Embedding a graph-like continuum in some surface

R. Christian* R. B. Richter† G. Salazar‡

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Abstract

We show that a graph-like continuum embeds in some surface if and only if it does not contain one of: a generalized thumbtack; or infinitely many $K_{3,3}$'s or K_5's that are either pairwise disjoint or all have just a single point in common.

Keywords: graph-like continuum, surface, embedding

1 Introduction

In recent years, a resurgence of interest in fundamental embeddability questions has emerged concerning embeddings of a Peano continuum P into surfaces. For example, see [7, 9, 10, 11]. For a fixed surface Σ, this question has recently been answered in the doctoral dissertation of the first author, where the following result appears. (We recall that a *surface* is a compact, connected, 2-manifold without boundary. A *Peano continuum* is a non-empty, compact, connected, locally connected, metric space. A generalized thumbtack will be defined later.)

Theorem 1.1 ([1]) *Let P be a Peano continuum and Σ a surface. Then P does not embed in Σ if and only if P contains one of the following:*

1. *a generalized thumbtack;*
2. *a finite graph that does not embed in Σ;*
3. *a surface of genus less than that of Σ; or*
4. *the disjoint union of Σ and a point.*

This result follows on (and its proof uses) the works of Claytor [2, 3], who proved the same result in the case Σ is the sphere. (See also [7, 8, 10].)

A *graph-like continuum* is a compact, connected, metric space G with a 0-dimensional subspace V (the *vertex-set*) so that $G-V$ consists of components, each of which is open in G, is homeomorphic to $\mathbb{R}$, and has a closure homeomorphic to either the unit circle S^1 or the closed interval $[0,1]$. There is a more general concept of *graph-like space* which is as defined above, except G need not be either compact or metric. These concepts were introduced by Thomassen and Vella [12]. We will not be concerned with the more general spaces, but they arise, for example, in the context of infinite graphic matroids (N. Bowler, personal communication). When G is compact, 0-dimensional is equivalent to totally disconnected.

*Perth, Australia.

†Department of Combinatorics and Optimization, University of Waterloo. Waterloo, Ontario, Canada. Supported by NSERC.

‡Instituto de Fisica, Universidad Autonoma de San Luis Potosi. San Luis Potosi, Mexico. Supported by CONACYT Grant 106432.

A graph-like continuum is an example of a Peano continuum. The Freudenthal compactification of a connected, locally finite graph is an example of a graph-like continuum; there are many others that can be derived from infinite graphs. There are also many that cannot be so derived.

This work is devoted to determining which graph-like continua embed in some surface. We shall refer to a finite graph G as being *contained in* a Peano continuum P if there is a subspace of P that is homeomorphic to the natural graph-like continuum associated with G (each edge is a homeomorph of a compact interval, with the vertices of G describing the natural identifications of the various endpoints of these intervals.) Obviously, any graph-like continuum that contains $K_{3,\infty}$ or infinitely many disjoint $K_{3,3}$'s cannot embed in any surface.

There is one other example of a graph-like continuum that does not embed in any surface: the generalized thumbtack. The *thumbtack space* $\mathfrak{T}$ consists of the unit disc $\{(x,y,0) \mid x^2+y^2 \leq 1\}$ in 3-dimensional space, together with the unit interval $\{(0,0,z) \mid 0 \leq z \leq 1\}$. It is standard and easy that no neighbourhood of $(0,0,0)$ in $\mathfrak{T}$ is contained in an open disc and, therefore, does not embed in any surface; however, $\mathfrak{T}$ is not a graph-like continuum. We now describe graph-like continua that model its non-embeddability property.

