ON THE DECAY OF CROSSING NUMBERS OF SPARSE GRAPHS

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Abstract. Richter and Thomassen proved that every graph has an edge e such that the crossing number $\text{cr}(G-e)$ of $G-e$ is at least $(2/5)\text{cr}(G)-O(1)$. Fox and Cs. Tóth proved that dense graphs have large sets of edges (proportional in the total number of edges) whose removal leaves a graph with crossing number proportional to the crossing number of the original graph; this result was later strenghtened by Černý, Kynčl and G. Tóth. These results make our understanding of the decay of crossing numbers in dense graphs essentially complete. In this paper we prove a similar result for large sparse graphs in which the number of edges is not artificially inflated by operations such as edge subdivisions. We also discuss the connection between the decay of crossing numbers and expected crossing numbers, a concept recently introduced by Mohar and Tamon.

1. Introduction

The *crossing number* $\text{cr}(G)$ of a graph G is the minimum number of pairwise crossings of edges in a drawing of G in the plane. A graph G is k-*crossing-critical* if $\text{cr}(G) \geqslant k$, but $\text{cr}(G-e) < k$ for every edge e of G. Since loops are totally irrelevant for crossing number purposes, all graphs under consideration are loopless.

1.1. The decay of crossing numbers.

In this paper we are concerned with the effect of edge removal in the crossing number of a graph (following Fox and Tóth [10], this is referred to as the *decay of crossing numbers*). Richter and Thomassen [22] proved that every graph G has some edge e such that $\text{cr}(G-e) \geqslant (2/5)\text{cr}(G) - 37/5$. They conjectured that there always exist an edge e such that $\text{cr}(G-e) \geqslant \text{cr}(G) - c\sqrt{\text{cr}(G)}$, for some universal constant c. This conjecture was proved by Fox and Tóth [10] for dense graphs.

Fox and Tóth actually proved a much stronger result: the existence of a large subset of edges whose removal leaves a graph whose crossing number

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is at least a proportion of the crossing number of the original graph. More precisely, they proved that for every fixed $\epsilon > 0$, there is a constant $n_0 = n_0(\epsilon)$ such that if G is a graph with $n > n_0$ vertices and $m > n^{1+\epsilon}$ edges, then G has a subgraph G' with at most $(1 - \frac{\epsilon}{24})m$ edges such that $cr(G') \geqslant (\frac{1}{28} - o(1))\mathrm{cr}(G)$.

This result was further strenghtened by Černý, Kynčl and G. Tóth [5], who proved that for every $\epsilon, \gamma > 0$ there is an $n_0 = n_0(\epsilon, \gamma)$ such that if G is a graph with $n > n_0$ vertices and $m > n^{1+\epsilon}$ edges, then G has a subgraph G' with at most $(1 - \frac{\epsilon\gamma}{1224})m$ edges such that $cr(G') \geqslant (1-\gamma)\mathrm{cr}(G)$.

1.2. **The decay of crossing numbers of sparse graphs.** Due to the Fox-Tóth and the Černý-Kynčl-Tóth results, our understanding of the decay of crossing numbers of dense graphs is essentially complete. The situation for sparse graphs is quite different. Although the Richter and Thomassen result is fully general, it only guarantees the existence of a single edge whose deletion leaves a graph with crossing number substantially large. As pointed out in [10], by combining the following two facts one obtains an improvement to the Richter-Thomassen result for graphs with n vertices and $m > 8.1n$ edges: (i) every graph with $m \geqslant \frac{103}{16}n$ satisfies $\mathrm{cr}(G) \geqslant 0.032\frac{m^3}{n^2}$ [20]; and (ii) for any graph G and any edge e of G, $\mathrm{cr}(G-e) \geqslant \mathrm{cr}(G) - m + 1$ [21].

In this paper we investigate the decay of crossing numbers of sparse graphs. We are particularly interested in establishing results as similar as possible as those in [10] and [5]: the existence of large sets of edges whose removal leaves a graph whose crossing number is at least some (constant) fraction of the crossing number of the original graph.

In contrast with dense graphs, in a sparse graph it is possible to artificially increase the number of edges of a graph, while maintaining its crossing number, without adding any substantial topological feature. Consider, for instance, a graph consisting of a large planar grid plus an additional edge e joining two vertices far apart; subdivide this additional edge r times (for some integer $r > 0$) to get a path P, and let G denote the resulting graph. For any given $\alpha > 0$, we can make r sufficiently large so that any set of at least $\alpha|E(G)|$ edges of G contains at least an edge of P. That is, for any set E_0 of at least $\alpha|E(G)|$ edges of G, the crossing number of $G - E_0$ is 0.

This example shows that no general result can possibly be established if we allow the number of edges to be artificially inflated. In particular, degree 2 vertices need to be precluded from the graphs under consideration. This is a particular instance of a more general way to spuriously increase the number of edges, by substituting a set of (possibly just one) edges joining the same two vertices by a plane connected graph, as we now describe.

We first recall the definition of a bridge. Let G be a graph, and let u, v be distinct vertices of G. Following Tutte, a *uv-bridge* is either a single edge joining u and v, together with u and v (in which case it is *trivial*), or a subgraph of G obtained by adding to a connected component K of $G\backslash\{u, v\}$

all the edges attaching K to u or v, together with their ends. A uv-bridge is *uv-planar* if it can be embedded in the plane with u and v in the same face.

Suppose that u, v are distinct vertices incident with the same face in a connected plane graph H with $|V(H)| > 2$, and let k be the maximum number of pairwise edge-disjoint uv-paths in H. We say that (H, u, v) is a *uv-blob of width k*. Now consider a graph G, and let u, v be vertices of G, joined by $k \geqslant 1$ edges. It is easy to see that we may substitute the edges joining u and v by an arbitrarily large uv-blob of width k, leaving the crossing number (and the criticality of G, if G is critical) unchanged. Conversely, if G is a graph with a vertex cut $\{u, v\}$, and for some $\{u, v\}$-bridge H we have that (H, u, v) is a uv-blob of width k, then G may be simplified, leaving its crossing number (and its criticality, if G is critical) unchanged, by substituting H by k parallel uv-edges.

Note that the concept of uv-blob captures, in particular, the operation of edge subdivision. Indeed, a subdivided edge is simply a uv-blob of width 1, all of whose vertices, other than u and v, have degree 2.

1.3. The main result.

Since we are interested in proving the existence of large sets of edges (linear in the crossing number) with a special property (their removal does not decrease the crossing number arbitrarily), we need to preclude the existence of $\{u, v\}$-bridges (for any pair u, v of vertices) that are uv-blobs, since they inflate the number of edges of a graph, while adding no topologically interesting structure whatsoever to the graph itself.

As it happens, such objects are the *only* structure that needs to be avoided. A graph is *irreducible* if there do not exist vertices u, v and a $\{u, v\}$-bridge H such that (H, u, v) is a uv-blob. We prove that if G is irreducible, then a large set of its edges (linear in the crossing number) may be removed, and still leave a graph whose crossing number is at least a fraction of the crossing number of the original graph. More precisely:

Theorem 1. *For each $\epsilon > 0$ and each positive integer k there exist $m_0 := m_0(\epsilon, k)$ and $\gamma := \gamma(\epsilon)$ with the following property. Every 2-connected irreducible graph G with $\mathrm{cr}(G) = k$ and at least m_0 edges has a set E_0 of at least γk edges such that $\mathrm{cr}(G - E_0) > (1/2 - \epsilon)\mathrm{cr}(G)$.*

Trivially, 3-connected graphs are irreducible, so in particular Theorem 1 applies to all 3-connected graphs.

We also apply our techniques to improve (for sufficiently large graphs) the Richter and Thomassen result on crossing-critical graphs. Richter and Thomassen proved in [22] that every graph G has an edge e such that $\mathrm{cr}(G - e) \geqslant (2/5)\mathrm{cr}(G) - 37/5$.

In order to improve on this result, again we need to be careful not to allow the artificial inflation in the number of edges. However, we do not need the full condition of irreducibility: it suffices to require that each vertex is adjacent to at least 3 other vertices. A slight variant of this requirement (namely *X-minimality*) was introduced by Ding, Oporowski, Thomas, and

Vertigan in [6], with the same motivation of not allowing a graph with given crossing number (in their case, a 2-crossing-critical graph) to spuriously grow its number of edges.

