^2 = 2$ blocks into $k_1/2$ blocks of sizes as equal as possible). Thus the total number of such blocks is k_{t+1}; indeed, if $t = 0$, there are $2 \cdot k_1/2 = k_1$ such blocks, and in the case $t > 0$ there are $2k_t^2 \cdot k_t/2 = k_t^3 = k_{t+1}$ such blocks. Finally, randomly reorder the vertices inside each of these k_{t+1} blocks, and denote the resulting block system $A_1^{t+1}, \dots, A_{k_{t+1}}^{t+1}$. The final ordering $\mathcal{A}^{t+1}$ is simply the concatenation $A_1^{t+1}, \dots, A_{k_{t+1}}^{t+1}$.

Conclusion. After finishing Step d, we have an ordering $\mathcal{A}^d$ of the vertices along the spine. This *final ordering* $\mathcal{A}^d$ is the one we shall use to embed all the edges in $M^1 \cup \dots \cup M^d$.

Analysis of the algorithm: expected number of pages

The key step (Claim B below) is to estimate the number of pages in which M^{t+1} can be embedded. To achieve this, we first estimate the size of the blocks A_i^{t+1}, as follows.

Claim A. *Let* $t \in \{0, 1, \dots, d-2\}$. *Then w.h.p.* $\max\{|A_\ell^{t+1}|\}_{\ell \in [k_{t+1}]} \leq 2^{2^t} n/k_{t+1}$.

Proof. We proceed by induction on t. In the case $t = 0$, the first step of the algorithm, we simply partition the vertices into two blocks H_0^1 and T_0^1 (the M^1-heads and the M^1-tails), and then partition each of these blocks into $k_1/2$ parts as equal as possible, thus obtaining $A_1^1, \dots, A_{k_1}^1$. Thus each A_i^1 has size $n/k_1 < 2^{2^0} n/k_1$. Thus the statement holds for $t = 0$.

Suppose now that $t \geq 1$. Recall that $H_i^{t+1}(j) := \{u \in A_i^t \cap H^{t+1} \ : \ M^{t+1}(u) \in A_j^t\}$. For each $\ell \in [k_{t+1}]$, there exist $i, j \in [k_t]$ such that the block A_ℓ^{t+1} is obtained by subdividing into $k_t/2$ parts, as equal as possible, either the block $H_i^{t+1}(j)$ or the block $T_i^{t+1}(j)$. Thus it suffices to show that w.h.p. $\max_{i,j \in [k_t]}\{|T_i^{t+1}(j)|\} \leq (2^{2^t} n/k_{t+1})(k_t/2)$ and $\max_{i,j \in [k_t]}\{|H_i^{t+1}(j)|\} \leq (2^{2^t} n/k_{t+1})(k_t/2)$. We show the first inequality, as the proof for the second one is totally analogous.

By the inductive hypothesis, the probability $|A_i^t|/n$ that a vertex u is in A_i^t is w.h.p. at most $(2^{2^{t-1}} n/k_t)/n = 2^{2^{t-1}}/k_t$. Since such a u is equally likely to be in H^{t+1} as in T^{t+1}, the probability that u is in $A_i^t \cap H^{t+1}$ is then w.h.p. at most $2^{2^{t-1}}/2k_t$. Now the probability that $M^{t+1}(u)$ is in A_j^t is $|A_j^t|/n$, which is w.h.p. at most $2^{2^{t-1}}/k_t$. Thus $|H_i^{t+1}(j)|$ is w.h.p. at most $2^{2^t}/2k_t^2$. Thus the probability that a vertex is in A_i^t is w.h.p. at most $2^{2^t}/k_t^2$, and so the size of A_i^t is w.h.p. at most $2^{2^t} n/k_t^2$. A concentration argument using Chernoff's inequality then shows that w.h.p. $\max_{i,j \in [k_t]}\{|H_i^{t+1}(j)|\} \leq 2 \cdot (2^{2^t} n/k_t^2) = (2^{2^t} n/k_{t+1})(k_t/2)$, as required. $\square$

Claim B. *For each* $t \in \{0, 1, \dots, d-2\}$, *w.h.p.* M^{t+1} *can be embedded into at most* $6k_t \cdot 2^{2^{t-1}} \sqrt{n/k_{t+1}}$ *pages.*

Proof. The core of the proof is to estimate (upper bound), for each $i \in [k_t]$, the number of pages in which one can embed w.h.p. the M^{t+1}-edges whose head is in the block B_i^{t+1}.

