Book drawings of complete bipartite graphs

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Abstract

We recall that a *book with k pages* consists of a straight line (the *spine*) and k half-planes (the *pages*), such that the boundary of each page is the spine. If a graph is drawn on a book with k pages in such a way that the vertices lie on the spine, and each edge is contained in a page, the result is a *k-page book drawing* (or simply a *k-page drawing*). The *pagenumber* of a graph G is the minimum k such that G admits a k-page embedding (that is, a k-page drawing with no edge crossings). The *k-page crossing number* $\nu_k(G)$ of G is the minimum number of crossings in a k-page drawing of G. We investigate the pagenumbers and k-page crossing numbers of complete bipartite graphs. We find the exact pagenumbers of several complete bipartite graphs, and use these pagenumbers to find the exact k-page crossing number of $K_{k+1,n}$ for $k \in \{3, 4, 5, 6\}$. We also prove the general asymptotic estimate $\lim_{k\to\infty} \lim_{n\to\infty} \nu_k(K_{k+1,n})/(2n^2/k^2) = 1$. Finally, we give general upper bounds for $\nu_k(K_{m,n})$, and relate these bounds to the k-planar crossing numbers of $K_{m,n}$ and K_n.

Keywords: 2-page crossing number, book crossing number, complete bipartite graphs, Zarankiewicz conjecture

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1 Introduction

In [5], Chung, Leighton, and Rosenberg proposed the model of embedding graphs in books. We recall that a *book* consists of a line (the *spine*) and $k \geq 1$ half-planes (the *pages*), such that the boundary of each page is the spine. In a *book embedding*, each edge is drawn on a single page, and no edge crossings are allowed. The *pagenumber* (or *book thickness*) $p(G)$ of

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a graph G is the minimum k such that G can be embedded in a k-page book [2, 5, 11, 13]. Not surprisingly, determining the pagenumber of an arbitrary graph is NP-Complete [5].

In a *book drawing* (or *k-page drawing*, if the book has k pages), each edge is drawn on a single page, but edge crossings are allowed. The *k-page crossing number* $\nu_k(G)$ of a graph G is the minimum number of crossings in a k-page drawing of G.

Instead of using a straight line as the spine and halfplanes as pages, it is sometimes convenient to visualize a k-page drawing using the equivalent *circular model*. In this model, we view a k-page drawing of a graph $G = (V, E)$ as a set of k circular drawings of graphs $G^{(i)} = (V, E^{(i)})$ $(i = 1, \ldots, k)$, where the edge sets $E^{(i)}$ form a k-partition of E, and such that the vertices of G are arranged identically in the k circular drawings. In other words, we assign each edge in E to exactly one of the k circular drawings. In Figure 1 we illustrate a 3-page drawing of $K_{4,5}$ with 1 crossing.

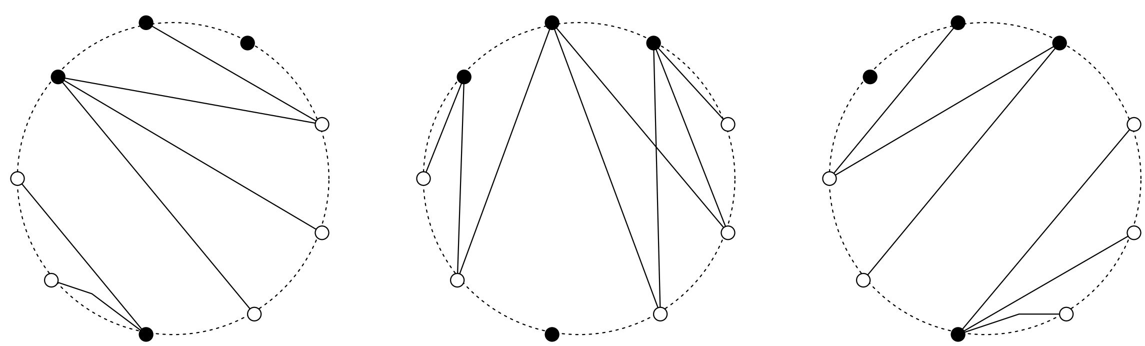

Figure 1: A 3-page drawing of $K_{4,5}$ with 1 crossing. Vertices in the chromatic class of size 4 are black, and vertices in the chromatic class of size 5 are white. We have proved (Theorem 3) that the pagenumber of $K_{4,5}$ is 4, and so it follows that $\nu_3(K_{4,5}) \geq 1$. Now this 1-crossing drawing implies that $\nu_3(K_{4,5}) \leq 1$, and so it follows that $\nu_3(K_{4,5}) = 1$.

Very little is known about the pagenumbers or k-page crossing numbers of interesting families of graphs. Even computing the pagenumber of planar graphs is a nontrivial task; Yannakakis proved [24] that four pages always suffice, and sometimes are required, to embed a planar graph. It is a standard exercise to show that the pagenumber $p(K_n)$ of the complete graph K_n is $\lceil n/2 \rceil$. Much less is known about the k-page crossing numbers of complete graphs. A thorough treatment of k-page crossing numbers (including estimates for $\nu_k(K_n)$), with general lower and upper bounds, was offered by Shahrokhi et al. [21]. In [9], de Klerk et al. recently used a variety of techniques to compute several exact k-page crossing numbers of complete graphs, as well as to give some asymptotic estimates.

Bernhart and Kainen [2] were the first to investigate the pagenumbers of complete bipartite graphs, giving lower and upper bounds for $p(K_{m,n})$. The upper bounds in [2] were then improved by Muder, Weaver, and West [18]. These upper bounds were further improved by Enomoto, Nakamigawa, and Ota [12], who derived the best estimates known to date. Much less is known about the k-page crossing number of $K_{m,n}$.

1.1 1-page drawings of $K_{m,n}$

Although calculating the 1-page crossing number of the complete graph K_n is trivial, this is by no means the case for the complete bipartite graph $K_{m,n}$. Still, our knowledge about $\nu_1(K_{m,n})$ is almost completely satisfactory, due to the following result by Riskin [20].

Theorem 1 (Riskin [20]). *If $m \big| n$ then $\nu_1(K_{m,n}) = \frac{1}{12}n(m-1)(2mn-3m-n)$, and this minimum value is attained when the m vertices are distributed evenly amongst the n vertices.*

1.2 2-page drawings of $K_{m,n}$

Zarankiewicz's Conjecture states that the (usual) crossing number $\text{cr}(K_{m,n})$ of $K_{m,n}$ equals $Z(m,n) := \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$, for all positive integers m, n. Zarankiewicz [25] found drawings of $K_{m,n}$ with exactly $Z(m,n)$ crossings, thus proving $\text{cr}(K_{m,n}) \le Z(m,n)$. These drawings can be easily adapted to 2-page drawings (without increasing the number of crossings), and so it follows that $\nu_2(K_{m,n}) \le Z(m,n)$.

Since $\text{cr}(G) \le \nu_2(G)$ for any G, Zarankiewicz's Conjecture implies the (in principle, weaker) conjecture $\nu_2(K_{m,n}) = Z(m,n)$. Zarankiewicz's Conjecture has been verified (for $\text{cr}(K_{m,n})$, and thus also for $\nu_2(K_{m,n})$) for $\min\{m,n\} \le 6$ [14], and for the special cases $(m,n) \in \{(7,7),(7,8),(7,9),(7,10),(8,8),(8,9),\ (8,10)\}$ [23]. Recently, de Klerk and Pasechnik [8] used semidefinite programming techniques to prove that $\lim_{n\to\infty} \nu_2(K_{7,n})/Z(7,n) = 1$.

1.3 k-page drawings of $K_{m,n}$ for $k \ge 3$: lower bounds

As far as we know, neither exact results nor estimates for $\nu_k(K_{m,n})$ have been reported in the literature, for any $k \ge 3$. Indeed, all the nontrivial results known about $\nu_k(K_{m,n})$ are those that can be indirectly derived from the thorough investigation of Shahrokhi, Sýkora, Székely, and Vrt'o on multiplanar crossing numbers [22].