Theorem 2. *For each positive integer k, there is an integer $m_1 := m_1(k)$ with the following property. Let G be a 2-connected graph in which each vertex is adjacent to at least 3 vertices. If $\mathrm{cr}(G) = k$ and G has at least m_1 edges, then G has an edge e such that $\mathrm{cr}(G - e) > (2/3)\mathrm{cr}(G) - 10^8$.*

We conclude this section with a brief overview of the proofs of Theorems 1 and 2, and of the rest of this paper.

As in [5], [10], and [22], we make essential use of the embedding method. This technique consists of finding a set E_0 of edges in a graph G, and for each $e = uv \in E_0$ a set of pairwise edge-disjoint uv-paths $\mathscr{P}(e)$, with the aim of drawing $G - E_0$ (with $\mathrm{cr}(G - E_0)$ crossings) and then embedding each $e \in E_0$ very closely to some path in $\mathscr{P}(e)$. The idea is to choose the set E_0 so that the embedding can be done without adding too many crossings.

Richter and Thomassen proved the existence of an edge $e = uv$ (so that $E_0 = \{e\}$) with the property that there is a uv-path (that avoids e) of length at most 4, all of whose internal vertices have degree less than 12. Fox and Tóth, and Černý-Kynčl-Tóth used the density of G to show the existence of a large set E_0 of edges, such that each edge $e = uv$ of E_0 has a large collection $\mathscr{P}(e)$ of short edge-disjoint paths, and such that the collections $\mathscr{P}(e)$ are pairwise edge-disjoint.

In our current setup (sparse graphs) for all we know the graphs under consideration may have maximum degree 3, and so in general we cannot expect to find collections $\mathscr{P}(e)$ of more than two edge-disjoint paths, for each $e \in E_0$. We prove that, indeed, each graph under consideration has large set E_0 of edges such that each $e = uv \in E_0$ has two short uv-paths $P(e), Q(e)$ whose internal vertices have bounded degree, and if $e \neq f$ then $P(e) \cup Q(e)$ and $P(f) \cup Q(f)$ are edge-disjoint. As it happens, $P(e)$ and $Q(e)$ are not necessarily edge-disjoint, but this turns out to be unimportant. To be slightly more precise, let us mention that each graph $\Xi = e \cup P(e) \cup Q(e)$ has the property that $P(e)$ and $Q(e)$ have length at most ℓ, and the degree of their internal vertices is less than Δ. Following the lively notation in [5], we call each Ξ an (ℓ, Δ)-*earring*.

Most of the rest of this paper is devoted to proving the result described in the previous paragraph. We start by establishing, in Section 2, several assorted statements on planar graphs; these are, in one way or another, elementary consequences of Euler's formula. The existence of a large set of edge-disjoint (ℓ, Δ)-earrings (for certain values of ℓ and Δ) is proved in Section 3 for planar graphs, and in Section 4 for irreducible nonplanar graphs.

In Section 5 we establish the version of the embedding method that we need. The proofs of Theorems 1 and 2 are in Section 6.

In Section 7 we discuss the connection between the decay of crossing numbers and the concept, recently introduced by Mohar and Tamon [18], of expected crossing numbers. Finally, in Section 8 we present some concluding remarks and open questions.

2. Assorted lemmas on planar graphs

A *branch* in a graph is a path whose endpoints have degree at least 3, and all whose internal vertices have degree 2.

Lemma 3. *Let $G = (V, E)$ be a planar graph with minimum degree at least* 2*, and let $B \subseteq V$ be a set of vertices of degree at least* 3*. Suppose that the number of branchs with both endpoints in B is at most s. Then there are at least $|V|/2 - s/2 - (3/2)|B|$ edges with both endpoints in $V \backslash B$.*

Proof. Let $W := V \backslash B$. To help comprehension, we color white (respectively, black) the vertices in W (respectively, B). A branch is *black* if its endpoints are both black. A white vertex is *black-covered* if all its adjacent vertices are black. A black-covered vertex is *of Type I* if it has degree 2; otherwise (that is, if it has degree $\geqslant 3$) it is *of Type II.*

Since there are no black vertices of degree 2, then no black branch can contain more than one Type I vertex. Thus there are at most s Type I vertices.

Let W' denote the set of black-covered vertices of Type II, and let G' denote the subgraph of G induced by the edges incident with a vertex in W'. This is a bipartite graph with bipartition (W', B'), for some $B' \subseteq B$. A standard Euler formula argument yields that $|E(G')| \leqslant 2|V(G')| - 4 = 2|W'| + 2|B'| - 4$. Since each vertex in W' has degree at least 3 (in G', as well as in G) it follows that $|E(G')| = \sum_{v \in W'} d(v) \geqslant 3|W'|$. Thus $3|W'| \leqslant 2|W'| + 2|B'| - 4 \leqslant 2|W'| + 2|B| - 4$, and so $|W'| \leqslant 2|B| - 4$. Thus, there are at most $2|B| - 4$ Type II vertices.

Therefore, the total number of black-covered vertices is at most $s+2|B|-4$. It follows that there are at least $|W| - s - 2|B| + 4 > |W| - s - 2|B|$ white vertices that are adjacent to at least one white vertex, and so there are at least $|W|/2 - s/2 - |B| = |V|/2 - s/2 - (3/2)|B|$ edges with both endpoints in W. □

The *length* of a face in a plane graph is the length of its boundary walk.

A *digon* in an embedded graph consists of two parallel edges, together with their common endpoints. If the endpoints are u and v, then it is a *uv-digon.* A plane embedding of a graph G is *clean* if for each pair of vertices u, v joined by parallel edges, there exist edges e, e' with endpoints u and v, such that the disc bounded by the digon formed by e and e' contains all edges parallel to e and e', and no other edges.

Lemma 4. *Let G be a connected plane graph in which each vertex is adjacent to at least* 3 *vertices. Suppose that the embedding of G is clean. Let $r \geqslant 0$*

be an integer. Let F be the set of faces of G, and let F' be the set of those faces whose length is at most $r+5$. Then $|F'| \geqslant \frac{r|F|+12}{r+3}$.

Proof. Let H be a graph obtained from G as follows: for each pair (u, v) of vertices joined by parallel edges, contract to a single all the parallel edges between u and v. Let F_H denote the set of faces of H, and let F'_H denote the set of faces of H with length at most $r+5$. Our first task is to show that $|F'_H| \geqslant \frac{r|F_H|+12}{r+3}$.

For each $f \in F_H$ the sum $w(f) := \sum_{v \sim f} 1/d(v)$ is the *weight* of f, where $d(v)$ denotes the degree of the vertex v and $v \sim f$ means that v is incident with f. (A vertex v contributes to $w(f)$ as many times as the boundary walk of f passes through v.) Since H is simple and has minimum degree at least 3, then, letting $l(f)$ denote the length of f, we have $l(f) \geqslant 3$ and $w(f) \leqslant l(f)/3$. It is easy to see that $|V(H)| = \sum_{f \in F_H} w(f)$ and $2|E(H)| = \sum_{f \in F_H} l(f)$. From the last two equations and Euler's formula it follows that $2 = \frac{1}{2} \sum_{f \in F_H} \{2w(f) - l(f) + 2\}$.

Since $w(f) \leqslant l(f)/3$, we have

$$12 \leqslant \sum_{f \in F_H} \{-l(f) + 6\} = \sum_{f \in F'_H} \{-l(f) + 6\} + \sum_{f \in F_H - F'_H} \{-l(f) + 6\}.$$

Since $l(f) \geqslant 3$ for each $f \in F_H$, then $-l(f)+6 \leqslant 3$ and thus $\sum_{f \in F'_H} \{-l(f)+6\} \leqslant 3|F'_H|$. If $f \in F_H - F'_H$ then $l(f) - 6 \geqslant r$, that is, $-l(f)+6 \leqslant -r$, and so $\sum_{f \in F_H - F'_H} \{-l(f)+6\} \leqslant -r(|F_H| - |F'_H|)$. Thus, $12 \leqslant 3|F'_H| - r(|F_H| - |F'_H|)$, and so $|F'_H| \geqslant \frac{r|F_H|+12}{r+3}$, as required.

Now as we inflate back H to G, each face in F'_H becomes a face in F'. The other faces in F' are precisely the $t := |E(G) \backslash E(H)|$ faces created in the inflation process, that is, those bounded by parallel edges. Thus $|F| = |F_H| + t$ and $|F'| = |F'_H| + t$. Thus $|F'| - t \geqslant \frac{r(|F|-t)+12}{r+3}$, and so $|F'| \geqslant \frac{r|F|+12}{r+3} + t(1 - \frac{r}{r+3}) \geqslant \frac{r|F|+12}{r+3}$. □

If D is a digon in a plane graph, then the open (respectively, closed) disc bounded by D will be denoted $\Delta(D)$ (respectively, $\overline{\Delta}(D)$). If D, D' are digons, then we write $D' \preceq D$ if $\Delta(D') \subseteq \Delta(D)$. We recall that a vertex of degree 0 is an *isolated vertex*.