So let $i \in [k_t]$ be fixed. The subblock of B_i^{t+1} that contains the vertices that are heads of M^{t+1}-edges is

$$H_i^{t+1}(i-1) \cdots H_i^{t+1}(1)\, H_i^{t+1}(k_t)\, H_i^{t+1}(k_t-1) \cdots H_i^{t+1}(i).$$

As we observed in Remark 12, if we had stopped in Substep (d) of Step t, then all the M^{t+1}-edges whose heads are in this block could be embedded in a single page. However, in Substep (e) of this same step, each of these k_t blocks $H_i^{t+1}(i-1), H_i^{t+1}(1), H_i^{t+1}(k_t), H_i^{t+1}(k_t-1), \ldots, H_i^{t+1}(i)$ gets partitioned into $k_t/2$ blocks of sizes as equal as possible; let us call them *subblocks*, and denote them $S_1^i, S_2^i, \ldots, S_{k_t^2/2}^i$, in the order in which they appear in B_i^{t+1}. Afterwards, the order of the elements within each subblock will be changed, but (this is the key property), in all subsequent steps the relative order of these subblocks is maintained. It follows that if $\{e_1, e_2, \ldots, e_{k_t^2/2}\}$ is a set of M^{t+1}-edges, where for each $j = 1, 2, \ldots, k_t^2/2$ the head vertex of e_j is in S_j^i, then $\{e_1, e_2, \ldots, e_{k_t^2/2}\}$ can be simultaneously embedded in one page in the final ordering.

For $j \in [k_t^2/2]$, let p_j^i be the minimum number of pages in which the whole set of M^{t+1}-edges whose head vertices are in S_j^i can be embedded in the final ordering. It follows from the observation in the previous paragraph that the whole set of M^{t+1}-edges whose head vertices are in B_i^{t+1} can be embedded in $\max\{p_1^i, p_2^i, \ldots, p_{k_t^2/2}^i\}$ pages. We conclude that the entire M^{t+1} can be embedded in $k_t \cdot \big(\max\{p_j^i\}_{i\in[k_t], j\in[k_t^2/2]}\big)$ pages.

Let us now estimate p_j^i, for an arbitrary $j \in [k_t^2/2]$. After defining S_j, in further steps the order of the vertices within S_j^i is changed, possibly several times: first a random reordering is done, and the subsequent reorderings depend only on the matchings $M^{t+2}, M^{t+3}, \ldots, M^d$. Since these matchings are random independent matchings, we may then assume (for the purpose of estimating p_j^i) that the vertices in S_j^i appear in the final ordering in a random order. Thus it follows from Corollary 10 that w.h.p. $p_j^i \le 6\sqrt{|S_j^i|}$.

Now each S_j^i is A_ℓ^{t+1} for some $\ell \in [k_{t+1}]$. Thus w.h.p. the entire M^{t+1} can be embedded in $k_t \cdot \big(\max\{6\sqrt{|A_\ell^{t+1}|}\}_{\ell\in[k_{t+1}]}\big)$ pages. Using Claim A, we conclude that w.h.p. the entire M^{t+1} can be embedded in $6k_t \cdot 2^{2^{t-1}}\sqrt{n/k_{t+1}}$ pages. □

Proof of Theorem 3. Applying Claim B with $t = 0$, we obtain that w.h.p. M^1 can be embedded into at most $6k_0 \cdot 2^{2^{t-1}}\sqrt{n/k_1} = 6 \cdot 2^{2^{t-1}}\sqrt{n/k_1}$ pages. Applying the same claim with $0 < t \le d-2$, we obtain that w.h.p. M^r can be embedded into at most $6k_t \cdot 2^{2^{t-1}}\sqrt{n/k_{t+1}} = 6 \cdot 2^{2^{t-1}}\sqrt{n/k_t}$ pages. Finally, recall that in Step d we omit Substep (e). Thus, as observed immediately after Remark 12, all the M^d edges can be embedded in a book with k_{d-1}^2 pages.