A *web centred at* w is a graph-like continuum W that contains pairwise disjoint cycles (that is, homeomorphs of S^1) $C_1, C_2, C_3 \dots$ so that: (i) for each $i = 2, 3, \dots,$ $T - C_i$ has two components $K_{i,<}$ and $K_{i,>}$, with $K_{i,<}$ containing $C_1 \cup C_2 \cup \cdots \cup C_{i-1}$ and $K_{i,>}$ containing $C_{i+1}, C_{i+2}, \dots$; and (ii) for each $i = 2, 3, \dots,$ either $|\operatorname{cl}(K_{i,<}) \cap \operatorname{cl}(K_{i,>})| \geq 3$ or there are $x_<, y_< \in \operatorname{cl}(K_{i,<}) \cap C_i$ and $x_>, y_> \in \operatorname{cl}(K_{i,>}) \cap C_i$ so that $x_<, x_>, y_<, y_>$ are all distinct and occur in this cyclic order in C_i (this is the definition of *overlapping* C_i*-bridges*); and (iii) the C_i converge to w (that is, every neighbourhood of w contains all but finitely many of the C_i).

A *generalized thumbtack* is the union of a web W centred at w plus an additional single edge that is disjoint from W except that w is one end of the edge. Our main theorem is the following.

Theorem 1.2 (Main Theorem) *Let G be a graph-like continuum. Then one of the following occurs:*

1. *G embeds in some surface; or*
2. *G contains a generalized thumbtack; or*
3. *G contains infinitely many disjoint $K_{3,3}$'s or K_5's; or*
4. *G contains infinitely many $K_{3,3}$'s or K_5's that have precisely one point in common, to which they converge.*

It follows easily from Theorem 1.1 that if P is a Peano continuum, then either:

(i) there exists a surface in which P embeds; or

(ii) P contains a generalized thumbtack; or

(iii) P contains an infinite sequence $G_1, G_2, \dots,$ of finite graphs so that, for each surface Σ, some G_i does not embed in Σ.

We are interested in replacing the last condition with a finite list of obstructions. For graph-like continua, our main result provides such a list, but we do not know how to obtain a comparable result for Peano spaces.

In this context, Robertson and Seymour (personal communication) used the Graph Minors Structure Theorem to prove an interesting theorem. For every integer $k > 0$, consider the graphs consisting of either: k disjoint $K_{3,3}$'s; k disjoint K_5's; k $K_{3,3}$'s having precisely a vertex in common; k K_5's having precisely a vertex in common; k $K_{3,3}$'s having precisely an edge in common; and k K_5's having precisely and edge in common. Their result is that, for every k, there is a G_i from (iii) that has one of the six graphs listed above as a minor.

Because G is connected, Outcome 3 of Theorem 1.2 improves to either a "star" of $K_{3,3}$'s or K_5's (that is, all connected by disjoint arcs to a single point, to which they converge) or a "comb" of $K_{3,3}$'s or K_5's (that is, all connected by disjoint arcs to a single arc, again everything converging to a single point). This is quite analogous to the "Star-Comb Lemma" [4, Lemma 8.2.2].

Our main theorem is reminiscent of Levinson's Theorem [6], that an infinite, locally finite, vertex transitive graph is either planar or has infinite genus. See [5, Ch. 6].

In [7], it was observed that a generalized thumbtack does not embed in any surface. Claytor [3] shows (in different terms) that containing a generalized thumbtack is equivalent to containing one of two particular generalized thumbtacks (see also [8]).

2 Proof of the main theorem

Let G be a graph-like continuum with vertex set V. An *edge* is a component of $G-V$. For any partition (U,W) of V into closed sets, the *cut* $\delta(U,W)$ is the set of all edges having one end in U and one end in W. The following fact is central (it is proved in greater generality in [13]).

Lemma 2.1 [13, Theorem 12] *Any cut in a graph-like continuum is finite.* ■

Because cuts are finite, there are minimal, non-empty cuts; these are *bonds*. If $\delta(U,W)$ is a bond, then $G-\delta(U,W)$ has precisely two components, one containing all the vertices in U and the other containing all the vertices in W. We remark that a bond is a set of edges; often the partition (U,W) will not be explicitly required and so we may refer to a bond b, with the understanding that b determines and is determined by the partition (U,W) of V.