Proposition 5. *Let $G = (V, E)$ be a plane graph, and let Z be a set of isolated vertices of G. Suppose that for each digon D in G, the disc bounded by D contains at least one vertex in Z. Then G has at most $3|V \backslash Z| + |Z|$ edges.*

Proof. Let $Y := V \backslash Z$. To help comprehension, we colour the vertices in Y and Z black and green, respectively.

We prove the stronger statement that G has at most $3|Y| + |Z| - 6$ edges. We proceed by induction on the number of digons in G. In the base case G

has no digons, and so by Euler's Formula it has at most $3|Y| - 6$ edges, as required. For the inductive step, we assume that G has at least one digon, and let D be a $\preceq$-minimal digon in G.

Suppose first that D is also $\preceq$-maximal. Then let G' be the graph obtained from G by removing one edge of D and one green vertex contained in $\Delta(D)$. Now G' contains one fewer edge and one fewer green vertex than G. It is easy to see that the induction hypothesis can be applied to G', and so the inductive step follows.

Therefore we may assume that D is not $\preceq$-maximal. Among all digons that contain D, let D' be a $\preceq$-minimal one.

Suppose that D and D' have an edge e in common, and let $\overline{e}$ be the other edge of D. It is easy to see that the induction hypothesis can be applied to the graph obtained from G by removing $\overline{e}$ and a green vertex contained in $\Delta(D)$, and once again the inductive step follows. Thus we may assume that D and D' do not have an edge in common.

If $\Delta(D')$ contains a green vertex not contained in $\Delta(D)$, the situation is again straightforward: the induction hypothesis can be applied to the graph G' obtained by removing one edge of D and one green vertex contained in $\Delta(D)$, and the inductive step follows. Thus we may assume that every green vertex contained in $\Delta(D')$ is contained in $\Delta(D)$.

In this case, there are no digons other than D' and D contained in $\overline{\Delta}(D')$. Now let G' be the graph obtained by removing from G the black vertices and all the edges contained in $\Delta(D')$. Let Y' and Z' denote the sets of black and green vertices of G', respectively, and let E' denote the set of edges of G' (note that $Z' = Z$). We may clearly apply the induction hypothesis to G', obtaining that $|E'| \leqslant 3|Y'| + |Z| - 6$. Let $Y'' := Y \backslash Y'$, and $E'' := E \backslash E'$. Let x, y be the vertices of D'. Consider the graph G'' that consists of the vertices in $Y'' \cup \{x, y\}$ and the edges in E''. Since G'' has exactly one digon (namely D), the usual Euler formula argument yields $|E(G'')| \leqslant 3|V(G'')| - 5$. However, this inequality is tight only if G'' is maximally planar, that is, if no edge can be added between two nonadjacent vertices while maintaining planarity; thus, since x and y are not adjacent in G'', it follows that $|E(G'')| \leqslant 3|V(G'')| - 6$. Thus $|E''| \leqslant 3(|Y''| + 2) - 6$. That is, $|E| - |E'| \leqslant 3(|Y| - |Y'| + 2) - 6$, and so $|E| \leqslant 3|Y| + |Z| - 6$, as required. □

A set Z of vertices in a 2-connected planar graph G is an *anchor* if the following hold:

(1) no vertex in Z is part of a 2-vertex-cut in G; and
(2) if $\{u, v\}$ is a 2-vertex-cut in G, then every nontrivial uv-bridge contains a vertex in Z.

Lemma 6. *Let G be a 2-connected plane graph in which each vertex is adjacent to at least 3 distinct vertices, and let Z be an anchor of G. Let $Y \subseteq V(G) \backslash Z$, and let E_Y denote the set of edges of G with both endpoints in Y. Then the number of faces of G that are incident with exactly 2 vertices of Y is at most $3|Y| + |Z| + |E_Y|$.*

Proof. We may assume that $|Y| \geqslant 2$, as otherwise there is nothing to prove. Let F_2 denote the set of faces of G that are incident with exactly two vertices of Y.

We start by coloring red each edge in E_Y, and green each vertex in Z. Now for each $f \in F_2$, join the two vertices in Y incident with f by a simple *blue* arc contained (except, obviously, for its endpoints) in f. Let H denote the plane graph that consists of the vertices in Y plus all the red edges and the blue arcs (now seen as edges), as well as the set Z of green vertices. Note that the green vertices are isolated in H. We remark that $|F_2|$ is the number of blue edges in H.

Note that if D is a blue digon in H (that is, both edges of D are blue), with vertices u and v, then $\overline{\Delta}(D)$ contains a uv-bridge in G. This bridge may be trivial (in which case it is a red edge) or nontrivial (in which case, by hypothesis, $\Delta^o(D)$ contains a green vertex).

Finally, let K denote the graph that results from H by substituting each red edge by an isolated red vertex (placed in the interior of the red edge). Note that $|E(K)| = |F_2|$, that the vertex set of K is the union of Y with the set of all green or red vertices, and that there are $|Z|$ green and $|E_Y|$ red vertices.

The graph K has the property that for each (necessarily blue) digon D in K, $\Delta^o(D)$ contains either a green or a red vertex. Applying Proposition 5 we obtain that $|E(K)| \leqslant 3|Y| + |Z| + |E_Y|$. Thus $|F_2| \leqslant 3|Y| + |Z| + |E_Y|$, as required. □

If G is a plane graph, then we let G^o denote its dual.

Lemma 7. *Let G be a 2-connected plane graph, and let Z be an anchor of G. Suppose that the embedding of G is clean. Let F' be a set of faces of G of length at least 3. Then the number of branchs in G^o with both endpoints in F' is at most $3|F'| + |Z|$.*

Proof. Since the embedding is clean, we may as well assume (in the context of this lemma) that G has no parallel edges. It follows that all branchs with both endpoints in F' are actual edges in G^o. Thus our goal is to show that there are at most $3|F'| + |Z|$ edges in G^o with both endpoints in F'.

Regarding G and G^o as simultaneously embedded, remove everything except for F' (seen as a set of vertices in G^o), the edges (in G^o) joining two vertices in F', and the vertices in Z. The result is a graph G' in which each vertex in Z is isolated, and such that the disc bounded by every digon contains a vertex in Z. To see this last property, note that if e and f are the edges of a digon in G', then the edges in G corresponding to e and f are a 2-edge-cut in G; since Z is an anchor set of G, it then follows that the disc bounded by the digon must contain a vertex of Z in its interior.

Applying Proposition 5, we obtain that G' has at most $3|F'| + |Z|$ edges. This finishes the proof, since there is a bijection between the edges in G' and the edges in G^o with both endpoints in F'. □

3. Earrings in planar graphs

Černý, Kynčl and Tóth introduced the lively terminology of *earring of size* p to describe a graph consisting of an edge $e = uv$ plus a collection of p pairwise edge-disjoint, bounded-length uv-paths. In order to use the re-embedding method, the goal is to find many pairwise edge-disjoint earrings.

As we mentioned in Section 1, in our current context of sparse graphs, where (for all we know) the graphs under consideration may have maximum degree 3, the best we could hope for is to prove the existence of a large collection of earrings, each of size 2. As we also mentioned, in this discussion we do not need the two uv-paths of each earring to be edge-disjoint, but only a weaker condition (see (iii) in the following definition).

Let ℓ, Δ be positive integers. An (ℓ, Δ)-*earring* of a graph G is a subgraph of G that consists of a *base* edge $e = uv$ plus two distinct uv-paths P, Q (disjoint from e) with the following properties: (i) each of P and Q has at most ℓ edges; (ii) each internal vertex of P or Q has degree less than Δ; and (iii) if f is an edge in both P and Q, then $\{e, f\}$ is a 2-edge-cut of G.

An edge $e = uv$ in a 2-connected plane graph is an (ℓ, Δ)-*edge* if each of its two incident faces has length at most $\ell + 1$, and no vertex incident with these two faces, other than possibly u or v, has degree Δ or greater. If e is an (ℓ, Δ)-edge, then the subgraph that consists of e plus the cycles that bound its two incident faces, is an (ℓ, Δ)-earring, the (ℓ, Δ)-earring $\Xi(e)$ *associated to* e.

