

The Erdős-Sós Conjecture for Geometric Graphs

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Abstract

Let $f(n, k)$ be the minimum number of edges that must be removed from some complete geometric graph G on n points, so that there exists a tree on k vertices that is no longer a planar subgraph of G . In this paper we show that $\left(\frac{1}{2}\right) \frac{n^2}{k-1} - \frac{n}{2} \leq f(n, k) \leq 2 \frac{n(n-2)}{k-2}$. For the case when $k = n$, we show that $2 \leq f(n, n) \leq 3$. For the case when $k = n$ and G is a geometric graph on a set of points in convex position, we show that at least three edges must be removed.

1 Introduction

One of the most notorious problems in extremal graph theory is the Erdős-Sós Conjecture, which states that every simple graph with average degree greater than $k - 2$ contains every tree on k vertices as a subgraph. This conjecture was recently proved true for all sufficiently large k (unpublished work of Ajtai, Komlós, Simonovits, and Szemerédi).

In this paper we investigate a variation of this conjecture in the setting of geometric graphs. Recall that a *geometric graph* G consists of a set S of points in the plane (these are the vertices of G), plus a set of straight line segments, each of which joins two points in S (these are the edges of G). In particular, any set S of points in the plane in general position naturally induces a complete geometric graph. For brevity, we often refer to the edges of this graph simply as edges of S . If S is in convex position then G is a *convex geometric graph*. A geometric graph is *planar* if no two of its edges cross each other. An *embedding*

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of an abstract graph H into a geometric graph G is an isomorphism from H to a planar geometric subgraph of G . For $r \geq 0$, an r -edge is an edge of G such that in one of the two open semi-planes defined by the line containing it, there are exactly r points of G .

In this paper all point sets are in general position and G is a complete geometric graph on n points. It is well known that for every integer $1 \leq k \leq n$, G contains every tree on k vertices as a planar subgraph [3]. Even more, it is possible to embed any such tree into G , when the image of a given vertex is prespecified [5].

Let F be a subset of edges of G , which we call *forbidden edges*. If T is a tree for which every embedding into G uses an edge of F , then we say that F *forbids* T . In this paper we study the question of what is the minimum size of F so that there is a tree on k vertices that is forbidden by F . Let $f(n, k)$ be the minimum of this number taken over all complete geometric graphs on n points. As $f(2, 2) = 1$, $f(3, 3) = 2$, $f(4, 4) = 2$ and $f(n, 2) = \binom{n}{2}$, we assume throughout the paper that $n \geq 5$ and $k \geq 3$.

We show the following bounds on $f(n, k)$.

Theorem 1.1.

$$\left(\frac{1}{2}\right) \frac{n^2}{k-1} - \frac{n}{2} \leq f(n, k) \leq 2 \frac{n(n-2)}{k-2}$$

Theorem 1.2.

$$2 \leq f(n, n) \leq 3$$

In the case when G is a convex complete geometric graph, we show that the minimum number of edges needed to forbid a tree on n vertices is three.

An equivalent formulation of the problem studied in this paper is to ask how many edges must be removed from G so that it no longer contains *some* planar subtree on k vertices. A related problem is to ask how many edges must be removed from G so that it no longer contains *any* planar subtree on k vertices. For the case of $k = n$, in [6], it is proved that if any $n - 2$ edges are removed from G , it still contains a planar spanning subtree. Note that if the $n - 1$ edges incident to any vertex of G are removed, then G no longer contains a spanning subtree. In general, for $2 \leq k \leq n - 1$, in [1], it is proved that if any set of $\left\lceil \frac{n(n-k+1)}{2} \right\rceil - 1$ edges are removed from G , it still contains a planar subtree on k vertices. In the same paper it is also shown that this bound is tight—a geometric graph on n vertices and a subset of $\left\lceil \frac{n(n-k+1)}{2} \right\rceil$ of its edges are shown, so that when these edges are removed, every planar subtree has at most $k - 1$ vertices. In [4] the authors study the similingly unrelated problem of packing two trees into planar graphs. That is, given two trees on n vertices, the authors consider the question of when it is possible to find a planar graph having both of them as spanning trees and in which the trees are edge disjoint. However, although theirs is a combinatorial question rather than geometric, their Theorem 2.1 implies our Lemma 2.2. We provide a self contained proof of Lemma 2.2 for completeness.