It follows that w.h.p. $M^1 \cup M^2 \cup \cdots \cup M^d$ can be embedded into at most

$$
\begin{aligned}
6{\cdot}2\sqrt{\frac{n}{k_1}} + \sum_{t=1}^{d-2} 6{\cdot}2^{2^{t-1}}\sqrt{\frac{n}{k_t}} + k_{d-1}^2 &< 6{\cdot}2^{2^{d-1}}(d-1)\sqrt{\frac{n}{k_1}} + {k_1}^{2\cdot 3^{d-2}} \\
&= 6{\cdot}2^{2^{d-1}}(d-1)\sqrt{2n^{1-1/(1+4\cdot 3^{d-2})}} + n^{2\cdot 3^{d-2}/(1+4\cdot 3^{d-2})} \\
&= 6\sqrt{2}{\cdot}2^{2^{d-1}}(d-1)\cdot n^{\frac{1}{2}-\frac{1}{2+8\cdot 3^{d-2}}} + n^{\frac{1}{2}-\frac{1}{2+8\cdot 3^{d-2}}} \\
&= \left(6\sqrt{2}{\cdot}2^{2^{d-1}}(d-1)+1\right)\cdot n^{\frac{1}{2}-\frac{1}{2+8\cdot 3^{d-2}}}
\end{aligned}
$$

pages. □

Proof of Theorem 4. The proof for random bipartite d-regular graphs is virtually identical to the proof for d-regular graphs. The proof for this case is actually easier: we assign all the heads to one chromatic class, and all the tails to the other chromatic class, so that every vertex is either always a head or always a tail. □

5 Embedding a graph in $11\sqrt{m}$ pages: proof of Theorem 5

Let G be an unlabeled graph with n vertices and m edges. Assign an arbitrary orientation to each edge. Consider a random permutation of the vertices of G, and label them $1, 2, \ldots, n$. For each $i \in [n]$, let A_i be the set of outneighbors of i, written in decreasing order, and let S be the concatenation $A_1 A_2 \cdots A_n$. Thus S is a permutation of a multiset on $[n]$.

Theorem 5 is a consequence of the following:

Claim. *W.h.p. S has no strictly monotone increasing subsequence of length $(11/2)\sqrt{m}$.*

Deferring its proof for the moment, assume the Claim is true. Then w.h.p. S can be decomposed into $(11/2)\sqrt{m}$ (not necessarily strictly) monotone decreasing subsequences; the proof of this is essentially the same as the proof of Proposition 8. By Lemma 7 it follows that w.h.p. G can be embedded into $11\sqrt{m}$ pages. Since this event holds w.h.p. for a random permutation of the vertices, it follows that there exists a permutation of the vertices of G (spine order) for which a $11\sqrt{m}$-page embedding exists, thus proving Theorem 5.

Thus it only remains to prove the Claim.

Proof of Claim. Each element i of S is the head of a directed edge of G; the tail of this directed edge is the *precursor* $p(i)$ of i. A subsequence $i_1 i_2 \ldots i_r$ of S is *good* if $i_1, i_2, \ldots, i_r$, $p(i_1), p(i_2), \ldots, p(i_r)$ are all distinct. If there is an increasing subsequence of S of length ℓ, then clearly there is a good increasing subsequence of length $\ell/2$. So it suffices to show that w.h.p. there is no good increasing subsequence of length $k := (11/4)\sqrt{m}$.

There is a bijection between the set of good subsequences and the collection of all k-matchings (that is, matchings with k edges) of G. Thus it suffices to show that w.h.p. there is no k-matching whose corresponding good subsequence is increasing.