The following lemma is the main workhorse in this paper.

Lemma 8. *Let* $G = (V, E)$ *be a 2-connected planar graph in which each vertex is adjacent to at least* 3 *other vertices. Let* Z *be an anchor of* G*, where each vertex in* Z *has degree* 4*. Then* G *has at least* $10^{-10}|E| - 10^{-5}|Z|$ *pairwise edge-disjoint* $(5000, 500)$*-earrings.*

Proof. Throughout the proof, we make use of several constants that are either very small, very close to 1, or somewhat large. In order to simplify the whole discussion, we first proceed to introduce these constants. We let $\ell_0 = 5000$, $\Delta_0 = 500$, $\mathbf{c}_1 = 10^{-10}$, $\mathbf{c}_2 = 10^{-5}$, $\mathbf{c}_3 = 999/1000$, $\mathbf{c}_4 = 1/1000$, $\mathbf{c}_5 = 999$, $\mathbf{c}_6 = 36/5000$, and $\mathbf{z}_1 = 3(10^{-10})$.

It is a trivial observation that every planar graph has a clean plane embedding (clean embeddings are defined before Lemma 4). Throughout the proof we consider a fixed clean embedding of G in the plane. Let F denote the set of all faces of G, and let $t := |Z|$.

Claim 9. *It suffices to show that there are at least* $(2\ell_0(2\ell_0 + 1)+1)\cdot(\mathbf{z}_1|F| - \mathbf{c}_2 t)$ (ℓ_0, Δ_0)*-edges.*

Proof. Consider the graph H whose vertices are the (ℓ_0, Δ_0)-edges of G, with two distinct (ℓ_0, Δ_0)-edges e, f adjacent if $\Xi(e)$ and $\Xi(f)$ have some edge in common.

We note that H has maximum degree at most $2\ell_0(2\ell_0 + 1)$. This follows at once from the following two easy observations: (i) for each (ℓ_0, Δ_0)-edge

e, $\Xi(e)$ has at most $2\ell_0$ edges other than e; and (ii) each edge of G belongs to at most $2\ell_0+1$ (ℓ_0,Δ_0)-earrings of the form $\Xi(f)$ for some edge f.

Thus, $V(H)$ has a stable set of size at least $|V(H)|/(2\ell_0(2\ell_0+1)+1)$. Suppose that G has at least $(2\ell_0(2\ell_0+1)+1)\cdot(\mathbf{z}_1|F|-\mathbf{c}_2t)$ (ℓ_0,Δ_0)-edges; that is, $|V(H)| \geqslant (2\ell_0(2\ell_0+1)+1)\cdot(\mathbf{z}_1|F|-\mathbf{c}_2t)$. Then H has a stable set S of size at least $\mathbf{z}_1|F|-\mathbf{c}_2t$; that is, there is a collection of at least $\mathbf{z}_1|F|-\mathbf{c}_2t$ pairwise edge-disjoint (ℓ_0,Δ_0)-earrings.

Since G has minimum degree at least 3, a routine Euler formula argument yields that $|F| \geqslant |E|/3+2$. Thus there are at least $\mathbf{z}_1(|E|/3+2)-\mathbf{c}_2t > \mathbf{c}_1|E|-\mathbf{c}_2t$ pairwise edge-disjoint (ℓ_0,Δ_0)-earrings, as required in Lemma 8. □

Let W be the set of those vertices of G with degree at least Δ_0, and let F_W denote the set of faces of G that are incident with some vertex in W. For each integer $j \geqslant 1$, let F_j denote the set of those faces of G incident with exactly j vertices in W (and perhaps other vertices in $V\backslash W$), and let $f_j = |F_j|$. Note that F_W is the disjoint union $\bigcup_{i\geqslant 1} F_i$.

Let F_{long} (respectively, F_{short}) denote the collection of faces of G with length greater than (respectively, at most) ℓ_0+1, and let $f_{\mathrm{long}} := |F_{\mathrm{long}}|$ and $f_{\mathrm{short}} := |F_{\mathrm{short}}|$. It follows immediately from Lemma 4 that

$$f_{\mathrm{short}} \geqslant \mathbf{c}_3|F|. \tag{1}$$

Since F is the disjoint union of F_{long} and F_{short}, then $|F| = f_{\mathrm{long}} + f_{\mathrm{short}}$, and so $f_{\mathrm{short}} \geqslant \mathbf{c}_3(f_{\mathrm{long}}+f_{\mathrm{short}})$ implies $f_{\mathrm{short}} \geqslant (\mathbf{c}_3/(1-\mathbf{c}_3))f_{\mathrm{long}}$. Note that $\mathbf{c}_5 = \mathbf{c}_3/(1-\mathbf{c}_3)$. Therefore,

$$f_{\mathrm{short}} \geqslant \mathbf{c}_5 f_{\mathrm{long}}. \tag{2}$$

We note that $\sum_{u\in W} d(u) = \sum_{i\geqslant 1} i f_i$. A routine application of Euler's formula yields that $\sum_{i\geqslant 3} i f_i \leqslant 2(3|W|-6) = 6|W|-12$. Since all vertices of Z have degree 4 it follows that $W \subseteq V\backslash Z$, and so we can apply Lemma 6, to obtain $f_2 \leqslant 3|W|+t+|E_W|$. Combining these observations we obtain

$$f_1 \geqslant \sum_{u\in W} d(u) - 12|W| - 2|E_W| - 2t + 12. \tag{3}$$

Claim 10. *If $|F_W| > 24t + 24\mathbf{c}_4 f_{\mathrm{short}}$, then Lemma 8 follows.*

Proof. We establish four subclaims, and finally show that the proof follows easily from them.

SUBCLAIM A *If $|E_W| > 6|W|-12+\mathbf{c}_4 f_{\mathrm{short}}$, then Lemma 8 follows.*

Proof. If e_1, e_2, e_3 are parallel edges with common endpoints u, v, and e_2 is in the disc bounded by the digon formed by e_1 and e_3, then e_2 is a *sheltered* edge. By Euler's formula, a simple graph on $|W|$ vertices has at most $3|W|-6$ edges. Since the embedding of G is clean, it follows that the subgraph of G induced by W has at least $|E_W|-2(3|W|-6) = |E_W|-6|W|+12$ sheltered edges. The fact that G is clean also implies that each sheltered edge is a (ℓ_0, Δ_0)-edge, and so G has at least $|E_W| - 6|W| + 12$ (ℓ_0, Δ_0)-edges.

Suppose that $|E_W| > 6|W| - 12 + \mathbf{c}_4 f_{\text{short}}$. Then G has at least $\mathbf{c}_4 f_{\text{short}}$ (ℓ_0, Δ_0)-edges. Using (1), it follows that G has at least $\mathbf{c}_3\mathbf{c}_4|F|$ (ℓ_0, Δ_0)-edges. The result now follows from Claim 9, since $\mathbf{c}_3\mathbf{c}_4 > (2\ell_0(2\ell_0 + 1) + 1)\mathbf{z}_1$. □

Subclaim B *If* $(1/6)(\sum_{u\in W} d(u)) \leqslant 12|W| + 2t + 2|E_W| - 12$, *then* $|F_W| < 24t + 24\mathbf{c}_4 f_{\text{short}}$ *or else Lemma 8 follows.*

Proof. By Subclaim A, under the given hypothesis we may assume that $\sum_{u\in W} d(u) \leqslant 72|W| + 12t + (72|W| - 144 + 12\mathbf{c}_4 f_{\text{short}}) - 72 = 144|W| + 12t + 12\mathbf{c}_4 f_{\text{short}} - 216 < 144|W| + 12t + 12\mathbf{c}_4 f_{\text{short}}$.

Since each vertex in W has degree at least Δ_0, it follows that $\Delta_0|W| \leqslant \sum_{u\in W} d(u)$. Hence, $|W| < (12t + 12\mathbf{c}_4 f_{\text{short}})/(\Delta_0 - 144)$. On the other hand, obviously $|F_W| \leqslant \sum_{u\in W} d(u)$, and so $|F_W| < 144(12t + 12\mathbf{c}_4 f_{\text{short}})/(\Delta_0 - 144) + 12t + 12\mathbf{c}_4 f_{\text{short}}$. Since $144/(\Delta_0 - 144) \leqslant 1$, this implies $|F_W| < 12t + 12\mathbf{c}_4 f_{\text{short}} + 12t + 12\mathbf{c}_4 f_{\text{short}} = 24t + 24\mathbf{c}_4 f_{\text{short}}$. □

Subclaim C *If* $(1/6)(\sum_{u\in W} d(u)) \leqslant f_{\text{long}}$, *then* $|F_W| \leqslant 6 f_{\text{short}}/\mathbf{c}_5$.