We recall that a multiplanar drawing is similar to a book drawing, but involves unrestricted planar drawings. Formally, let $G = (V, E)$ be a graph. A *k-planar drawing* of G is a set of k planar drawings of graphs $G^{(i)} = (V, E^{(i)})$ $(i = 1, \ldots, k)$, where the edge sets $E^{(i)}$ form a k-partition of E. Thus, to obtain the k-planar drawing, we take the drawings of the graphs $G^{(i)}$, and (topologically) identify the k copies of each vertex. The *k-planar crossing number* $\text{cr}_k(G)$ of G is the minimum number of crossings in a k-planar drawing of G. A *multiplanar drawing* is a k-planar drawing for some positive integer k.

It is very easy to see that $\nu_k(G) \ge \text{cr}_{\lceil k/2 \rceil}(G)$, for every graph G and every nonnegative integer k. Thus lower bounds of multiplanar (more specifically, r-planar) crossing numbers immediately imply lower bounds of book (more specifically, $2r$-page) crossing numbers. A strong result by Shahrokhi, Sýkora, Székely, and Vrt'o is the exact determination of the

r-planar crossing number of $K_{2r+1,n}$ ([22, Theorem 3]:

$$\mathrm{cr}_r(K_{2r+1,n}) = \left\lfloor \frac{n}{2r(2r-1)} \right\rfloor \left(n - r(2r-1) \left(\left\lfloor \frac{n}{2r(2r-1)} \right\rfloor - 1 \right) \right).$$

Using this result and our previous observation $\nu_k(G) \ge \mathrm{cr}_{\lceil k/2 \rceil}(G)$, one obtains:

Theorem 2 (Follows from [22, Theorem 3])**.** *For every even integer k and every integer n,*

$$\nu_k(K_{k+1,n}) \ge \left\lfloor \frac{n}{k(k-1)} \right\rfloor \left(n - \frac{k}{2} \left(k-1 \right) \left(\left\lfloor \frac{n}{k(k-1)} \right\rfloor - 1 \right) \right). \qquad \square$$

Regarding general lower bounds, using the following inequality from [22, Theorem 5]

$$\mathrm{cr}_r(K_{m,n}) \ge \frac{1}{3(3r-1)^2} \binom{m}{2} \binom{n}{2}, \quad \text{for } m \ge 6r-1 \text{ and } n \ge \max\{6r-1, 2r^2\},$$

and the observation $\nu_k(K_{m,n}) \ge \mathrm{cr}_{\lceil k/2 \rceil}(K_{m,n})$, one obtains

$$\nu_k(K_{m,n}) \ge \frac{1}{3(3\lceil \frac{k}{2} \rceil - 1)^2} \binom{m}{2} \binom{n}{2}, \text{for } m \ge 6\lceil k/2 \rceil - 1 \text{ and } n \ge \max\{6\lceil k/2 \rceil - 1, 2\lceil k/2 \rceil^2\}. \tag{1}$$

We finally remark that slightly better bounds can be obtained in the case $k = 4$, using the bounds for biplanar crossing numbers by Czabarka, Sýkora, Székely, and Vrt'o [6, 7].

1.4 k-page drawings of $K_{m,n}$ for $k \ge 3$: upper bounds

We found no references involving upper bounds of $\nu_k(K_{m,n})$ in the literature. We note that since not every $\lceil k/2 \rceil$-planar drawing can be adapted to a k-page drawing, upper bounds for $\lceil k/2 \rceil$-planar crossing numbers do not yield upper bounds for k-page crossing numbers, and so the results on $(k/2)$-planar drawings of $K_{m,n}$ in [22] cannot be used to derive upper bounds for $\nu_k(K_{m,n})$.

Below (cf. Theorem 6) we shall give general upper bounds for $\nu_k(K_{m,n})$. We derive these bounds using a natural construction, described in Section 7.

2 Main results

In this section we state the main new results in this paper, and briefly discuss the strategies of their proofs.

2.1 Exact pagenumbers

We have calculated the exact pagenumbers of several complete bipartite graphs:

Theorem 3. *For each $k \in \{2,3,4,5,6\}$, the pagenumber of $K_{k+1,\lfloor (k+1)^2/4 \rfloor + 1}$ is $k+1$.*

The proof of this statement is computer-aided, and is based on the formulation of $\nu_k(K_{m,n})$ as a vertex coloring problem on an associated graph. This is presented in Section 3.

By the clever construction by Enomoto, Nakamigawa, and Ota [12], $K_{k+1,\lfloor (k+1)^2/4 \rfloor}$ can be embedded into k pages, and so Theorem 3 implies (for $k \in \{2,3,4,5,6\}$) that $\lfloor \frac{(k+1)^2}{4} \rfloor + 1$ is the smallest value of n such that $K_{k+1,n}$ does *not* embed in k pages. The case $k=2$ follows immediately from the nonplanarity of $K_{3,3}$; we have included this value in the statement for completeness.

2.2 The k-page crossing number of K_{k+1}, n: exact results and bounds

Independently of the intrinsic value of learning some exact pagenumbers, the importance of Theorem 3 is that we need these results in order to establish the following general result. We emphasize that we follow the convention that $\binom{a}{b} = 0$ whenever $a < b$.

Theorem 4. *Let $k \in \{2,3,4,5,6\}$, and let n be any positive integer. Define $\ell := \lfloor \frac{(k+1)^2}{4} \rfloor$ and $q := n \bmod \lfloor \frac{(k+1)^2}{4} \rfloor$. Then*

$$\nu_k(K_{k+1,n}) = q \cdot \binom{\frac{n-q}{\ell} + 1}{2} + (\ell - q) \cdot \binom{\frac{n-q}{\ell}}{2}.$$

In this statement we have included the case $k=2$ again for completeness, as it asserts the well-known result that the 2-page crossing number of $K_{3,n}$ equals $Z(3,n) = \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$.

Although our techniques do not yield the exact value of $\nu_k(K_{k+1,n})$ for other values of k, they give lower and upper bounds that imply sharp asymptotic estimates:

Theorem 5. *Let k, n be positive integers. Then*

$$2n^2 \left(\frac{1}{k^2 + 2000k^{7/4}} \right) - n < \nu_k(K_{k+1,n}) \le \frac{2n^2}{k^2} + \frac{n}{2}.$$

Thus

$$\lim_{k\to\infty} \left(\lim_{n\to\infty} \frac{\nu_k(K_{k+1,n})}{2n^2/k^2} \right) = 1.$$

To grasp how this result relates to the bound in Theorem 2, let us note that the corresponding estimate (lower bound) from Theorem 2 is $\lim_{k\to\infty}\big(\lim_{n\to\infty} \nu_k(K_{k+1,n})/(2n^2/k^2)\big) \ge 1/4$. Theorem 5 gives the exact asymptotic value of this quotient.

In a nutshell, the strategy to prove the lower bounds in Theorems 4 and 5 is to establish lower bounds for $\nu_k(K_{k+1,n})$ obtained under the assumption that $\nu_k(K_{k+1,s+1})$ cannot be k-page embedded (for some integer $s := s(k)$). These results put the burden of the proof of the lower bounds in Theorems 4 and 5 in finding good estimates of $s(k)$. For $k = 3, 4, 5, 6$ (Theorem 4), these come from Theorem 3, whereas for $k > 6$ (Theorem 5) these are obtained from [12, Theorem 5], which gives general estimates for such integers $s(k)$. The lower bounds for $\nu_k(K_{k+1,n})$ needed for both Theorems 4 and 5 are established in Section 4.

In Section 5 we prove the upper bounds on $\nu_k(K_{k+1,n})$ claimed in Theorems 4 and 5. To obtain these bounds, first we find a particular kind of k-page embeddings of $K_{k+1,\lfloor (k+1)^2/4 \rfloor}$, which we call *balanced* embeddings. These embeddings are inspired by, although not equal to, the embeddings described by Enomoto et al. in [12]. We finally use these embeddings to construct drawings of $\nu_k(K_{k+1,n})$ with the required number of crossings.

Using the lower and upper bounds derived in Sections 4 and 5, respectively, Theorems 4 and 5 follow easily; their proofs are given in Section 6.

2.3 General upper bounds for $\nu_k(K_{m,n})$

As we mentioned above, we found no general upper bounds for $\nu_k(K_{m,n})$ in the literature. We came across a rather natural way of drawing $K_{m,n}$ in k pages, that yields the general upper bound given in the following statement.