Let d_j denote the outdegree of vertex j. Then there are at most $\sum_{i_1,i_2,\ldots,i_k} d_{i_1}d_{i_2}\cdots d_{i_k}$ k-matchings of G, where the sum is over all k-sets of vertices of G. For each fixed k-matching, the probability that its corresponding good subsequence is increasing is $1/k!$. Thus it follows from the union bound that the probability that there is a good increasing subsequence of length k is at most

$$\frac{\sum_{i_1,i_2,\ldots,i_k} d_{i_1}d_{i_2}\cdots d_{i_k}}{k!} \le \frac{1}{k!}\cdot\frac{\left(\sum_{i=1}^n d_i\right)^k}{k!} = \frac{m^k}{(k!)^2} \le \frac{1}{e^2}\left(\frac{e}{k}\right)^{2k} m^k = \frac{1}{e^2}\left(\frac{e^2}{\left(\frac{11}{4}\right)^2}\right)^{\frac{11\sqrt{m}}{4}} = o(1). \qquad \square$$

6 Further results on decompositions of permutations

Before proceeding to the proof of Theorem 6, let us discuss general lower and upper bounds for $\mu(A)$.

For $k < (1.1)^{\sqrt{n}}$ a random column permutation gives that $\mu \le 3\sqrt{n}$. This follows from the proof of Lemma 9; indeed, for such a random column permutation each row w.h.p. can be decomposed into at most $3\sqrt{n}$ decreasing subsequences; routine concentration arguments show that the same holds for the whole collection of rows, as long as $k < (1.1)^{\sqrt{n}}$.

It is worth nothing that this $\mu \le 3\sqrt{n}$ is essentially best possible if the permutations are given deterministically, even for $k = 2$. Indeed, for the following $2 \times n$ matrix we have $\mu > \sqrt{n}$. Let one row be $1, 2, \ldots, n$ and let the other row be $n, n-1, \ldots, 1$. Then if a column permutation makes the first row decomposable into fewer than $\sqrt{n}$ decreasing subsequences, at least one of this subsequences has size greater than $\sqrt{n}$. The corresponding entries of this subsequence in the second row form an increasing subsequence of size greater than $\sqrt{n}$, from which it obviously follows that this row cannot be decomposed into fewer than $\sqrt{n}+1$ decreasing subsequences.

Proof of Theorem 6. We proceed by induction on k. The statement is trivial for $k = 1$. Let $t := (n/5)^{1/(1+a_{k-1})}$. For simplicity we shall assume that t is an integer and that t divides n. Denote $R_1, R_2, \ldots, R_k$ the rows of A.

For $i = 1, 2, \ldots, n/t$, let B_i be the subsequence of R_1 that contains the elements in $\{n - it + 1, \ldots, n - it + t\}$. We rearrange the columns of A so that R_1 now is $B_1B_2\cdots B_{n/t}$, and let A' denote the resulting matrix.

We need to show that in the resulting matrix A', w.h.p. each row can be decomposed into at most $3 \cdot n^{\frac{1}{2}-a_k}$ decreasing sequences. First we work with rows $2, \ldots, k$, and afterwards we deal with row 1.

For $i = 1, 2, \ldots, n/t$, let M_i be the $(k-1)\times t$ submatrix of A' that results by deleting the first row and taking the columns corresponding to the block B_i. Thus, the submatrix of A' consisting of rows $2, 3, \ldots, k$ is simply the concatenation of the matrices $M_1, M_2 \ldots, M_{n/t}$. For each fixed $i = 1, 2, \ldots, n/t$, we apply induction on M_i, and obtain that each of the rows of M_i w.h.p. can be decomposed into at most $t^{\frac{1}{2}-a_{k-1}}$ decreasing subsequences. For each i this event occurs w.h.p. with a concentration of $1 - 2^{n^c}$ for some constant c depending only on k. Thus the union bound can be applied, and so it follows that w.h.p. the columns of

A' can be rearranged to obtain a matrix A'' in which all the rows in all the M_is can be simultaneously decomposed into at most

$$\begin{aligned}\frac{n}{t}\cdot t^{\frac{1}{2}-a_{k-1}} &= n\cdot t^{-\frac{1}{2}-a_{k-1}} = n\cdot\left(\frac{n}{5}\right)^{(-\frac{1}{2}-a_{k-1})/(1+a_{k-1})}\\ &= \left(\frac{1}{5}\right)^{(-\frac{1}{2}-a_{k-1})/(1+a_{k-1})}\cdot n^{1+(-\frac{1}{2}-a_{k-1})/(1+a_{k-1})}\\ &\le 5^{2/3}\cdot n^{\frac{1}{2}-a_k} < 3\cdot n^{\frac{1}{2}-a_k}\quad(\text{since } a_{k-1}\le 1/2, \text{ then } \frac{(-\frac{1}{2}-a_{k-1})}{(1+a_{k-1})}\ge -2/3).\end{aligned}$$

decreasing subsequences.