Webs are obviously closely related to generalized thumbtacks. They are also related to vertices being incident with faces. The proof of [7, Lemma 3.3] for the sphere extends to any surface.

Lemma 2.2 *Let P be a 2-connected Peano continuum embeddable in the surface Σ. If W is a countable subset of P, then either P has an embedding in Σ so that each point of W is incident with a face of P, or P contains a web centred at some point of W.* ■

Our first observation toward proving our main theorem shows that every bond has a side that also does not embed in any surface.

Proposition 2.3 *Let G be a graph-like continuum that does not embed in any surface. If b is a bond in G, then either G has a generalized thumbtack or one of the two components of $G-b$ does not embed in any surface.*

Proof. Suppose H and J are the two components of $B-b$, and they embed in the surfaces Σ_H and Σ_J, respectively. There are only finitely many edges in b, so each of H and J has only finitely many vertices incident with edges in b. If any of these vertices is the centre of a web in either H or J, then this web combines with an incident edge from b to make a generalized thumbtack in G.

If none of the vertices in either H or J is the centre of a web in its sub-continuum, then Lemma 2.2 shows that H and J have embeddings in Σ_H and Σ_J, respectively, so that each vertex incident with an edge of b is incident with a face of the appropriate embedding. Now we may add, for each edge e of b, a cylinder joining Σ_H and Σ_J, attaching at each end in a face incident with the appropriate end of e. The edge e may then be added to the embedding. Since b is finite, the result is an embedding of G in some surface. ■

Another basic fact about graph-like continua is due to Thomassen and Vella.

Lemma 2.4 [12, Proof of Theorem 2.1] *A graph-like continuum has only countably many edges.* ■

We subdivide each loop of G; obviously, the resulting graph-like continuum embeds in a surface if and only if G does. Thus we may assume G has not loops.

Lemma 2.5 *Let u and v be any two vertices of G. Then there is a bond b of G so that u and v are in different components of $G - b$. In particular, every edge of G is in a bond.*

Proof. Because V is 0-dimensional, there is a partition of V into closed sets C_u and C_v containing u and v, respectively. Let K be the component of $G - \delta(C_u, C_v)$ containing u and let L be the component of $G - \delta(V \cap K, V \setminus K)$ containing v. Then $\delta(V \cap L, V \setminus L)$ is the desired bond. ■

We start by enumerating the edges as $e_1, e_2, \dots$ and letting b_1 be a bond containing e_1. Let H_1 and G_1 be the components of $G - b_1$, labelled so that G_1 does not embed in any surface. Note that e_1 is not in G_1.

For $i > 1$, let j be least so that $e_j \in G_{i-1}$. The inductive assumption is that G_{i-1} does not embed in any surface and that none of $e_1, e_2, \dots, e_{i-1}$ is in G_{i-1}; therefore, $j \geq i$. Let b_i be a bond in G_{i-1} containing e_j. Let H_i and G_i be the components of $G_{i-1} - b_i$, labelled so that G_i does not embed in any surface. Evidently, none of $e_1, e_2, \dots, e_i$ is in G_i and G_i does not embed in any surface.

The sequence $G_1, G_2, G_3, \dots$ consists of closed, connected subsets of G and $G_1 \supseteq G_2 \supseteq \cdots$. Therefore, $\bigcap_{i \geq 1} G_i$ is a closed, connected subset of G. Since $\bigcap_{i \geq 1} G_i$ has no edge, it is just a single vertex x.

We need one more observation before we start getting the conclusions.

Claim 1 *Let $i \in \{1, 2, \dots\}$ and let b be any bond in G_i. If L is the component of $G_i - b$ containing x, then there is a $j > i$ so that $G_j \subseteq L$.*

Proof. Since b is finite, there is a $j > i$ so that no edge of b is in G_j. Since $x \in G_j$ and G_j is connected, $G_j \subseteq L$. □

There is one easy case in which the result holds.