Proof. Suppose that $(1/6)(\sum_{u\in W} d(u)) \leqslant f_{\text{long}}$. The obvious inequality $|F_W| \leqslant \sum_{u\in W} d(u)$ then implies that $|F_W| \leqslant 6 \cdot f_{\text{long}}$. The required inequality follows from (2). □

Subclaim D *If* $(1/6)(\sum_{u\in W} d(u)) > 12|W| + 2t + 2|E_W| - 12$ *and* $(1/6)(\sum_{u\in W} d(u)) > f_{\text{long}}$, *then* $|F_W| \leqslant \mathbf{c}_6 f_{\text{short}}$ *or else Lemma 8 follows.*

Proof. We show that, under the given hypotheses, if $|F_W| > \mathbf{c}_6 f_{\text{short}}$, then there are at least $(\mathbf{c}_3\mathbf{c}_6/3)|F|$ (ℓ_0, Δ_0)-edges; the subclaim then follows from Claim 9, since $(\mathbf{c}_3\mathbf{c}_6)/3 \geqslant (2\ell_0(2\ell_0 + 1) + 1) \cdot \mathbf{z}_1$.

It follows that, under the current hypotheses,

$$f_{\text{long}} < (1/3) \sum_{u\in W} d(u) - 12|W| - 2t - 2|E_W| + 12. \tag{4}$$

Since $|F_1 \backslash F_{\text{long}}| \geqslant f_1 - f_{\text{long}}$, using (3) and (4) we obtain

$$|F_1 \backslash F_{\text{long}}| \geqslant \sum_{u\in W} d(u) - 12|W| - 2|E_W| - 2t + 12 - f_{\text{long}} > (2/3) \sum_{u\in W} d(u).$$

Since each face in F_1 is (by definition) incident with exactly one vertex in W, the inequality $|F_1 \backslash F_{\text{long}}| > (2/3) \sum_{u\in W} d(u)$ implies that at least 1/3 of the edges incident with W have their two incident faces in $F_1 \backslash F_{\text{long}}$. Note that all such edges are (ℓ_0, Δ_0)-edges. We conclude that there are at least $(1/3) \sum_{u\in W} d(u)$ (ℓ_0, Δ_0)-edges incident with W. Since obviously $\sum_{u\in W} d(u) \geqslant |F_W|$, this implies that there are at least $|F_W|/3$ (ℓ_0, Δ_0)-edges.

Using the assumption $|F_W| > \mathbf{c}_6 f_{\text{short}}$ and (1), it follows that there are at least $(\mathbf{c}_3\mathbf{c}_6/3)|F|$ (ℓ_0, Δ_0)-edges, as required. $\square$

We now complete the proof of Claim 10.

Since the hypotheses of Subclaims B, C, and D are exhaustive, it follows from these subclaims that either we may assume that $|F_W| < 24t + 24\mathbf{c}_4 f_{\text{short}}$, or $|F_W| \leqslant 6f_{\text{short}}/\mathbf{c}_5$, or we may assume that $|F_W| \leqslant \mathbf{c}_6 f_{\text{short}}$. Since $\max\{24\mathbf{c}_4, 6/\mathbf{c}_5, \mathbf{c}_6\} = 24\mathbf{c}_4$, it follows that we may assume that $|F_W| < 24t + 24\mathbf{c}_4 f_{\text{short}}$. $\square$

We now complete the proof of Lemma 8.

A face is *white* if it is either in $F_{\text{short}} \backslash F_W$ or has length exactly 2, and is *black* otherwise. We let $F_\circ$ (respectively, $F_\bullet$) denote the set of all white (respetively, black) faces. Let $f_\circ := |F_\circ|$, and $f_\bullet := |F_\bullet|$.

Now consider the dual G^o of G. The 2-connectivity of G implies that G^o is also 2-connected. Let us say that an edge in G^o is *white* if its endpoints are both white (faces in G).

The key (and completely straightforward) observation is that the edge of G associated to each white edge is an (ℓ_0, Δ_0)-edge. Our final goal is to prove that there are many white edges.

Every face in $F_\bullet$ is either in F_{long} or in F_W, and so $f_\bullet \leqslant f_{\text{long}} + |F_W|$. Using (2), Claim 10, and the obvious inequality $f_{\text{short}} \leqslant |F|$, we obtain

$$f_\bullet \leqslant 24t + (24\mathbf{c}_4 + 1/\mathbf{c}_5)|F|. \tag{5}$$

By Lemma 7, G^o has at most $3f_\bullet + t$ branchs with both endpoints black. Lemma 3 (applied to G^o) then implies that there are at least $|F|/2 - (3f_\bullet + t)/2 - (3/2)f_\bullet = |F|/2 - 3f_\bullet - t/2 \geqslant (1/2 - 3(24\mathbf{c}_4 + 1/\mathbf{c}_5))|F| - (145/2)t$ white edges.

As we have observed, the edge of G associated to each white edge is an (ℓ_0, Δ_0)-edge. Thus there are at least $(1/2 - 3(24\mathbf{c}_4 + 1/\mathbf{c}_5))|F| - (145/2)t$ (ℓ_0, Δ_0)-edges. Since $1/2 - 3(24\mathbf{c}_4 + 1/\mathbf{c}_5) \geqslant (2\ell_0(2\ell_0 + 1) + 1) \cdot \mathbf{z}_1$ and $145/2 \leqslant (2\ell_0(2\ell_0 + 1) + 1) \cdot \mathbf{c}_2$, then we are done by Claim 9. $\square$

4. Earrings in nonplanar graphs

Lemma 11. *Let $G = (V, E)$ be a 2-connected irreducible graph. Then G has at least $10^{-10}|E| - (10^{-5} + 2)\mathrm{cr}(G)$ pairwise edge-disjoint $(5000, 500)$-earrings.*

Proof. Let $\ell_0 := 5000$, $\Delta_0 := 500$, $\mathbf{c}_1 := 10^{-10}$, $\mathbf{c}_2 := 10^{-5}$, and $\mathbf{c}_7 := (10^{-5} + 2)$. Let $t := \mathrm{cr}(G)$, and let $\mathcal{D}$ be a drawing of G with exactly t crossings. Let H denote the plane graph that results by regarding the t crossings as degree 4 vertices (this is the *crossings-to-vertices conversion*), which we colour green to help comprehension (the other vertices of H, each of which corresponds to a vertex in G, are coloured black). We claim that (i) each vertex in H is adjacent to at least 3 other vertices; (ii) no green vertex is part of a 2-vertex-cut; (iii) H is 2-connected; and (iv) the set of green vertices is an anchor set for H.

We start by noting that (i) follows easily from the irreducibility of G, plus the observation that in any crossing-minimal drawing of any graph, the two edges involved in any crossing cannot have a common endpoint.

By way of contradiction, suppose that u, v are green vertices such that $\{u, v\}$ is a 2-vertex-cut in H. It is easy to see that then there are exactly two uv-bridges. Let B be any of these uv-bridges, and let H' denote the plane graph obtained from H by performing a Whitney switching on B around u and v. Now by reversing the crossings-to-vertices conversion, we obtain from H' a drawing of G in which the edge intersections corresponding to u and v are tangential, not crossings. Each of these two tangential edge intersections may be removed with a small perturbation, yielding a drawing of G with two fewer crossings than $\mathcal{D}$, contradicting the crossing-minimality of $\mathcal{D}$. This contradiction shows that $\{u, v\}$ cannot be a 2-vertex-cut in H. A similar contradiction is obtained from the assumption that H has a 2-vertex-cut with exactly one green vertex (in this case one obtains a drawing of G with one fewer crossing than $\mathcal{D}$). This proves (ii).

The 2-connectedness of G readily implies that no black vertex can be a cut vertex of H. On the other hand, a similar switching argument as in the proof of (ii) shows that no green vertex can be a cut vertex of H. This proves (iii).

Now let u, v be black vertices such that $\{u, v\}$ is a 2-vertex-cut in H, and let B be a nontrivial uv-bridge. If B does not contain any green vertex, then (B, u, v) is clearly a uv-blob of G. Since this contradicts the irreducibility of G, (iv) follows.