For the first row, each of the n/t blocks B_i in A' is a random permutation of its elements. Each of these B_i gets internally reshuffled (say into a block B_i'') to get A''; since this reshuffling depends only on $R_2,\dots,R_k$, each of which is a permutation obtained independently of each other and of R_1, it follows that within A'' each of the n/t blocks B_i'' is a random permutation of the elements in B_i. Each of these blocks has size t, and so by Lemma 9 w.h.p. each of them can be partitioned into $3\sqrt{t}$ decreasing subsequences. (Here we use a concentration argument analogous to the one we used above for rows $R_2,\dots,R_k$). Note that if $1\le i<j\le n/t$, then every element of B_i is strictly greater than every element of B_j. Thus we can choose one decreasing subsequence of each block, and we can concatenate them to obtain a decreasing sequence. We conclude that w.h.p. the entire first row of A'' can be partitioned into $3\sqrt{t}=3\cdot(n/5)^{1/2(1+a_{k-1})}=3\cdot(n/5)^{\frac{1}{2}-a_k}<3n^{\frac{1}{2}-a_k}$ decreasing sequences (here we used that $\frac{1}{2}-a_k<\frac{1}{2}$ for all $k\ge 2$, and so $(1/5)^{\frac{1}{2}-a_k}<1$).

Thus w.h.p. every row of M can be decomposed into at most $3\cdot n^{\frac{1}{2}-a_k}$ decreasing sequences, as needed. □

For general k, the only lower bound we can prove is $\Omega(n^{\frac{1}{2}-\frac{c}{k}})$, for some universal constant c. Interestingly enough, our proof follows indirectly from the results we have established for the pagenumber of random bipartite k-regular graphs. For suppose A is a $k\times n$ matrix, each of whose rows is a random permutation of $[n]$, chosen independently of each other. Then A can be regarded as encoding the information of a bipartite k-regular random graph with bipartition $(X,Y)=(\{x_1,x_2,\dots,x_n\},\{1,2,\dots,n\})$: the columns represent $x_1,x_2,\dots,x_n$, and the k entries of column i are the k vertices of $\{1,2,\dots,n\}$ that are adjacent to x_i. We claim that w.h.p. $\mu(A)>n^{\frac{1}{2}-\frac{1}{k}}$. Indeed, if $\mu(A)$ were smaller, then after some column rearranging each row could be decomposed into $n^{\frac{1}{2}-\frac{1}{k}}$ decreasing sequences, so the edges corresponding to each row could be embedded into $2\cdot n^{\frac{1}{2}-\frac{1}{k}}$ pages (place first the X vertices in the order given by the rearranged columns, then the Y vertices in the order $1,2,\dots,n$, and apply Lemma 7), so the whole graph could be embedded into at most $2k\cdot n^{\frac{1}{2}-\frac{1}{k}}$ pages. This contradicts that the pagenumber of the random bipartite k-regular graph on n vertices is at least $\sqrt{k}\cdot(n/\log n)^{\frac{1}{2}-\frac{1}{2(k-1)}}$ (Theorem 2).

For the particular case $k=2$ we use a different argument to show a lower bound of $\Omega(n^{1/4})$, in Lemma 14 below (compare with the $O(n^{1/3})$ bound given by Theorem 6). In the proof we make use of the following variant of the longest common pattern between two permutations.