Theorem 6. *Let k, m, n be nonnegative integers. Let $r := m$ mod k and $s := n$ mod k. Then*

$$\nu_k(K_{m,n}) \leq \frac{(m-r)(n-s)}{4k^2}(m-k+r)(n-k+s) \leq \frac{1}{k^2}\binom{m}{2}\binom{n}{2}.$$

The proof of this statement is given in Section 7.

2.4 k-page vs. $(k/2)$-planar crossing numbers

As we have already observed, for every even integer k, every k-page drawing can be regarded as a $(k/2)$-planar drawing. Thus, for every graph G, $\mathrm{cr}_{k/2}(G) \leq \nu_k(G)$.

Since there is (at least in principle) considerable more freedom in a $(k/2)$-planar drawing than in a k-page drawing, it is natural to ask whether or not this additional freedom can be translated into a substantial saving in the number of crossings. For small values of m or n, the answer is yes. Indeed, Beineke [1] described how to draw $K_{k+1,k(k-1)}$ in $k/2$ planes without crossings, but by Proposition 15, $K_{k+1,k^2/4+500k^{7/4}}$ cannot be k-page embedded; thus the $k/2$-planar crossing number of $K_{k+1,k(k-1)}$ is 0, whereas its k-page crossing number can be arbitrarily large. Thus it makes sense to ask about the asymptotic behaviour when k, m, and n all go to infinity. Letting $\gamma(k) := \lim_{m,n\to\infty} \mathrm{cr}_{k/2}(K_{m,n})/\nu_k(K_{m,n})$, we focus on the question: is $\lim_{k\to\infty} \gamma(k) = 1$?

Since we do not know (even asymptotically) the $(k/2)$-planar or the k-page crossing number of $K_{m,n}$, we can only investigate this question in the light of the current best bounds available.

In Section 8 we present a discussion around this question. We conclude that if the $(k/2)$-planar and the k-page crossing numbers (asymptotically) agree with the current best upper bounds, then indeed the limit above equals 1. We also observe that this is not the case for complete graphs: the currently best known $(k/2)$-planar drawings of K_n are substantially better (even asymptotically) than the currently best known k-page drawings of K_n.

3 Exact pagenumbers: proof of Theorem 3

We start by observing that for every integer n, the graph $K_{k+1,n}$ can be embedded in $k+1$ pages, and so the pagenumber $p(K_{k+1,\lfloor (k+1)^2/4\rfloor+1})$ of $K_{k+1,\lfloor (k+1)^2/4\rfloor+1}$ is at most $k+1$. Thus we need to show the reverse inequality $p(K_{k+1,\lfloor (k+1)^2/4\rfloor+1}) \geq k+1$, for every $k \in \{3,4,5,6\}$. It clearly suffices to show that $\nu_k(K_{k+1,\lfloor (k+1)^2/4\rfloor+1}) > 0$, for every $k \in \{3,4,5,6\}$.

These inequalities are equivalent to k-colorability of certain auxiliary graphs. To this end, we define an auxiliary graph $G_D(K_{m,n})$ associated with a 1-page (circular) drawing D of $K_{m,n}$ as follows. The vertices of $G_D(K_{m,n})$ are the edges of $K_{m,n}$, and two vertices are adjacent if the corresponding edges cross in the drawing D.

We immediately have the following result, that is essentially due to Buchheim and Zheng [4].

Lemma 7 (cf. Buchheim-Zheng [4]). *One has $\nu_k(K_{m,n}) > 0$ if and only if the chromatic number of $G_D(K_{m,n})$ is greater than k for all circular drawings D of $K_{m,n}$.*

As a consequence we may decide if $\nu_k(K_{m,n}) > 0$ by considering all possible circular drawings D of $K_{m,n}$, and computing the chromatic numbers of the associated auxiliary graphs $G_D(K_{m,n})$. The number of distinct circular drawings of $K_{m,n}$ may be computed using the classical *orbit counting lemma*, often attributed to Burnside, although it was certainly already known to Frobenius.

Lemma 8 (Orbit counting lemma). *Let a finite group $\mathcal{G}$ act on a finite set Ω. Denote by Ω^g, for $g \in \mathcal{G}$, the set of elements of Ω fixed by g. Then the number N of orbits of $\mathcal{G}$ on Ω is the average, over $\mathcal{G}$, of $|\Omega^g|$, i.e.*

$$N = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} |\Omega^g|.$$

We will apply this lemma by considering that a circular drawing of $K_{m,n}$ is uniquely determined by the ordering of the m blue and n red vertices on a circle. We therefore define the finite set Ω as the set of all $\binom{m+n}{n}$ such orderings. Now consider the usual action of the dihedral group $\mathcal{G} := D_{m+n}$ on the set Ω. For our purposes two orderings are the same, i.e. correspond to the same circular drawing of $K_{m,n}$, if they belong to the same orbit of $\mathcal{G}$. We therefore only need to count the number of orbits by using the last lemma. The final result is as follows. (We omit the details of the counting argument, as it is a straightforward exercise in combinatorics.)

Lemma 9. *Let m and n be positive integers and denote $d = gcd(m, n)$. The number of distinct circular drawings of $K_{m,n}$ equals:*

$$\frac{1}{2(m+n)} \begin{cases} \frac{m+n}{2}\left(\binom{\frac{m+n}{2}}{n/2} + \binom{\frac{m+n-2}{2}}{m/2} + \binom{\frac{m+n-2}{2}}{n/2}\right) + \sum_{k=0}^{d-1} \binom{\frac{m+n}{o(k)}}{\frac{m}{o(k)}} & (m,\, n \text{ even}), \\ (m+n)\binom{\frac{m+n-1}{2}}{n/2} + \sum_{k=0}^{d-1} \binom{\frac{m+n}{o(k)}}{\frac{m}{o(k)}} & (m \text{ odd},\, n \text{ even}), \\ (m+n)\binom{\frac{m+n-2}{2}}{(m-1)/2} + \sum_{k=0}^{d-1} \binom{\frac{m+n}{o(k)}}{\frac{m}{o(k)}} & (m,\, n \text{ odd}), \end{cases}$$

where $o(k)$ is the minimal number between 1 and d such that $k \cdot o(k) \equiv 0 \mod d$. In other words, $o(k)$ is the order of the subgroup generated by k in the additive group of integers mod d.

In what follows we will present computer-assisted proofs that the chromatic number of $G_D(K_{m,n})$ is greater than k, for specific integers k, m, n. We do not need to compute the chromatic number exactly if we can prove that it is lower bounded by a value strictly greater than k. A suitable lower bound for our purposes is the Lovász ϑ-number.

Lemma 10 (Lovász [17])**.** *Given a graph $G = (V, E)$ and the value*

$$\vartheta(G) := \max_{X \succeq 0} \left\{ \sum_{i,j \in V} X_{ij} \;\middle|\; X_{ij} = 0 \text{ if } (i,j) \in E,\ trace(X) = 1,\ X \in \mathbb{R}^{V \times V} \right\},$$

one has

$$\omega(\bar{G}) \leq \vartheta(G) \leq \chi(\bar{G}),$$

where $\omega(\bar{G})$ and $\chi(\bar{G})$ are the clique and chromatic numbers of the complement $\bar{G}$ of G, respectively.

The $\vartheta(G)$-number may be computed for a given graph G by using semidefinite programming software. For our computation we used the software DSDP [3].

Corollary 11. *If, for given positive integers m, n and k, $\vartheta(\overline{G_D(K_{m,n})}) > k$ for all circular drawings D of $K_{m,n}$, then $\nu_k(K_{m,n}) > 0$.*

If, for a given circular drawing D, we find that $\vartheta(\overline{G_D(K_{m,n})}) = k$, then we compute the chromatic number of $G_D(K_{m,n})$ exactly, by using satisfiability or integer programming software. For our computation we used the satisfiability solver Akmaxsat [15], and for the integer programming formulation the solver XPRESS-MP [16]. The formulation of the chromatic number as the solution of a maximum satisfiability problem is described in [10, §3.3]. The integer programming formulation we used is the following.