We can thus apply Lemma 8 to H, and obtain that H has a collection $\mathcal{E}$ of at least $\mathbf{c}_1|E(H)| - \mathbf{c}_2 t$ pairwise edge-disjoint (ℓ_0, Δ_0)-earrings. If any such earring contains a green vertex, then it obviously contains at least two edges incident with a green vertex. Since these earrings are pairwise edge-disjoint, it immediately follows that $\mathcal{E}$ has a subcollection $\mathcal{E}'$, with $|\mathcal{E}'| \geqslant |\mathcal{E}| - 2t$ pairwise edge-disjoint (ℓ_0, Δ_0)-earrings that do not contain any green vertex. That is, each earring in $\mathcal{E}'$ is an (ℓ_0, Δ_0)-earring of G.

Therefore, $\mathcal{E}'$ is a collection at least $|\mathcal{E}| - 2t \geqslant \mathbf{c}_1|E(H)| - (\mathbf{c}_2 + 2)t$ pairwise edge-disjoint (ℓ_0, Δ_0)-earrings in G. Since $|E(H)| \geqslant |E|$, it follows that $|\mathcal{E}'| \geqslant \mathbf{c}_1|E| - (\mathbf{c}_2 + 2)t = \mathbf{c}_1|E| - \mathbf{c}_7 t$. □

5. The embedding method: adding edges with few crossings

Our main goal is to show that every (sufficiently large) irreducible graph has a large collection of edges whose removal leaves a graph with large crossing number. The first main ingredient is the existence of a large collection of pairwise edge-disjoint (ℓ, Δ)-earrings (for some fixed ℓ and Δ); this is Lemma 8. The second main ingredient is the *embedding method*, which was used under similar circumstances by Richter and Thomassen [22], Fox and

Tóth [10], and Černý, Kynčl and Tóth [5] (see also [13, 24, 26]). We use the embedding method to prove the following.

Lemma 12. *Let G be a graph, and let ℓ, Δ, and r be positive integers. Suppose that G has a collection of r pairwise edge-disjoint (ℓ, Δ)-earrings. Then G has a set E_0 of r edges such that* $\mathrm{cr}(G - E_0) > (1/2)\mathrm{cr}(G) - (1/2)(\Delta\ell + \ell^2)r$.

Proof. Let $\Xi_1, \Xi_2, \ldots, \Xi_r$ be a collection of pairwise edge-disjoint (ℓ, Δ)-earrings in G. For $i = 1, 2, \ldots, r$, let $e_i = u_i v_i$ be the base edge of Ξ_i, and let P_i, Q_i be the $u_i v_i$-paths such that $\Xi_i = P_i \cup Q_i \cup \{e_i\}$. We shall show that $E_0 := \{e_1, e_2, ..., e_r\}$ satisfies the required property.

Let $t := \mathrm{cr}(G - E_0)$, and let $\mathcal{D}$ be a drawing of $G - E_0$ with t crossings. The strategy is to extend $\mathcal{D}$ to a drawing of G by drawing e_i very close to either P_i or Q_i, for $i = 1, 2, \ldots, r$. Our aim is to show that this can be done while adding relatively few crossings.

We analyze several types of crossings of P_i and Q_i, for $i = 1, 2, \ldots, r$. A crossing in $\mathcal{D}$ is (i) *of Type 1* if one edge is in P_i and the other edge is in Q_i, for some $i \in \{1, \ldots, r\}$; (ii) *of Type 2A* if one edge is in $P_i \cup Q_i$ and the other edge is in $P_j \cup Q_j$, for some $i \neq j$, $i, j \in \{1, \ldots, r\}$; and (iii) *of Type 2B* if one edge is in $P_i \cup Q_i$ for some $i \in \{1, \ldots, r\}$ and the other in $E(G)\backslash \bigcup_{j=1}^{r}(P_j \cup Q_j)$. Note that if a crossing $\times$ involving an edge of $\bigcup_{i=1}^{r} P_i \cup Q_i$ is neither of Type 1, nor 2A, nor 2B, then the edges involved in $\times$ must be both in P_i or both in Q_i, for some $i \in \{1, 2, .., r\}$. As we shall see, this last type of crossing is irrelevant to our discussion.

For $i = 1, 2, \ldots, r$ and $k \in \{1, 2\}$, let $\chi_k(P_i)$ (respectively, $\chi_k(Q_i)$) denote the number of crossings of Type k that involve an edge in P_i (respectively, Q_i).

In every crossing-minimal drawing of any graph, no pair of edges cross each other more than once. Since each of P_i and Q_i has at most ℓ edges, it follows that

$$\chi_1(P_i) \leqslant \ell^2, \text{for } i = 1, \ldots, r. \tag{6}$$

Now let $\mathscr{R}$ be the set of all sequences $(R_1, R_2, \ldots, R_r)$, with $R_i \in \{P_i, Q_i\}$ for $i = 1, 2, \ldots, r$, and consider the sum $\Sigma := \sum_{R \in \mathscr{R}} \big(\sum_{i=1}^{r} \chi_2(R_i)\big)$.

We claim that a crossing of Type 2A contributes in exactly 2^r to Σ. To see this, first note that such a crossing involves an edge of an $R_i \in \{P_i, Q_i\}$ and an edge of an $R_j \in \{P_j, Q_j\}$ for some $i \neq j$. Let T_i (respectively, T_j) be the element in $\{P_i, Q_i\}\backslash R_i$ (respectively, $\{P_j, Q_j\}\backslash R_j$). There are 2^{r-2} sequences in $\mathscr{R}$ that include both R_i and R_j, and so for each such sequence, the crossing contributes in 2 to Σ. There are 2^{r-2} sequences in $\mathscr{R}$ that include R_i and do not include R_j, and so for each such sequence, the crossing contributes in 1 to Σ. Analogously, there are 2^{r-2} sequences in $\mathscr{R}$ that include R_j and do not include R_i, and so for each such sequence, the crossing contributes in 1 to Σ. Therefore each crossing of Type 2A contributes in $2 \cdot 2^{r-2} + 2^{r-2} + 2^{r-2} = 2^r$ to Σ, as claimed. Note that this reasoning assumes that no crossing of Type

2A is in both P_i and Q_i for the same i. This is immediate if P_i and Q_i are edge-disjoint, but we recall from our definition of earring that P_i and Q_i may share edges. However, the validity of our reasoning follows since (again, by the definition of earring) any edge $f \in E(P_i) \cap E(Q_i)$ is a cut edge of $G - e_i$, from which it follows that f cannot be crossed in any optimal drawing of $G - E_0$.

We also note that a crossing of Type 2B contributes to Σ in exactly 2^{r-1}. Indeed, such a crossing involves (for some fixed i) an edge of R_i and an edge that belongs to no R_j; it contributes in 1 to $\chi(R_i)$, and there are 2^{r-1} sequences in $\mathscr{R}$ that include R_i. (As in the previous paragraph, we remark that we are making use of the valid assumption that no crossing is in both P_i and Q_i for the same i).

In conclusion, each crossing of Type 2A or 2B contributes to Σ in at most 2^r. Since only crossings of Types 2A and 2B contribute to Σ, and $\mathcal{D}$ has t crossings in total, we conclude that $\sum_{R \in \mathscr{R}} \big(\sum_{i=1}^{r} \chi_2(R_i) \big) \leqslant 2^r t$. Since $|\mathscr{R}| = 2^r$, it follows that for some sequence $(R_1, R_2, \ldots, R_r) \in \mathscr{R}$, $\sum_{i=1}^{r} \chi_2(R_i) \leqslant t$. By relabeling (exchanging) P_i and Q_i if necessary, we may assume without any loss of generality that $R_i = P_i$ for each $i = 1, 2, \ldots, r$, and so

$$\sum_{i=1}^{r} \chi_2(P_i) \leqslant t. \tag{7}$$

Now note that some P_i may have self-crossings. However, for each i there is a simple curve α_i, contained in P_i, joining u_i and v_i. The definition of crossings of types 1, 2A, and 2B obviously extend to the crossings on each α_i, and so (6) and (7) imply that $\chi_1(\alpha_i) \leqslant \ell^2$ for $i = 1, 2, \ldots, r$, and $\sum_{i=1}^{r} \chi_2(\alpha_i) \leqslant t$. Moreover (this is the effect of having obtained α_i by avoiding the self-crossings of its corresponding P_i), for $i = 1, 2, \ldots, r$, each crossing of α_i is of one of these types.