Suppose that $\lambda = \lambda_1\lambda_2\cdots\lambda_r$ and $\nu = \nu_1\nu_2\cdots\nu_r$ are permutations of (possibly distinct) subsets of $[n]$. We say that λ and ν are *order equivalent* if for all $i, j \in [r]$ we have $\lambda_i < \lambda_j$ if and only if $\nu_i < \nu_j$. Now given two permutations $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, $\pi = \pi_1\pi_2\cdots\pi_n$ of $[n]$, define $L(\sigma, \pi)$ as the length of the longest subsequence $i_1 < i_2 < \cdots < i_r$ such that $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_r}$ is order equivalent to $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_r}$ (this parameter is related to the length of the *longest common pattern* of σ and π [6, 11]).

It is not difficult to prove the following, using the same ideas as in the proof of Lemma 9).

Proposition 13. *If σ, π are random permutations of $[n]$, then w.h.p. $L(\sigma, \pi) = O(n^{1/2})$.* □

We now state and prove our non-trivial lower bound on $\mu(A)$ for the case $k = 2$.

Lemma 14. *Let A be a $2 \times n$ matrix, each of whose rows is a random permutation of $[n]$, chosen independently of each other. Then w.h.p. $\mu(A) = \Omega(n^{1/4})$.*

Proof. Suppose that there is a reordering of the columns of A such that each of the resulting row permutations σ', π' can be decomposed into $\frac{n^{1/4}}{t}$ decreasing subsequences, for some $t := t(n)$. Then for some $r \geq n^{1/2} \cdot t^2$ there exist $i_1 < i_2 < \cdots < i_r$ such that $\sigma'_{i_1}\sigma'_{i_2}\cdots\sigma'_{i_r}$ and $\pi'_{i_1}\pi'_{i_2}\cdots\pi'_{i_r}$ are both decreasing. This implies that $L(\sigma, \pi) \geq n^{1/2}\cdot t^2$. Since by Proposition 13 w.h.p. $L(\sigma, \pi) = O(n^{1/2})$, we conclude that w.h.p. $\mu(A) = \Omega(n^{1/4})$. □

7 Concluding Remarks

As we observed in the Introduction, Malitz [19] noted that his bound $\Omega(\sqrt{d} \cdot n^{1/2-1/d})$ for the pagenumber of (some) d-regular graphs is tight for $d > \log n$, and asked if it was also tight for $d < \log n$. Theorem 1 answers this in the negative, and Theorem 3 shows that the pagenumber of the typical d-regular graph is $o(n^{1/2})$. We have no reason to expect that the lower bound we established in Theorem 1 is tight, but we believe that this bound is closer to being tight than the upper bound in Theorem 3, as follows:

Conjecture 15. *There is a universal constant $c > 0$ such that for each fixed $d \geq 3$ the pagenumber of the random d-regular graph on n vertices is w.h.p.*

$$\Theta\left(n^{\frac{1}{2}-\frac{c}{d}}\right).$$

A possible approach is the following. The edge set of a d-regular graph can be covered by at most $(d+1)$ matchings. Start with a random ordering of the vertices on the spine, and perform the same sequence of reorderings of the vertices as in the proof of Theorem 3. The technical issue that we have not managed to overcome is that one must have that during the reordering process, a sufficient amount of "randomness" should remain, so that a good bound on the number of pages could be obtained.

We also believe that with some additional ideas the following could be proved.

Conjecture 16. *For each d there is an $a_d > 0$ such that the pagenumber of every d-regular graph on n vertices is at most $n^{1/2-a_d}$.*

Even though we do not have a full rigorous proof yet, we think we can establish this conjecture for the case of bipartite graphs.

As we mentioned Section 6, problems on subsequences of permutations are of great interest in combinatorics. Regarding the bounds for $\mu(A)$ (cf. Theorem 6 and Lemma 14), we suspect that for $k = 2$ w.h.p. we have $\mu(A) = \Theta(n^{1/3})$. For $k \geq 3$ we do not have any sensible guess as to which one of the upper bound (Theorem 6) and the lower bound (see the discussion after the proof of Theorem 6) is closer to the answer.

We do conjecture that the bound in Proposition 13 is tight:

Conjecture 17. *If σ, π are random permutations of $[n]$, then w.h.p. $L(\sigma, \pi) = \Theta(n^{1/2})$.*

In view of the asymptotic tightness of the related results reported in [11], we feel this conjecture should be reasonably straightforward to settle, but so far it has eluded our efforts.

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