For given $G = (V, E)$ with adjacency matrix A, and set of colors $C = \{1, \dots, k\}$, define the binary variables

$$x_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is assigned color } j, \\ 0 & \text{else,} \end{cases} \qquad (i \in V,\ j \in C),$$

and consider the integer programming feasibility problem:

$$\text{Find an } x \in \{0,1\}^{V \times C} \text{ such that } \sum_{j \in C} x_{ij} = 1 \ \forall i \in V,\ \sum_{i \in V} A_{pi} x_{ij} \le |E|(1 - x_{pj}) \ \forall p \in V, j \in C. \tag{2}$$

Lemma 12. *A given graph $G = (V, E)$ is k-colorable if and only if the integer program* (2) *has a solution.*

We may therefore solve (2) with $G = G_D(K_{m,n})$, for each circular drawing D of $K_{m,n}$, to decide if $\nu_k(K_{m,n}) > 0$.

Finally we describe the results we obtained by using the computational framework described in this section.

Case $k = 3$: proof of $\nu_3(K_{4,5}) > 0$.

By Lemma 9, there are 10 distinct circular drawings D of $K_{4,5}$. For each D we showed numerically that $\vartheta(\overline{G_D(K_{4,5})}) > 3$. The required result now follows from Corollary 11.

Case $k = 4$: proof of $\nu_4(K_{5,7}) > 0$.

By Lemma 9, there are 38 distinct circular drawings D of $K_{5,7}$. For all but one D we showed numerically that $\vartheta(\overline{G_D(K_{5,7})}) > 4$. The remaining case was settled by showing $\chi(G_D(K_{5,7})) > 4$ using the satisfiability reformulation. The required result now follows from Corollary 11 and Lemma 7.

Case $k = 5$: proof of $\nu_5(K_{6,10}) > 0$.

By Lemma 9, there are 210 distinct circular drawings D of $K_{6,10}$. For all but one D we showed numerically that $\vartheta(\overline{G_D(K_{6,10})}) > 5$. The remaining case was settled by showing $\chi(G_D(K_{6,10})) > 5$ using the satisfiability reformulation. The required result now follows from Corollary 11 and Lemma 7.

Case $k = 6$: proof of $\nu_6(K_{7,13}) > 0$.

By Lemma 9, there are 1980 distinct circular drawings D of $K_{7,13}$. For all but one D we showed numerically that $\vartheta(\overline{G_D(K_{7,13})}) > 6$. The remaining case was settled by showing $\chi(G_D(K_{7,13})) > 4$ using the integer programming reformulation (2). The required result now follows from Corollary 11, Lemma 12, and Lemma 7.

4 k-page crossing numbers of $K_{k+1,n}$: lower bounds

Our aim in this section is to establish lower bounds for $\nu_k(K_{k+1,n})$. Our strategy is as follows. First we find (Proposition 13) a lower bound under the assumption that $K_{k+1,s+1}$ cannot be k-page embedded (for some integer $s := s(k)$). Then we find values of s such that $K_{k+1,s+1}$ cannot be k-page embedded; these are given in Propositions 14 (for $k \in \{2,3,4,5,6\}$) and 15 (for every k). We then put these results together and establish the lower bounds required in Theorem 4 (see Lemma 16) and in Theorem 5 (see Lemma 17).

Proposition 13. *Suppose that $K_{k+1,s+1}$ cannot be k-page embedded. Let n be a positive integer, and define $q := n$ mod s. Then*

$$\nu_k(K_{k+1,n}) \geq q \cdot \binom{\frac{n-q}{s}+1}{2} + (s-q) \cdot \binom{\frac{n-q}{s}}{2}.$$

Proof. It is readily verified that if $n \leq s$ then the right hand side of the inequality in the proposition equals 0, and so in this case the inequality trivially holds. Thus we may assume that $n \geq s+1$.

Let D be a k-page drawing of $K_{k+1,n}$. Construct an auxiliary graph G as follows. Let $V(G)$ be the set of n degree-$(k+1)$ vertices in $K_{k+1,n}$, and join two vertices u, v in G by an edge if there are edges e_u, e_v incident with u and v (respectively) that cross each other in D.

Since $K_{k+1,s+1}$ cannot be embedded in k pages, it follows that G has no independent set of size $s+1$. Equivalently, the complement graph $\overline{G}$ of G has no clique of size $s+1$. Turán's theorem asserts that $\overline{G}$ cannot have more edges than the Turán graph $T(n,s)$, and so G has at least as many edges as the complement $\overline{T}(n,s)$ of $T(n,s)$. We recall that $\overline{T}(n,s)$ is formed by the disjoint union of s cliques, q of them with $(n-q)/s+1$ vertices, and $s-q$ of them with $(n-q)/s$ vertices. Thus

$$|E(G)| \geq q \cdot \binom{\frac{n-q}{s}+1}{2} + (s-q) \cdot \binom{\frac{n-q}{s}}{2}.$$

Since clearly the number of crossings in D is at least $|E(G)|$, and D is an arbitrary k-page drawing of $K_{k+1,n}$, the result follows. □

Proposition 14. *For each $k \in \{2,3,4,5,6\}$, $K_{k+1,\lfloor\frac{(k+1)^2}{4}\rfloor+1}$ cannot be k-page embedded.*

Proof. This is an immediate consequence of Theorem 3. □

Proposition 15. *For each positive integer k, the graph $K_{k+1,k^2/4+500k^{7/4}}$ cannot be k-page embedded.*

Proof. Define $g(n) := \min\{m \mid \text{the pagenumber of } K_{m,n} \text{ is } n\}$. Enomoto et al. proved that $g(n) = n^2/4 + O(n^{7/4})$ ([12, Theorem 5]). Our aim is simply to get an explicit estimate of the $O(n^{7/4})$ term (without making any substantial effort to optimize the coefficient of $n^{7/4}$).

In the proof of [12, Theorem 5], Enomoto et al. gave upper bounds for three quanitites m_1, m_2, m_3, and proved that $g(n) \le m_1 + m_2 + m_3$. They showed $m_1 \le n^{3/4}(n-r)$ (for certain $r \le n$), $m_2 \le (n^{1/4}+1)(2n^{1/4}+2)(n-1)$, and $m_3 \le (n^{1/4}+1)(n^{1/4}+2)(2n^{1/4}+3)(n-1) + r'(n-r')$ (for certain $r' \le r$).

Noting that $r' \le r$, the inequality $m_1 \le n^{3/4}(n-r)$ gives $m_1 \le n^{3/4}(n-r')$. Elementary manipulations give $m_2 < 2n(n^{1/4}+1)^2 = 2n(n^{1/2}+2n^{1/4}+1) \le 2n(4n^{1/2}) = 8n^{3/2} < 8n^{7/4}$, and $m_3 < n(2n^{1/4}+3)^3 + r'(n-r') < n(5n^{1/4})^3 + r'(n-r') = 125n^{7/4} + r'(n-r')$. Thus we obtain $g(n) \le m_1 + m_2 + m_3 < 133n^{7/4} + (n^{3/4}+r')(n-r')$. An elementary calculus argument shows that $(n^{3/4}+r')(n-r')$ is maximized when $r' = (n-n^{3/4})/2$, in which case $(n^{3/4}+r')(n-r') = (1/4)(n^{3/4}+n)^2 = (1/4)(n^{3/2}+2n^{7/4}+n^2) < (1/4)(n^2+3n^{7/4})$.

Thus we get $g(n) < 133n^{7/4} + n^2/4 + (3/4)n^{7/4} < n^2/4 + 134n^{7/4}$, and so $g(k+1) < (k+1)^2/4 + 134(k+1)^{7/4} < k^2/4 + k/2 + 1/4 + 134(2k)^{7/4} < k^2/4 + 136(2k)^{7/4} = k^2/4 + 136 \cdot 2^{7/4} \cdot k^{7/4} < k^2/4 + 500k^{7/4}$. This means, from the definition of g, that $K_{k+1,k^2/4+500k^{7/4}}$ cannot be k-page embedded. □

Lemma 16. *For each $k \in \{2,3,4,5,6\}$, and every integer n,*

$$\nu_k(K_{k+1,n}) \ge q \cdot \binom{\frac{n-q}{\ell}+1}{2} + (\ell - q) \cdot \binom{\frac{n-q}{\ell}}{2},$$

where $\ell := \lfloor (k+1)^2/4 \rfloor$ and $q := n \bmod \lfloor (k+1)^2/4 \rfloor$.