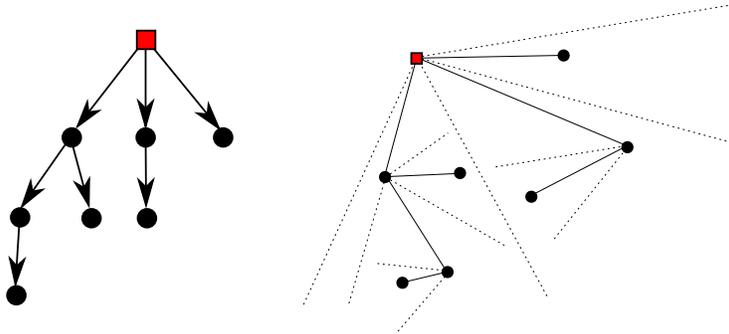


Figure 1: An embedding of a tree using the algorithm.

A previous version of this paper appeared in the conference proceedings of EUROCG'12 [2].

2 Spanning Trees

In this section we consider the case when $k = n$. Let T be a tree on n vertices. Consider the following algorithm to embed T into G . Choose a vertex v of T and root T at v . For every vertex of T choose an arbitrary order of its children. Suppose that the neighbors of v are u_1, \dots, u_m , and let n_1, \dots, n_m be the number of nodes in their corresponding subtrees. Choose a convex hull point p of G and embed v into p . Sort the remaining points of G counter-clockwise by angle around p . Choose $m + 1$ rays centered at p so that the wedge between two consecutive rays is convex and between the i -th ray and the $(i + 1)$ -th ray there are exactly n_i points of G . Let S_i be this set of points. For each u_i choose a convex hull vertex of S_i visible from p and embed u_i into this point. Recursively embed the subtrees rooted at each u_i into S_i . Note that this algorithm provides an embedding of T into G . We will use this embedding frequently throughout the paper. See Figure 1.

For every integer $n \geq 2$ we define a tree T_n as follows: If $n = 2$, then T_n consists of only one edge; if n is odd, then T_n is constructed by subdividing once every edge of a star on $\frac{n-1}{2}$ vertices; if n is even and greater than 2, then T_n is constructed by subdividing an edge of T_{n-1} . These trees are particular cases of *spider trees*. See Figure 2.

We prove the lower bound of $f(n, n) \geq 2$ of Theorem 1.2.

Theorem 2.1. *If G has only one forbidden edge, then any tree on n vertices can be embedded into G , without using the forbidden edge.*

Proof. Let e be the forbidden edge of G . Let T be a tree on n vertices. Choose a root for T . Sort the children of each node of T , by increasing size of their corresponding subtree. Embed T into G with the embedding algorithm, choosing

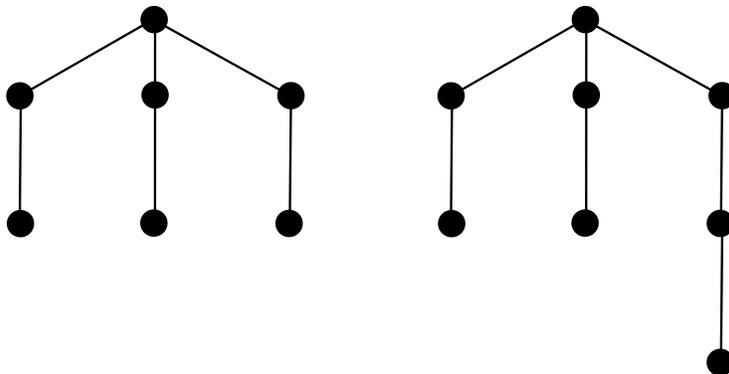


Figure 2: T_7 and T_8 .

at all times the rightmost point as the root of the next subtree. Suppose that e is used in this embedding. Let $e := (p, q)$ so that u is embedded into p and v is embedded into q (note that u and v are vertices of T).