The idea is to draw each e_i very close to its corresponding α_i. There are two kinds of crossings on the resulting drawings of e_i, $i = 1, \ldots, r$. Some crossings occur as we traverse e_i and pass very close to a crossing of α_i. The inequalities in the previous paragraph imply that there are, in total, at most $\ell^2 r + t$ crossings of this first kind. The second kind of crossing occurs as we pass very close to a vertex in α_i, and cross some edges incident with this vertex. Since each such vertex is an internal vertex of some P_i (that is, has degree $< \Delta$) and there are at most $\ell - 1$ internal vertices in each P_i, we conclude that each e_i has fewer than $\Delta\ell$ crossings of this second kind. Thus in total there are fewer than $\Delta\ell r$ crossings of the second kind.

We conclude that all the edges $e_1, e_2, \ldots, e_r$ may be added to the drawing $\mathcal{D}$ of $G - E_0$ by introducing fewer than $(\Delta\ell + \ell^2)r + t$ crossings. Since $t = \mathrm{cr}(G - E_0)$, it follows that $\mathrm{cr}(G) < 2\mathrm{cr}(G - E_0) + (\Delta\ell + \ell^2)r$ or, equivalently, $\mathrm{cr}(G - E_0) > (1/2)\mathrm{cr}(G) - (1/2)(\Delta\ell + \ell^2)r$. $\square$

If we are interested in removing only one edge (as we are in Theorem 2), we can improve the 1/2 coefficient in Lemma 12 to 2/3, as the following statement shows.

Lemma 13. *Let G be a graph, and let ℓ and Δ be positive integers. Suppose that G has an (ℓ, Δ)-earring. Then G has an edge e such that* $\mathrm{cr}(G-e) > (2/3)\mathrm{cr}(G) - (2/3)(\Delta\ell + \ell^2)$.

Proof. The proof is essentially the same as the proof of Lemma 13, with the following favourable exception. If we consider only one earring, then $r = 1$, and so there are no crossings of Type 2A. Each crossing of Type 2B contributes to Σ in at most 1, and so $\chi_2(P_1) + \chi_2(Q_1) \leqslant t$. By exchanging P_1 and Q_1 if necessary, we may assume that $\chi_2(P_1) \leqslant t/2$.

In parallel to the last paragraph of the proof of Lemma 13, in the present case we conclude that the edge e_1 may be added to the drawing $\mathcal{D}$ of $G - E_0 = G - e_1$ by introducing fewer than $(\Delta\ell + \ell^2) + t/2$ crossings. Since $t = \mathrm{cr}(G - e_1)$, it follows that $\mathrm{cr}(G) < (3/2)\mathrm{cr}(G - e_1) + \Delta\ell + \ell^2$ or, equivalently, $\mathrm{cr}(G - e_1) > (2/3)\mathrm{cr}(G) - (2/3)(\Delta\ell + \ell^2)$. □

6. Proof of Theorems 1 and 2

Proof of Theorem 1. Let $\ell_0 := 5000$ and $\Delta_0 := 500$, $\mathbf{c}_1 := 10^{-10}$, and $\mathbf{c}_7 := (10^{-5} + 2)$. Let k be a positive integer and let $\epsilon > 0$. Define $\gamma := \epsilon/((1/2)(\Delta_0\ell_0 + \ell_0^2))$ and $m_0 := ((\mathbf{c}_7 + \gamma)k)/\mathbf{c}_1$. Let $G = (V, E)$ be a 2-connected irreducible graph with $\mathrm{cr}(G) = k$ and at least m_0 edges.

Lemma 11 implies that G has a collection of at least $\mathbf{c}_1|E| - \mathbf{c}_7 k$ pairwise edge-disjoint (ℓ_0, Δ_0)-earrings. Since $|E| \geqslant ((\mathbf{c}_7 + \gamma)k)/\mathbf{c}_1$, it follows that G has a collection of at least γk pairwise edge-disjoint (ℓ_0, Δ_0)-earrings. Thus, by Lemma 12, G has a collection E_0 of at least γk edges such that $\mathrm{cr}(G - E_0) > (1/2)\mathrm{cr}(G) - (1/2)(\Delta_0\ell_0 + \ell_0^2)\gamma k = (1/2)\mathrm{cr}(G) - \epsilon k = ((1/2) - \epsilon)\mathrm{cr}(G)$. □

If u, v are vertices of a graph G, a *double uv-path* is a subgraph of G that consists of a uv-path with all its edges doubled.

Proof of Theorem 2. Let $\ell_0 := 5000$, $\Delta_0 := 500$, $\mathbf{c}_1 := 10^{-10}$, and $\mathbf{c}_7 := (10^{-5} + 2)$. Let k be a positive integer, and let $m_1 := (\mathbf{c}_7 k)/\mathbf{c}_1 + 1$. We prove that if $G = (V, E)$ is a 2-connected graph in which each vertex is adjacent to at least 3 vertices, $\mathrm{cr}(G) = k$, and G has at least m_1 edges, then G has an edge e such that $\mathrm{cr}(G - e) > (2/3)\mathrm{cr}(G) - 10^8$.

Suppose first that G is not irreducible, and let (B, u, v) be a minimal blob in G, (that is, G has no blob (B', u', v') such that B' is a subgraph of B). The minimality of B implies that B has no cut edges, and so its width $w(B)$ is at least 2. It is easy to see that if every edge of B is in a 2-edge-cut separating u and v, then B is a double uv-path. This clearly contradicts the X-minimality of G, and so we conclude that there is an edge e in B such that the uv-blob (in $G - e$) $B - e$ has width at least 2.

By way of contradiction, suppose that $\mathrm{cr}(G-e) < (2/3)\mathrm{cr}(G)$. It is straightforward to see that there is a crossing-minimal drawing $\mathcal{D}$ of $G-e$ in which the set E' of edges crossed in $B-e$ form a smallest *uv-edge cut* (that is, a minimum size edge cut in $B-e$ separating u and v), with each edge in E' crossed the same number (say s) of times. In particular, $\mathrm{cr}(G-e) \geqslant |E'|s \geqslant 2s$. The planarity of $B-e$ (with u, v in the same face) implies that: (i) if e is in distinct components of $(B-e)-E'$, then e can be added to $\mathcal{D}$ by introducing exactly s crossings; and (ii) otherwise, e can be added to $\mathcal{D}$ without introducing any crossings. In either case, the result is a drawing of G with at most $\mathrm{cr}(G-e)+s$ crossings, and so $\mathrm{cr}(G) \leqslant \mathrm{cr}(G-e)+s$. The assumption $\mathrm{cr}(G) > (3/2)\mathrm{cr}(G-e)$ then implies $\mathrm{cr}(G-e) < 2s$, contradicting that $\mathrm{cr}(G-e) \geqslant 2s$. Thus $\mathrm{cr}(G-e) \geqslant (2/3)\mathrm{cr}(G) > (2/3)\mathrm{cr}(G) - 10^8$.

Suppose finally that G is irreducible. Lemma 11 then implies that G has at least $\mathbf{c}_1|E| - \mathbf{c}_7 k$ pairwise edge-disjoint (ℓ_0, Δ_0)-earrings. Since $|E| \geqslant (\mathbf{c}_7 k)/\mathbf{c}_1 + 1$, it follows that G has at least one (ℓ_0, Δ_0)-earring. Thus, by Lemma 13, G has an edge e such that $\mathrm{cr}(G-e) > (2/3)\mathrm{cr}(G) - (2/3)(\Delta\ell + \ell^2) > (2/3)\mathrm{cr}(G) - 10^8$. □

7. Bounded decay and expected crossing numbers

The pioneering work of Richter and Thomassen, as well as our work in this paper, are naturally described as "bounded decay" results: the existence of sets of edges whose removal does not decrease arbitrarily the crossing number. The papers by Fox and Tóth [10] and by Černý, Kynčl and Tóth [5] concern themselves with "almost no decay" results: the existence of sets of edges whose removal results in a very small decrease of the crossing number.

As an additional motivation to bounded decay results, we discuss in this section a connection with expected crossing numbers, a concept recently introduced by Mohar and Tamon [18, 19].

7.1. **Expected crossing numbers and decay of crossing numbers.** Given a drawing $\mathcal{D}$ of a graph $G = (V, E)$, and a *weight function* $w : E \to \mathbb{R}_+$, define the *crossing weight* $\mathrm{cr}(\mathcal{D}, w)$ as $\sum_{\{e,f\} \in \mathbb{X}(\mathcal{D})} w(e)w(f)$, where $\mathbb{X}(\mathcal{D})$ is the set of all pairs of edges that cross each other in $\mathcal{D}$. The pair (G, w) is a *weighted graph*, and the *weighted crossing number* of (G, w) is $\mathrm{cr}(G, w) := \min_{\mathcal{D}} \mathrm{cr}(\mathcal{D}, w)$, where the minimum is taken over all drawings $\mathcal{D}$ of G. Now take the weights on the edges to be independently identically distributed random variables, with uniform distributions on the interval $[0, 1]$. The expected value of $\mathrm{cr}(G, w)$ under this distribution is the *expected crossing number* of G, and is denoted $\mathbb{E}(\mathrm{cr}(G))$.