Proof. It follows immediately from Propositions 13 and 14. □

Lemma 17. *For all positive integers k and n,*

$$\nu_k(K_{k+1,n}) > 2n^2\left(\frac{1}{k^2+2000k^{7/4}}\right) - n.$$

Proof. By Proposition 15 it follows that $K_{k+1,k^2/4+500k^{7/4}}$ cannot be k-page embedded. Thus, if we let $s := k^2/4 + 500k^{7/4} - 1$ and $q := n \bmod s$, it follows from Proposition 13 that $\nu_k(K_{k+1,n}) \ge q \cdot \binom{\frac{n-q}{s}+1}{2} + (s-q) \cdot \binom{\frac{n-q}{s}}{2} \ge s \cdot \binom{\frac{n-q}{s}+1}{2} > (n-q)^2/2s$. Thus we have

$$\nu_k(K_{k+1,n}) > \frac{(n-q)^2}{2s} > \frac{(n-s)^2}{2s} > \frac{n^2}{2s} - n > 2n^2\left(\frac{1}{k^2+2000k^{7/4}}\right) - n. \qquad \square$$

5 k-page crossing numbers of $K_{k+1,n}$: upper bounds

In this section we derive an upper bound for the k-page crossing number of $K_{k+1,n}$, which will yield the upper bounds claimed in both Theorems 4 and 5.

To obtain this bound we proceed as follows. First, we show in Proposition 18 that if for some s the graph $K_{k+1,s}$ admits a certain kind of k-page embedding (what we call a *balanced* embedding), then this embedding can be used to construct drawings of $\nu_k(K_{k+1,n})$ with a certain number of crossings. Then we prove, in Proposition 19, that $K_{k+1,\lfloor (k+1)^2/4 \rfloor}$ admits a balanced k-page embedding for every k. These results are then put together to obtain the required upper bound, given in Lemma 20.

5.1 Extending balanced k-page embeddings to k-page drawings

We consider k-page embeddings of $K_{k+1,s}$, for some integers k and s. To help comprehension, color the $k+1$ degree-s vertices *black*, and the s degree-$(k+1)$ vertices *white*. Given such an embedding, a white vertex v, and a page, the *load* of v in this page is the number of edges incident with v that lie on the given page.

The pigeon-hole principle shows that in an k-page embedding of $K_{k+1,s}$, for each white vertex v there must exist a page with load at least 2. A k-page embedding of $K_{k+1,s}$ is *balanced* if for each white vertex v, there exist $k-1$ pages in which the load of v is 1 (and so the load of v in the other page is necessarily 2).

Proposition 18. *Suppose that $K_{k+1,s}$ admits a balanced k-page embedding. Let $n \geq s$, and define $q := n$ mod s. Then*

$$\nu_k(K_{k+1,n}) \leq q \cdot \binom{\frac{n-q}{s}+1}{2} + (s-q) \cdot \binom{\frac{n-q}{s}}{2}.$$

Proof. Let Ψ be a balanced k-page embedding of $K_{k+1,s}$, presented in the circular model. To construct from Ψ a k-page drawing of $K_{k+1,n}$, we first "blow up" each white point as follows.

Let $t \geq 1$ be an integer. Consider a white point r in the circle, and let N_r be a small neighborhood of r, such that no point (black or white) other than r is in N_r. Now place $t-1$ additional white points on the circle, all contained in N_r, and let each new white point be joined to a black point b (in a given page) if and only if r is joined to b in that page. We say that the white point r has been *converted into a t-cluster.*

To construct a k-page drawing of $K_{k+1,n}$, we start by choosing (any) q white points, and then convert each of these q white points into an $((n-q)/s+1)$-cluster. Finally, convert each of the remaining $s-q$ white points into an $((n-q)/s)$-cluster. The result is evidently an k-page drawing D of $K_{k+1,n}$.

We finally count the number of crossings in D. Consider the t-cluster C_r obtained from some white point r (thus, t is either $(n-q)/s$ or $(n-q)/s+1$), and consider any page π_i. It is clear that if the load of r in π_i is 1, then no edge incident with a vertex in C_r is crossed in π_i. On the other hand, if the load of r in π_i is 2, then it is immediately checked that the number of crossings involving edges incident with vertices in C_r is exactly $\binom{t}{2}$. Now the load of r is 2 in exactly one page (since Ψ is balanced), and so it follows that the total number of crossings in D involving edges incident with vertices in C_r is $\binom{t}{2}$. Since to obtain D, q white points were converted into $((n-q)/s+1)$-clusters, and $s-q$ white points were converted into $((n-q)/s)$-clusters, it follows that the number of crossings in D is exactly

$$q \cdot \binom{\frac{n-q}{s}+1}{2} + (s-q) \cdot \binom{\frac{n-q}{s}}{2}. \qquad \square$$

5.2 Constructing balanced k-page embeddings

Enomoto, Nakamigawa, and Ota [12] gave a clever general construction to embed $K_{m,n}$ in s pages for (infinitely) many values of m, n, and s. In particular, their construction yields k-page embeddings of $K_{k+1,\lfloor (k+1)^2/4 \rfloor}$. However, the embeddings obtained from their technique are not balanced (see Figure 2). We have adapted their construction to establish the following.

Proposition 19. *For each positive integer k, the graph $K_{k+1,\lfloor (k+1)^2/4 \rfloor}$ admits a balanced k-page embedding.*

Proof. We show that for each pair of positive integers s, t such that t is either s or $s+1$, the graph $K_{s+t,st}$ admits a balanced $(s+t-1)$-page embedding. The proposition then follows: given k, if we set $s := \lfloor (k+1)/2 \rfloor$ and $t := \lceil (k+1)/2 \rceil$, then $t \in \{s, s+1\}$, and clearly $k+1 = s+t$ (and so $k = s+t-1$) and $\left\lfloor \frac{(k+1)^2}{4} \right\rfloor = st$.

To help comprehension, we color the $s+t$ degree-st vertices *black*, and we color the st degree-$(s+t)$ vertices *white*. We describe the required embedding using the circular model. Thus, we start with $s+t-1$ pairwise disjoint copies of a circle; these copies are the pages $0, 1, \ldots, s+t-2$. In the boundary of each copy we place the $s+t+st$ vertices, so that the vertices are placed in an identical manner in all $s+t-1$ copies. Each edge will be drawn in the interior of the circle of exactly one page, using the straight segment joining the corresponding vertices.

We now describe how we arrange the white and the black points on the circle boundary. We use the black-and-white arrangement proposed by Enomoto et al. [12]. We refer the reader to Figure 3. First we place the $s+t$ black points $b_0, b_1, \ldots, b_{s+t-1}$ in the circle boundary, in this clockwise cyclic order. Now for each $i \in \{0, 1, \ldots, t-1\}$, we insert between the vertices b_{s+i} and b_{s+i+1} a collection $w_{is}, w_{is+1}, \ldots, w_{is+s-1}$ of white vertices, also listed in the clockwise cyclic order in which they appear between b_{s+i} and b_{s+i+1} (operations on the indices of the black vertices are modulo $s+t$). For any i, j such that $0 \le i \le j < st$, we let $W[i:j]$ denote the set of white vertices $\{w_i, w_{i+1}, \ldots, w_j\}$. For $i = \{0, 1, \ldots, t-1\}$,