Suppose that the subtree rooted at v has at least two nodes. In the algorithm, we embedded this subtree into a set of at least two points. We chose a convex hull point (q), of this set visible from p to embed v . In this case we may choose another convex hull point visible from p to embed v and continue with the algorithm. Note that (p, q) is no longer used in the final embedding.

Suppose that v is a leaf, and that v has a sibling v' whose subtree has at least two nodes. Then we may change the order of the children of u so that e is no longer used in the embedding, or if it is, then v' is embedded into q , but then we proceed as above.

Suppose that v is a leaf, and that all its siblings are leaves. The subtree rooted at u is a star. We choose a point distinct from p and q in the point set where this subtree is embedded, and embed u into this point. Afterwards we join it to the remaining points. This produces an embedding that avoids e .

Assume then, that v is a leaf and that it has no siblings. We distinguish the following cases:

1. **u has no siblings.** In this case, the subtree rooted at the parent of u is a path of length two. It is always possible to embed this subtree without using e . See Figure 3.
2. **u has a sibling u' whose subtree is not an edge.** We may change the order of the siblings of u , with respect to their parent, so that the subtree rooted at u' will be embedded into the point set containing p and q . In the initial order—increasing by size of their corresponding subtrees— u' is after u . We may assume that in the new ordering, the order of the siblings of u before it, stays the same. Therefore p is the rightmost point of the set into which the subtree rooted at u' will be embedded. Embed u' into

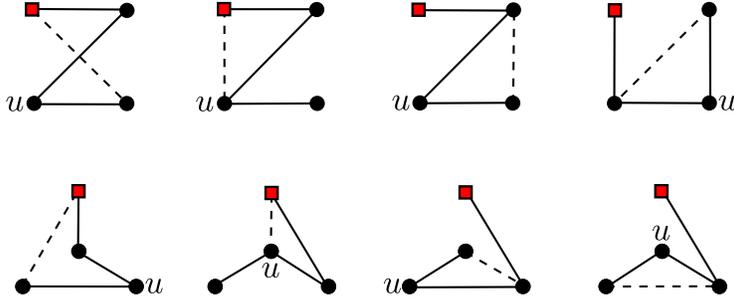


Figure 3: The embedding of a path of length three. The grandparent of u is highlighted and the forbidden edge is dashed.

p . Either we find an embedding not using e , or this embedding falls into one of the cases considered before.

3. u has at least one sibling, all whose corresponding subtrees are edges

Suppose that u has no grandparent; then T is equal to T_n and n is odd. Let w be the parent of u . Embed w into p . Let p_1, \dots, p_{n-1} be the points of G different from p sorted counter-clockwise by angle around p ; choose p_1 so that the angle between two consecutive points is less than π . Let $u_1, \dots, u_{(n-1)/2}$ be the neighbors of w . Embed each u_i into p_{2i-1} and its child into p_{2i} . If q equals p_{2j-1} for some j then embed u_j into p_{2j} and its child into p_{2j-1} . This embedding avoids e .

Suppose that w is the grandparent of u and let p' be the point into which w is embedded. Let S be the point set into which the subtree rooted at the parent of u is embedded. Note that S has an odd number of points. We replace the embedding as follows. Sort S counter-clockwise by angle around p' . Call a point *even* if it has an even number of points before it in this ordering. Call a point *odd* if it has an odd number of points before it in this ordering. If e is incident to an odd point, then we embed the parent of u into this point. The remaining subtree rooted at u can be embedded without using e . If the endpoints of e are both even, between them there is an odd point. We embed the parent of u into this point. The remaining vertices can be embedded without using e (see Figure 4).