Claim 2 *If, for infinitely many i, $G_i \setminus x$ contains either $K_{3,3}$ or K_5, then G contains infinitely many pairwise disjoint $K_{3,3}$'s or K_5's.*

Proof. For every i, $G_i \setminus x$ contains either a $K_{3,3}$ or K_5; let J_i be any one of these. Since J_i and x are both closed in G_i and G_i is normal, there is a bond b_i in G_i so that J_i and x are in different components of $G_i - b_i$. By Claim 1, there is a $j > i$ so that G_j is separated by b_i from J_i. This implies that there is an infinite set of pairwise disjoint $K_{3,3}$'s or K_5's in G. □

In view of Claim 2, we may assume that there are only finitely many i for which $G_i \setminus x$ contains either $K_{3,3}$ or K_5. In this case, the non-planarity of G_i implies G_i contains either a generalized thumbtack or a subspace J_i that is either a $K_{3,3}$ or a K_5. We are done if any G_i contains a generalized thumbtack, so we may assume the latter. The asumption implies that, for some i_0, if $i \geq i_0$, then $x \in J_i$. Again, without loss of generality, we may further assume $G = G_{i_0}$, so that no $G_i \setminus x$ contains a $K_{3,3}$ or K_5.

For each i, let J_i be a copy of either $K_{3,3}$ or K_5 in G_i. Infinitely often, J_i will be the same one of $K_{3,3}$ and K_5. Let I be an infinite set so that, for all $i \in I$, the J_i are pairwise homeomorphic. Furthermore, we may assume that the status of x in J_i either as vertex or in the interior of an edge is the same for all $i \in I$.

We know that, for each $i \in I$, $x \in J_i$. There are two ways x can appear in J_i: either as a vertex or in the interior of an edge. Let $V_i = V(J_i) \cup \{x\}$ (so, for example, if J_i is $K_{3,3}$ and x is in the interior of an edge, then $|V_i| = 7$). There are 2, 3, or 4 open arcs in $J_i - V_i$ having x in their closures. Let this number be k_i and arbitrarily label the arcs incident with x as $1, 2, \dots, k_i$.

Let B_i denote the set of components of $J_i - V_i$ that are incident with x and set $L_i = J_i - (\{x\} \cup \bigcup_{e \in B_i} e)$. Then L_i is a closed subspace of G_i that is disjoint from x and, therefore, it is separated from x by a finite bond. Claim 1 implies there is an infinite sequence $i_0 < i_1 < i_2 < \cdots$ so that, for each $j > 0$, $L_{i_{j-1}}$ is disjoint from G_{i_j}. In particular, the L_{i_j} are pairwise disjoint. To reduce the notation, we will use the index j in place of i_j, so L_{i_j} becomes L_j, J_{i_j} becomes J_j, etc.

For each $j < j'$, J_j and $J_{j'}$ have x in common. The intersection can only be at x and in the edges in B_j. For each $i = 1, 2, \dots, k_j$, let $y_{i,j,j'}$ be the first intersection with $J_{j'}$ of the edge i incident with x in J_j as we travel from L_j to x. There are several possibilities for $y_{i,j,j'}$: it is in $L_{j'}$; it is in the edge $i' \in \{1, 2, \dots, k_{j'}\}$; or it is at x. Crucially, there is, in total, a bounded number of possibilities for all the intersections $y_{i,j,j'}$.

By Ramsey's Theorem, there is an infinite set A of indices so that, for any $j, j', j'' \in A$, the intersections are all the same. For example, if $y_{i,j,j'}$ is in the edge i' from $B_{j'}$, then $y_{i,j,j''}$ and $y_{i,j',j''}$ are also in the edge i', but this edge i' is in $B_{j''}$. Note that all the k_j are the same value, which we set to be k.

Let n be the number of $y_{i,j,j'}$ that are not x.