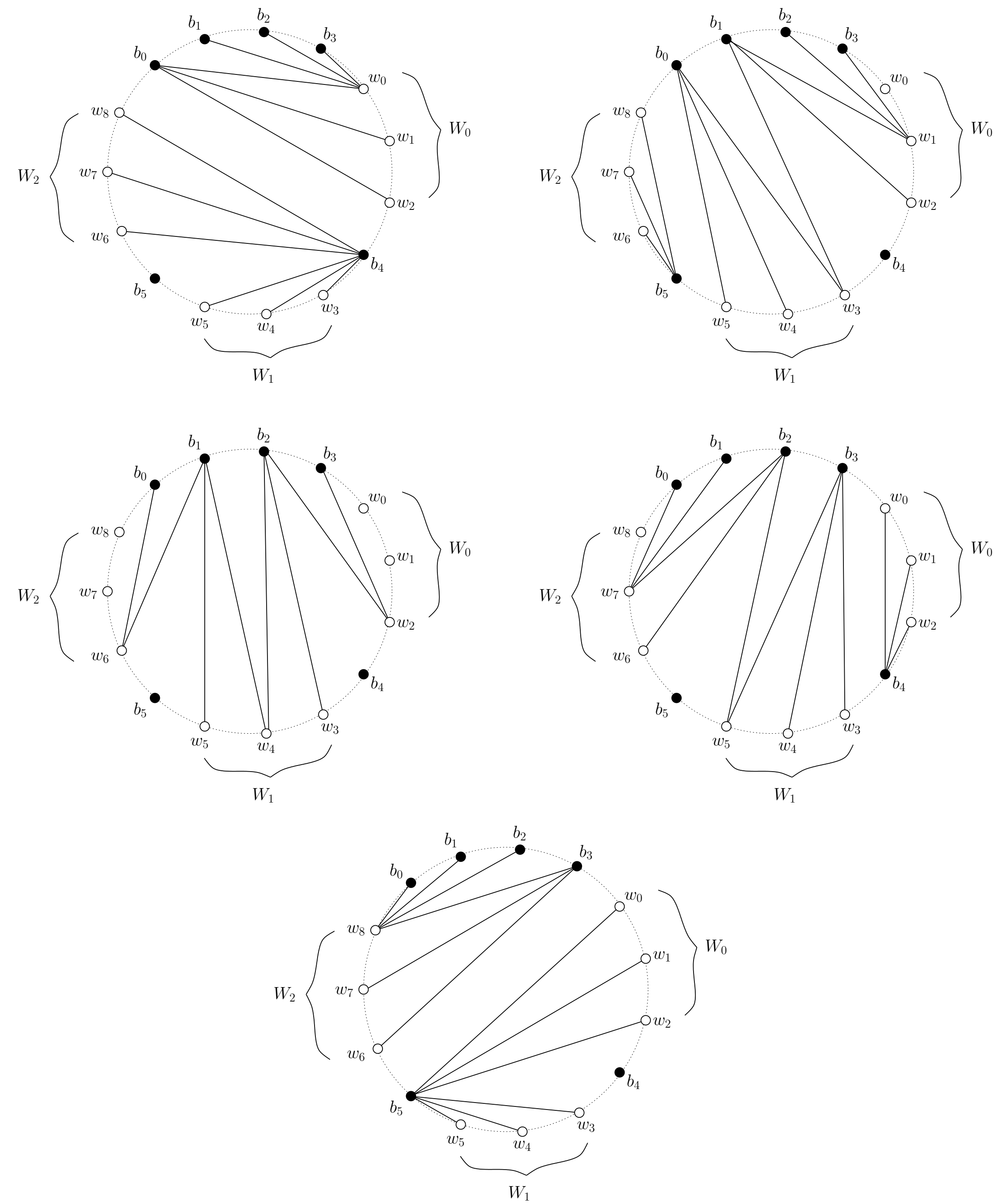

Figure 2: The 5-page embedding of $K_{6,9}$ obtained from the general construction of Enomoto, Nakamigawa, and Ota. This embedding is not balanced (for instance, the white vertex w_0 has degree 4 in Page 0).

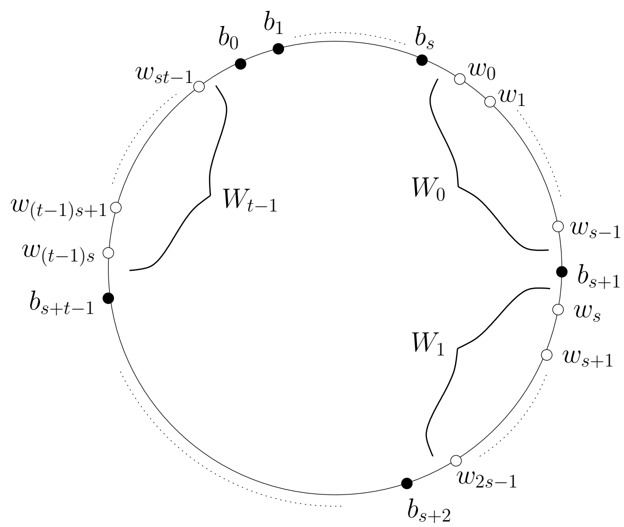

Figure 3: Layout of the vertices of $K_{s+t,st}$.

we call the set $W[is : is + s - 1] = \{w_{is}, w_{is+1}, \dots, w_{is+s-1}\}$ a *white block*, and denote it by W_i. Thus the whole collection of white vertices $w_0, w_1, \dots, w_{st-1}$ is partitioned into t blocks $W_0, W_1, \dots, W_{t-1}$, each of size s. Note that the black vertices $b_0, b_1, \dots, b_s$ occur consecutively in the circle boundary (that is, no white vertex is between b_i and b_{i+1}, for $i \in \{0, 1, \dots, s-1\}$). On the other hand, for $i = s+1, s+2, \dots, s+t-1$, the black vertex b_i occurs between two white vertices: loosely speaking, b_i is sandwiched between the white blocks W_{i-1} and W_i (operations on the indices of the white blocks are modulo t).

Now we proceed to place the edges on the pages. We refer the reader to Figures 4 and 5 for illustrations of the edges distributions for the cases $k = 5$ and 6. We remark that: (i) operations on page numbers are modulo $s+t-1$; (ii) operations on block indices are modulo t; (iii) operations on the indices of black vertices are modulo $s + t$; and (iv) operations on the indices of white vertices are modulo st.

For $r = 0, 1, \dots, s-1$, place the following edges in page r:

Type I For $i = r+1, r+2, \dots, t$, the edges joining b_{s+i} to all the vertices in the white block W_{t+r-i} (note that $b_{s+t} = b_0$).

Type II For $0 < i < r+1$, the edges joining b_i to all the vertices in $W[rs - i(s-1) : rs - (i-1)(s-1)]$.

Type III The edges joining b_{r+1} to all the vertices in $W[0 : r]$.

For $r = s, s+1, \dots, s+t-2$, place the following edges in page r:

Type IV For $i = 0, 1, \dots, r-s+1$, b_{s+i} to all the vertices in the white block $W_{r-s-i+1}$.