□

The upper bound of $f(n, n) \leq 3$ of Theorem 1.2 follows directly from Lemma 3.1. Now we prove in Lemma 2.2 and Theorem 2.3, that if G is a convex geometric graph, at least three edges are needed to forbid some tree on n vertices.

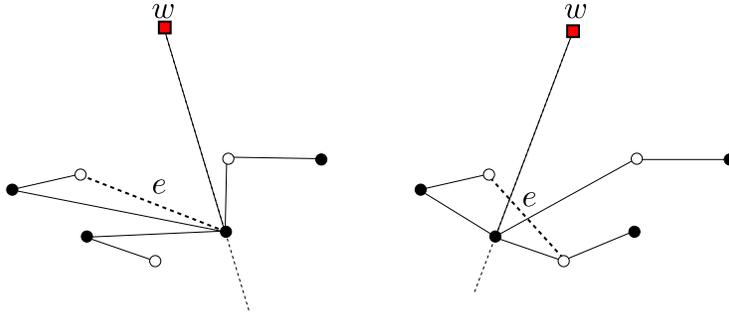


Figure 4: The two sub-cases, when u has a grandparent w , and all the subtrees of its children are edges. Odd points are painted in black and even points in white. The forbidden edges are dashed.

Lemma 2.2. *Let T be a tree on n vertices. If G is a convex geometric graph, then T can be embedded into G using less than $\frac{n}{2}$ convex hull edges of G .*

Proof. If T is a star, then any embedding of T into G uses only two convex hull edges. If T is a path then it can be embedded into G using at most two convex hull edges. Therefore, we may assume that T is neither a star nor a path.

Since T is not a path, it has a vertex of degree at least three. Choose this vertex as the root. Since T is not a star, the root has a child whose subtree has at least two nodes. Sort the children of T so that this node is first. Embed T into G with the embedding algorithm.

Let u and v be vertices of T , so that u is the parent of v . Suppose that the subtree rooted at v has at least two nodes. Then in the embedding algorithm we have at least two choices to embed v once the ordering of the children of u has been chosen. At least one of which is such that (u, v) is not embedded into a convex hull edge. Therefore, we may assume that the embedding is such that all the convex hull edges used are incident to a leaf.

Since the first child of the root is not a leaf, there is at most one convex hull edge incident to the root in the embedding. Note that any vertex of T , other than the root, is incident to at most one convex hull edge in the embedding. If $n/2$ or more convex hull edges are used, then there are at least $n/2$ non-leaf vertices, each adjacent to a leaf. These vertices must be all the vertices in T and there are only $n/2$ such pairs (n must also be even). Therefore every non-leaf vertex has at most one child which is a leaf. In particular the root has at most one child which is a leaf. Since the root was chosen of degree at least three it has a child which is not a leaf nor the first child; we place this vertex last in the ordering of the children of the root. The leaf adjacent to the root can no longer be a convex hull edge and the embedding uses less than $n/2$ convex hull edges. \square

Theorem 2.3. *If G is a convex geometric graph and has at most two forbidden*

edges, then any tree on n vertices can be embedded into G , without using a forbidden edge.

Proof. Let f_0 be an embedding given by Lemma 2.2, of T into G . For $0 \leq i \leq n$, let f_i be the embedding produced by rotating f_0 , i places to the right. Assume that in each of these rotations at least one forbidden edge is used, as otherwise we are done. Let e_1, \dots, e_m be the edges of T that are mapped to a forbidden edge in some rotation. Assume that the two forbidden edges are an l -edge and an r -edge respectively.