In what follows, we will refer to the sequence $(J_i)_{i \geq 0}$ that has all the J_i the same one of $K_{3,3}$ and K_5, all contain x in the same way, and, for $i < j$, the way $(J_i - x)$ intersects $(J_j - x)$, is always the same (in the above sense) as an *infinite genus sequence with parameters k and n.*

Claim 3 *For any infinite genus sequence with parameters k and n, $n < k$.*

Proof. Otherwise, consider the finite graph N consisting of L_j, the segments of each $i \in \{1, 2, \dots, k\}$ from L_j to $y_{i,j,j'}$, $L_{j'}$, and the segments of each $i' \in \{1, 2, \dots, k\}$ from $L_{j'}$ to any $y_{i,j,j'}$ they contain. Contracting $N \cap J_{j'}$ to a vertex yields a homeomorph of J_j. Since any graph that contracts to either $K_{3,3}$ or K_5 contains a subdivision of either $K_{3,3}$ or K_5, we have the contradiction that $G_j - x$ contains either $K_{3,3}$ or K_5. □

Claim 4 *If there is an infinite genus sequence with parameters k and $n = k - 1$, then there is an infinite genus sequence with parameters $k = 2$ and $n = 0$.*

Proof. Proceed as in the proof of Claim 3 to get N, but this time N includes the edge i for which $y_{i,j,j'} = x$, plus an edge from $L_{j'}$ to x that does not meet any other $L_{j''}$ with $j'' > j'$. Contracting $N \cap J_{j'}$ again yields a homeomorph of J_j, so $N \cap J_{j'}$ contains a subspace M homeomorphic to either $K_{3,3}$ or K_5 that has x in the interior of some edge. This can be repeated infinitely often to get a sequence that has the desired properties. □

Claim 5 *If an infinite genus sequence has parameters $k = 4$ and $n = 2$, then there is an infinite genus sequence with parameters $k = 3$ and $n = 1$.*

Proof. The hypothesis implies each L_j is a K_4, there are two $y_{i,j,j+1}$ in $J_{j+1} \setminus x$, and two $y_{i,j,j+1}$ are equal to x. Let a_j, b_j, c_j, d_j be the four vertices of L_j, labelled so that a_j and b_j are connected directly to x, without going through $J_{j+1} - x$. Delete the edges $a_j b_j$ and $c_j d_j$, and use L_{j+1} and x as vertices to find a $K_{3,3}$ in $N \cup J_{j+1}$. In this $K_{3,3}$, $k = 3$ and $n = 1$, so this is easily repeated to produce a sequence with this property. □

Claim 6 *There is an infinite genus sequence with* $n \leq 1$*.*

Proof. In view of Claim 2, we have assumed $k \geq 1$. Since x is not an isolated vertex, $k \geq 2$. Choose the sequence to minimize k and, given the minimal k, minimize n. If $k = 4$, then Claim 3 implies $n \leq 3$, while Claim 4 implies (given that the minimum k is 4) $n < 3$. Claim 5 and the minimality of k implies $n \neq 2$, so in this case $n \leq 1$.

Similarly and more simply, if $k = 3$, then Claims 3 and 4 imply $n \leq 1$. Likewise, If $k = 2$, then Claim 3 implies $n \leq 1$. □

Claim 7 *There is an infinite genus sequence with parameters* k *and* $n = 0$*.*

Proof. Claim 6 shows there is a sequence J_j with $n \leq 1$. We assume that $n = 1$. In this case, there is a $y_{i,j,j+1}$ in $J_{j+1} \setminus x$. In $J_{j+1} \setminus x$ there is an arc A from $y_{i,j,j+1}$ to a point of L_{j+1} that is connected directly to x without going through L_{j+2}. Let J'_j be the resulting homeomorph of J_j. This construction may be repeated infinitely often, yielding a sequence with the same k, but having $n = 0$. □

Let J_i be an infinite genus sequence with parameters k and $n = 0$. Obviously, any two J_i's have only x in common, completing the proof of Theorem 1.2.

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