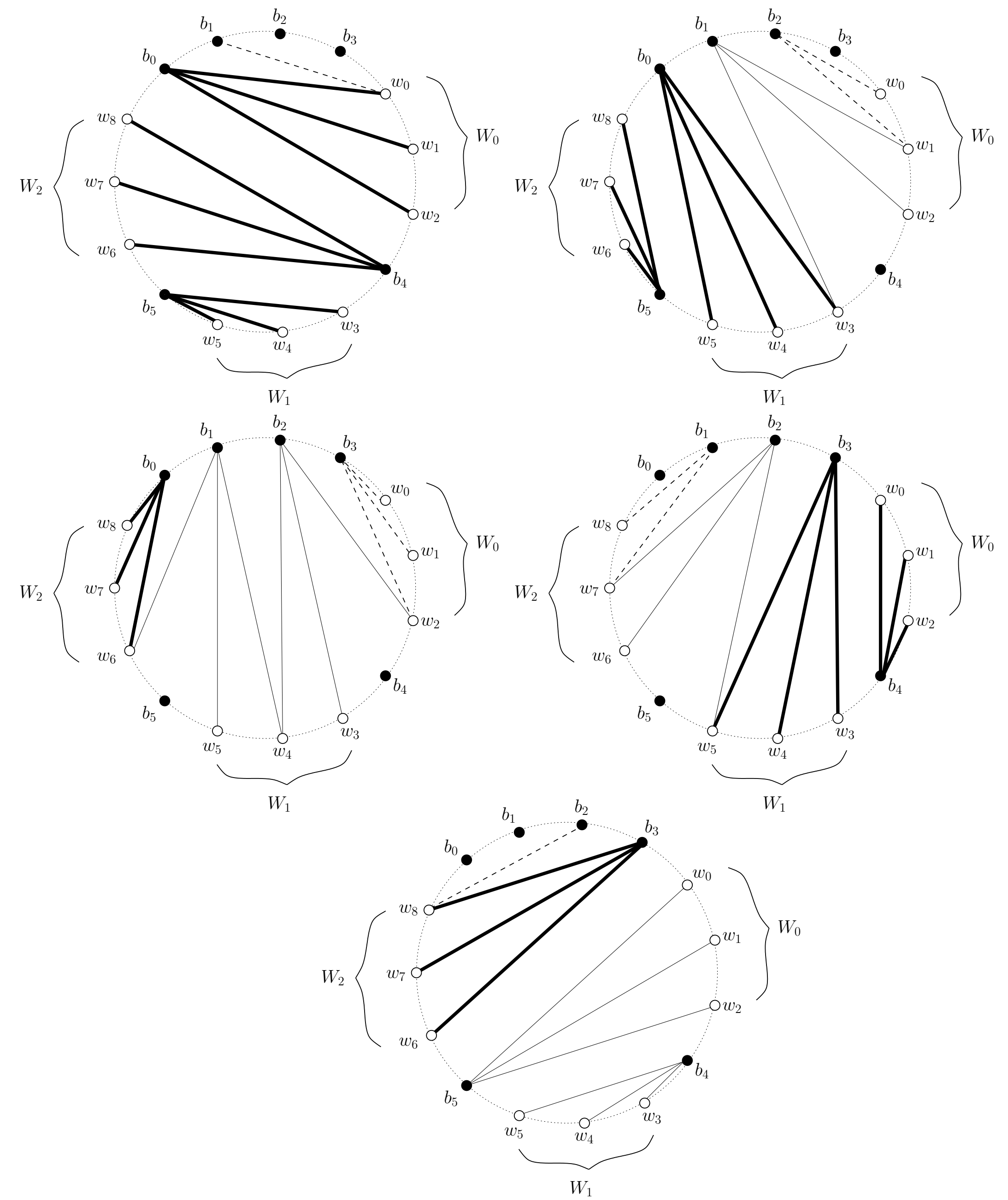

Figure 4: A balanced 5-page embedding of $K_{6,9}$. In this case $k = 5$, and so $s = t = 3$. Pages $0, 1, 2, 3$ and 4 are the upper left, upper right, middle left, middle right, and lower circle, respectively. For Pages $0, 1$, and 2, we have edges of Types I, II, and III, whereas for Pages 3 and 4, we have edges of Types IV, V, and VI. Edges of Types I and IV are drawn with thick segments; edges of Types II and V are drawn with thinner segments; and edges of Types III and VI are drawn with dashed segments.

TYPE V For $0 < i < s - r + t - 1$, the edges joining b_{s-i} to all the vertices in $W[(i + r - s + 1)s - i : (i + r - s + 1)s - i + (s - 1)]$.

TYPE VI The edges joining b_{r-t+1} to all the vertices in $W[st - t + r - s + 1 : st - 1]$.

It is a tedious but straightforward task to check that this yields an $(s+t-1)$-page embedding of $K_{s+t,st}$. Moreover, since every white vertex has load at least 1 in every page, it follows immediately that the embedding is balanced. □

5.3 The upper bound

Lemma 20. *For all positive integers k and n,*

$$\nu_k(K_{k+1,n}) \leq q \cdot \binom{\frac{n-q}{\ell} + 1}{2} + (\ell - q) \cdot \binom{\frac{n-q}{\ell}}{2},$$

where $\ell := \lfloor (k+1)^2/4 \rfloor$ and $q := n \bmod \lfloor (k+1)^2/4 \rfloor$.

Proof. It follows immediately by combining Propositions 18 and 19. □

6 Proofs of Theorems 4 and 5

We first observe that Theorem 4 follows immediately by combining Lemmas 16 and 20.

Now to prove Theorem 5, we let $\ell := \lfloor (k+1)^2/4 \rfloor$ and $q := n \bmod \lfloor (k+1)^2/4 \rfloor$, and note that it follows from Lemma 20 that

$$\begin{aligned}\nu_k(K_{k+1,n}) &\leq q \cdot \binom{\frac{n-q}{\ell} + 1}{2} + (\ell - q) \cdot \binom{\frac{n-q}{\ell}}{2} \leq \ell \cdot \binom{\frac{n-q}{\ell} + 1}{2} = \frac{\ell}{2} \cdot \left(\frac{n-q}{\ell} + 1\right)\left(\frac{n-q}{\ell}\right) \\ &= \frac{n-q}{2} \cdot \left(\frac{n-q}{\ell} + 1\right) \leq \frac{n}{2} \cdot \left(\frac{n}{\ell} + 1\right) = \frac{n^2}{2\ell} + \frac{n}{2} \leq \frac{n^2}{2(k^2/4)} + \frac{n}{2} = \frac{2n^2}{k^2} + \frac{n}{2}.\end{aligned}$$

Combining this with Lemma 17, we obtain

$$2n^2\left(\frac{1}{k^2 + 2000k^{7/4}}\right) - n < \nu_k(K_{k+1,n}) \leq \frac{2n^2}{k^2} + \frac{n}{2},$$

proving Theorem 5.

7 A general upper bound for $\nu_k(K_{m,n})$: proof of Theorem 6

We now describe a quite natural construction to draw $K_{m,n}$ in k pages, for every $k \geq 3$. Actually, our construction also works for the case $k = 2$, and for this case the upper bounds obtained coincide with the best known upper bound for $\nu_2(K_{m,n})$.

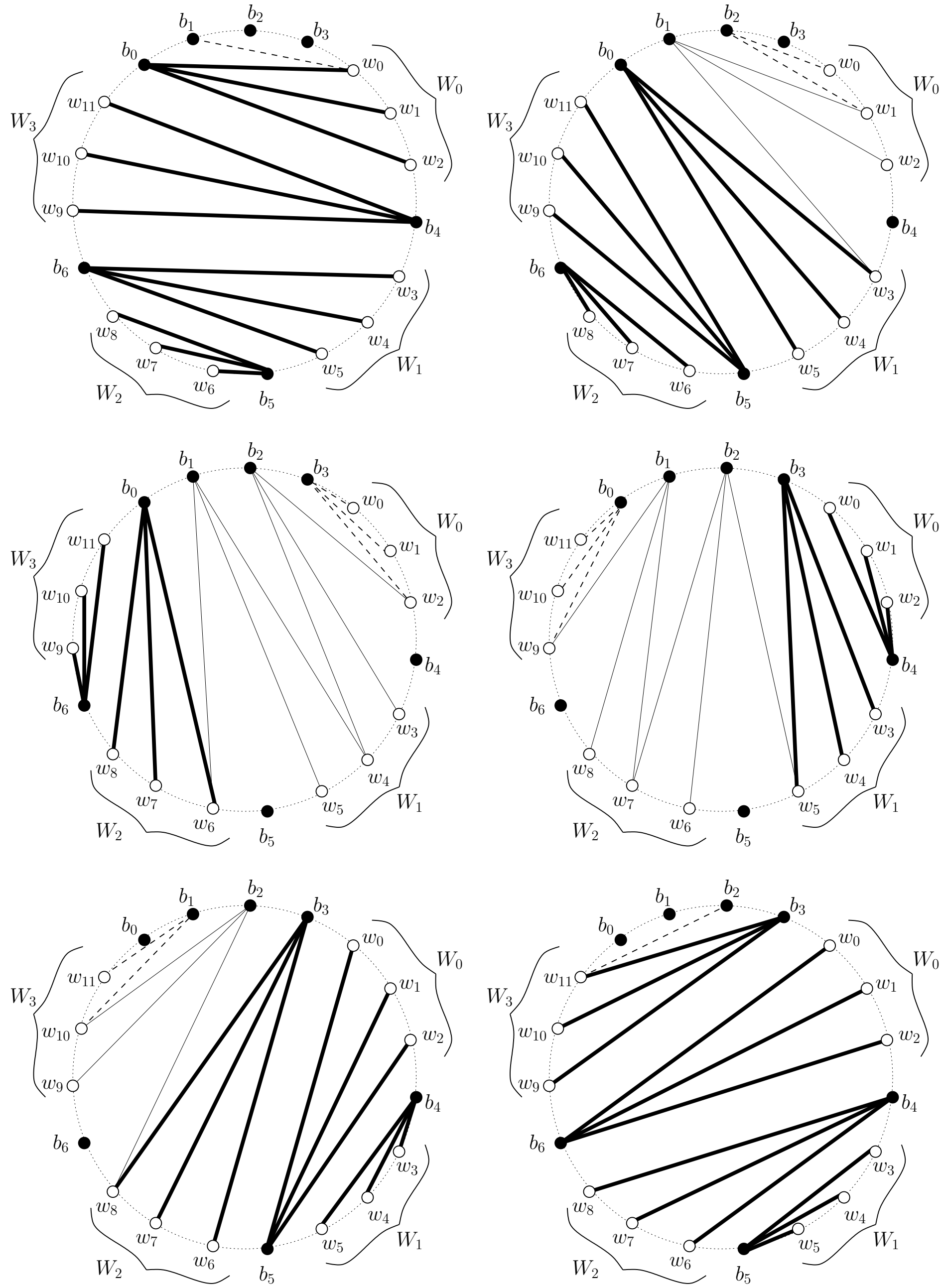

Figure 5: A balanced 6-page embedding of $K_{7,12}$. In this case $k = 6$, and so $s = 3$ and $t = 4$. Pages $0, 1, 2, 3, 4$, and 5 are the upper left, upper right, middle left, middle right, lower left, and lower right circles, respectively. For Pages $0, 1$, and 2, we have edges of Types I, II, and III, whereas for Pages $3, 4$, and 5, we have edges of Types IV, V, and VI. Edges of Types I and IV are drawn with thick segments; edges of Types II and V are drawn with thinner segments; and edges of Types III and VI are drawn with dashed segments.