Suppose that $l \neq r$. Then, each edge of T can be embedded into a forbidden edge at most once in all of the n rotations. Thus $m \geq n$. This is a contradiction, since T has $n - 1$ edges.

Suppose that $l = r$. Then, each of the e_i is mapped twice to a forbidden edge. Thus $m \geq n/2$. By Lemma 2.2, f_0 uses less than $n/2$ convex hull edges. Therefore, l and r must be greater than 0. But a set of $n/2$ or more r -edges, with $r > 0$, must contain a pair of edges that cross. And we are done, since f_0 is an embedding. \square

3 Bounds on $f(n, k)$

In this section we prove Theorem 1. First we show the upper bound.

Lemma 3.1. *If G is a convex geometric graph, then forbidding three consecutive convex hull edges of G forbids the embedding of T_n .*

Proof. Recall that T_n comes from subdividing a star, let v be the non leaf vertex of this star. Let $(p_1, p_2), (p_2, p_3), (p_3, p_4)$ be the forbidden edges, in clockwise order around the convex hull of G . Note that in any embedding of T_n into G , an edge incident to a leaf of T_n , must be embedded into a convex hull edge. Thus, the leaves of T_n nor its neighbors can be embedded into p_2 or p_3 , without using a forbidden edge. Thus, v must be embedded into p_2 or p_3 . Without loss of generality assume that v is embedded into p_2 . But then, the embedding must use (p_2, p_3) or (p_3, p_4) \square

Lemma 3.2. *If G is a convex geometric graph, then forbidding any three pairs of consecutive convex hull edges of G forbids the embedding of T_n .*

Proof. Let p_1, p_2 and p_3 be the vertices in the middle of the three pairs of consecutive forbidden edges of G . Note that a leaf of T_n , nor its neighbor can be embedded into p_1, p_2 or p_3 , without using a forbidden edge. But at most two points do not fall into this category. \square

Lemma 3.3. $f(n, k) \leq 2 \frac{n(n-2)}{k-2}$

Proof. Let G be a complete convex geometric graph. We forbid every r -edge of G for $r = 0, \dots, \left\lceil 2 \frac{n-2}{k-2} - 2 \right\rceil$. Note that, in total we are forbidding at most $n \left(\left\lceil 2 \frac{n-2}{k-2} - 2 \right\rceil + 1 \right) \leq 2 \frac{n(n-2)}{k-2}$ edges. As every subset of points of G is in

convex position, it suffices to show that every induced subgraph H of G on k vertices is in one of the two configurations of Lemma 3.1 and 3.2.

Assume then, that H does not contain three consecutive forbidden edges in its convex hull nor three pairs of consecutive forbidden edges in its convex hull. H has at most two (non-adjacent) pairs of consecutive forbidden edges in its convex hull. Therefore every forbidden edge of H in its convex hull—with the exception of at most two—must be preceded by an ℓ -edge (of G), with $\ell > \left\lceil 2 \frac{n-2}{k-2} - 2 \right\rceil$. H contains at least $\frac{k-2}{2}$ of these edges. The points separated by these edges amount to more than $\frac{k-2}{2} \left\lceil 2 \frac{n-2}{k-2} - 2 \right\rceil \geq n - k$ points of G . This is a contradiction, since together with the k points of H this is strictly more than n . □

Now, we show the lower bound of Theorem 1.1.

Lemma 3.4. $f(n, k) \geq \left(\frac{1}{2}\right) \frac{n^2}{k-1} - \frac{n}{2}$

Proof. Let F be a set of edges whose removal from G forbids some k -tree. Let $H := G \setminus F$. Note that H contains no complete K_k as a subgraph, otherwise any k -tree can be embedded in this subgraph [3]. By Turán’s Theorem [7], H cannot contain more than $\binom{k-2}{k-1} \frac{n^2}{2}$ edges. Thus F must have size at least $\left(\frac{1}{2}\right) \frac{n^2}{k-1} - \frac{n}{2}$. □

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