Proof of Theorem 6. For simplicity, we color the m vertices black, and the n vertices white. Let p, q, r, s be the nonnegative integers defined by the conditions $m = kp + r$ and $0 \le r \le k - 1$, and $n = kq + s$ and $0 \le s \le k - 1$ (note that the definitions of r and s coincide with those in the statement of Theorem 6). Our task is to describe a drawing of $K_{m,n}$ with exactly $(m - r)(n - s)(m - k + r)(n - k + s)/(4k^2)$ crossings.

We start our construction by dividing the set of black vertices into k groups $B_0, B_1, \ldots, B_{k-1}$, so that $k - r$ of them (say the first $k - r$) have size p, and the remaining r have size $p + 1$. Then we divide the set of white vertices into k groups $W_0, W_1, \ldots, W_{k-1}$, such that $k - s$ of them (say the first $k - s$) have size q, and the remaining s have size $q + 1$.

Then (using the circular drawing model) we place the groups alternately on a circumference, as in $B_0, W_0, B_1, W_1, \ldots, B_{k-1}, W_{k-1}$. Now for $i = 0, 1, 2, \ldots, k - 1$, we draw in page i the edges joining all black points in B_j to all white points in W_s if and only if $j + s = i$ (operations are modulo k).

A straightforward calculation shows that the total number of crossings in this drawing is $(k-r)(k-s)\binom{p}{2}\binom{q}{2} + (k-r)s\binom{p}{2}\binom{q+1}{2} + r(k-s)\binom{p+1}{2}\binom{q}{2} + rs\binom{p+1}{2}\binom{q+1}{2}$, and an elementary manipulation shows that this equals $(m - r)(n - s)(m - k + r)(n - k + s)/(4k^2)$. Thus $\nu_k(K_{m,n}) \le (m - r)(n - s)(m - k + r)(n - k + s)/(4k^2)$, as claimed.

Finally, note that since obviously $m - r \le m$, $n - s \le n$, $m - k + r \le m - 1$, and $n - k + s \le n - 1$, it follows that $\nu_k(K_{m,n}) \le (1/4k^2)m(m-1)n(n-1) = (1/k^2)\binom{m}{2}\binom{n}{2}$. $\square$

8 Concluding remarks

It seems worth gathering in a single expression the best lower and upper bounds we now have for $\nu_k(K_{m,n})$. Since $\nu_k(K_{m,n})$ may exhibit an exceptional behaviour for small values of m and n, it makes sense to express the asymptotic forms of these bounds. The lower bound (coming from [22, Theorem 5]) is given in (1), whereas the upper bound is from Theorem 6.

$$\frac{1}{3(3\lceil \frac{k}{2} \rceil - 1)^2} \le \lim_{m,n\to\infty} \frac{\nu_k(K_{m,n})}{\binom{m}{2}\binom{n}{2}} \le \frac{1}{k^2}. \qquad (3)$$

As we have observed (and used) above, $\mathrm{cr}_{k/2}(K_{m,n}) \le \nu_k(K_{m,n})$, and it is natural to ask whether $\nu_k(K_{m,n})$ is strictly greater than $\mathrm{cr}_{k/2}(K_{m,n})$ (we assume k even in this discussion). At least in principle, there is much more freedom in $k/2$-planar drawings than in k-page drawings. Thus remains the question: can this additional freedom be used to (substantially) save crossings?

With this last question in mind, we now carry over an exercise which reveals the connections between the book and multiplanar crossing numbers of complete and complete bipartite graphs.

It is not difficult to prove that the constants

$$\begin{aligned}
\textsc{BookBipartite} &:= \lim_{k\to\infty} k^2 \cdot \left(\lim_{m,n\to\infty} \frac{\nu_k(K_{m,n})}{\binom{m}{2}\binom{n}{2}} \right), \\
\textsc{BookComplete} &:= \lim_{k\to\infty} k^2 \cdot \left(\lim_{n\to\infty} \frac{\nu_k(K_n)}{\binom{n}{4}} \right), \\
\textsc{MultiplanarBipartite} &:= \lim_{k\to\infty} k^2 \cdot \left(\lim_{m,n\to\infty} \frac{\mathrm{cr}_k(K_{m,n})}{\binom{m}{2}\binom{n}{2}} \right), \\
\textsc{MultiplanarComplete} &:= \lim_{k\to\infty} k^2 \cdot \left(\lim_{n\to\infty} \frac{\mathrm{cr}_k(K_n)}{\binom{n}{4}} \right),
\end{aligned}$$

are all well-defined.

In view of (3), we have

$$\frac{4}{27} \le \textsc{BookBipartite} \le 1. \tag{4}$$

Using the best known upper bound for $\mathrm{cr}_k(K_{m,n})$ (from [22, Theorem 8]), we obtain

$$\textsc{MultiplanarBipartite} \le \frac{1}{4}. \tag{5}$$

We also invoke the upper bound $\mathrm{cr}_k(K_n) \le (1/64)k(n+k^2)^4/(k-1)^3$, which holds whenever k is a power of a prime and $n \ge (k-1)^2$ (see [22, Theorem 7]). This immediately yields

$$\textsc{MultiplanarComplete} \le \frac{3}{8}. \tag{6}$$

We also note that the observation $\mathrm{cr}_{k/2}(K_{m,n}) \le \nu_k(K_{m,n})$ immediately implies that

$$\textsc{MultiplanarBipartite} \le \frac{1}{4} \cdot \textsc{BookBipartite}. \tag{7}$$

Finally, applying the Richter-Thomassen counting argument [19] for bounding the crossing number of K_{2n} in terms of the crossing number of $K_{n,n}$ (their argument applies unmodified to k-planar crossing numbers), we obtain

$$\textsc{MultiplanarComplete} \ge \frac{3}{2} \cdot \textsc{MultiplanarBipartite}. \tag{8}$$

Suppose that the multiplanar drawings of Shahrokhi et al. [22] are asymptoticall optimal. In other words, suppose that equality holds in (5). Using (7), we obtain $\textsc{BookBipartite} \ge 1$, and by (4) then we get $\textsc{BookBipartite} = 1$. Moreover (again, assuming equality holds in (5)), using (8), we obtain $\textsc{MultiplanarComplete} \ge 3/8$, and so in view of (6) we get $\textsc{MultiplanarComplete} = 3/8$. Summarizing:

Observation 21. *Suppose that the multiplanar drawings of $K_{m,n}$ of Shahrokhi et al.* [22] *are asymptotically optimal, so that* MULTIPLANARBIPARTITE $= \frac{1}{4}$. *Then*

$$\text{BOOKBIPARTITE} = 1, \quad \text{and}$$
$$\text{MULTIPLANARCOMPLETE} = \frac{3}{8}. \qquad \square$$

In other words, under this scenario (the multiplanar drawings of $K_{m,n}$ in [22] being asymptotically optimal), the additional freedom of $(k/2)$-planar over k-page drawings of $K_{m,n}$ becomes less and less important as the number of planes and pages grows. In addition, under this scenario the k-planar crossing number of K_n also gets (asymptotically) determined.

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