Let us say that a family $\mathscr{G}$ of graphs is *robust* (or, more precisely, ϵ-*robust*) if there exist a constant $\epsilon := \epsilon(\mathscr{G})$ and an $n(\mathscr{G})$ such that $\mathbb{E}(\mathrm{cr}(G)) \geqslant \epsilon \cdot \mathrm{cr}(G)$ for every graph G in $\mathscr{G}$ with at least $n(\mathscr{G})$ vertices.

Mohar and Tamon proved in [18] that $\mathbb{E}(\mathrm{cr}(K_n))$ is $\Theta(n^4)$. From this it follows immediately that the family of all complete graphs is robust. Moreover, it follows from their Crossing Lemma for Expectations (Theorem 5.2

in [18]) that for each fixed $\gamma > 0$, the family of graphs with at least $\gamma \cdot n^2$ edges is also robust (more precisely, ϵ-robust, where ϵ might depend on γ). It is thus natural to inquire about the robustness of families of sparser graphs.

Our aim in this subsection is to unveil and exploit the close connection between robustness and several results and conjectures, presented in [5], on the decay of crossing numbers.

In [5], Černý, Kynčl and Tóth proved the following: for each $\epsilon > 0$, there exist $\delta, \gamma > 0$ such that every sufficiently large graph G with n vertices and $m \geqslant n^{1+\epsilon}$ edges has a subgraph G' with at most $(1-\delta)m$ edges such that $\text{cr}(G') \geqslant \gamma \cdot \text{cr}(G)$. This impressive "almost no decay" statement is best possible, in the sense that (as shown in [5]) one cannot require that *every* subgraph with $(1-\delta)m$ edges has crossing number at least $\gamma \cdot \text{cr}(G)$. In this vein, Černý, Kynčl and Tóth also investigated the following closely related problem.

Let us say that a family $\mathscr{G}$ of graphs is *stable* (or, more precisely, (δ, γ)-*stable*) if there exist positive constants $\delta := \delta(\mathscr{G})$, $\gamma := \gamma(\mathscr{G})$, and $n(\mathscr{G})$ such that for every graph $G \in \mathscr{G}$ with at least $n(\mathscr{G})$ vertices (and m edges), a positive fraction of all subgraphs of G with $(1-\delta)m$ edges has crossing number at least $\gamma \cdot \text{cr}(G)$. The requirement may be equivalently formulated as follows: if G' is a random subgraph of G obtained by deleting independently each edge with probability δ, then w.h.p. $\text{cr}(G') \geqslant \gamma \cdot \text{cr}(G)$.

In the earlier version [4] of [5], it was conjectured that for each $\epsilon > 0$, the family of graphs with $\Theta(n^{1+\epsilon})$ edges is stable. In [5], it was shown that this is false for $\epsilon < 1/3$ (we have slightly refined the construction in [5], and shown that it does not hold either for $\epsilon = 1/3$; see Theorem 17). The conjecture remains open for denser graphs:

Conjecture 14. *There exists an $\overline{\epsilon} \in (1/3, 1)$ such that, for each $\epsilon \in (\overline{\epsilon}, 1]$, the family of graphs with $\Theta(n^{1+\epsilon})$ edges is stable.*

(See also a weaker version put forward in [5]).

Before moving on to explore the close relationship between Conjecture 14 and the robustness of dense graphs, we note the stability of random graphs:

Remark 15. *The family of all random graphs $G(n,p)$ with $p > 2/n$, is stable.*

Proof. We start by noting that $\mathbb{E}(\text{cr}(G(n,p)) \leqslant p^2 \text{cr}(K_n) \leqslant (1/10)p^2 n^4$. From the other side, Spencer and G. Tóth ([25], Section 4) proved that there is a $c > 0$ such that for n sufficiently large the lower bound $\mathbb{E}(\text{cr}(G(n, 2/n))) > cn^2$ holds. Standard sparsening of $G(n,p)$ (keeping each edge with probability $2/(pn)$) gives that for $p > 2/n$, $\mathbb{E}(\text{cr}(G(n,p))) > (c/4)p^2 n^4$. Using these bounds, together with the observation that if each edge of a $G(n,p)$ is removed with probability ϵ then we obtain a $G(n, (1-\epsilon)p)$, the remark follows. □

The key connection between expected crossing number (robustness) and the decay of crossing numbers (stability) is the following observation:

Proposition 16. *If a family $\mathscr{G}$ of graphs is stable, then it is robust. More precisely: if $\mathscr{G}$ is (δ,γ)-stable, then it is $\delta^2\gamma$-robust.*

Proof. Suppose that $\mathscr{G}$ is a (δ,γ)-stable family of graphs. Let G be a (sufficiently large) graph in $\mathscr{G}$, and let w be a random weight assignment (sampled from the uniform distribution) on the edges of G. Our aim is to show that the expected value of $\mathrm{cr}(G,w)$ is at least $\delta^2\gamma\cdot\mathrm{cr}(G)$.

Let G' be the subgraph of G that results by deleting the edges that receive a weight smaller than δ under w. Let $\mathcal{D}$ be a drawing of G that minimizes $\mathrm{cr}(G,w)$, and let $\mathcal{D}'$ be the restriction of G to G'. Clearly $\mathcal{D}'$ has at most $\mathrm{cr}(G,w)/\delta^2$ crossings, and so $\mathrm{cr}(G')\leqslant\mathrm{cr}(\mathcal{D}')\leqslant\mathrm{cr}(G,w)/\delta^2$. Thus $\mathrm{cr}(G,w)\geqslant\delta^2\mathrm{cr}(G')$.

Note that G' may be equivalently regarded as a graph obtained from G by deleting each edge independently with probability δ. Since $\mathscr{G}$ is (δ,γ)-stable, it follows that w.h.p. $\mathrm{cr}(G')\geqslant\gamma\cdot\mathrm{cr}(G)$. Therefore the expected value of $\mathrm{cr}(G,w)$ is at least $\delta^2\gamma\cdot\mathrm{cr}(G)$, as required. □

We now proceed with a concrete illustration of how the results and techniques on the decay of crossing numbers (specifically, those developed in [5]) find an immediate application in expected crossing numbers.

As we observed above, Černý, Kynčl and Tóth [5] proved that, for each $\epsilon\in(0,1/3)$, the family of graphs with $\Theta(n^{1+\epsilon})$ edges is *not* stable. We have slightly refined the construction in [5], and extended it to cover the case $\epsilon=1/3$.

Theorem 17 (Non-stability of graphs with $\Theta(n^{4/3})$ edges)**.** *For every $\delta,\gamma>0$ there exist $c:=c(\delta,\gamma)$ and $n_0:=n_0(\delta,\gamma)$ such that there exist infinitely many graphs G with $n>n_0$ vertices and $c\cdot n^{4/3}<m<n^{4/3}$ edges, that satisfy the following. If G' is a random subgraph of G obtained by deleting independently each edge with probability δ, then w.h.p.*

$$\mathrm{cr}(G')<\gamma\cdot\mathrm{cr}(G).$$

We omit the proof of this result, since it closely resembles the proof of our next statement. Theorem 18 shows the non-robustness of graphs with $\Theta(n^{4/3})$ edges, and illustrates how the non-stability results and techniques in [5] can be extended to prove the non-robustness of graphs with $\Theta(n^{1+\epsilon})$ edges for each $\epsilon\in(0,1/3)$.

Theorem 18 (Non-robustness of graphs with $\Theta(n^{4/3})$ edges)**.** *For every $\gamma>0$ there exist $c:=c(\gamma)$ and $n_0:=n_0(\gamma)$ such that there are infinitely many graphs G with $n>n_0$ vertices and $c\cdot n^{4/3}<m<n^{4/3}$ edges, and*

$$\mathbb{E}(\mathrm{cr}(G))<\gamma\cdot\mathrm{cr}(G).$$

Proof. For readability purposes, we shall omit explicitly taking the integer part of several quantities involved. The integrality requirement will be, in every case, obvious from the context.

We may assume without loss of generality that γ is small enough so that $e^{-1200/\gamma} < \gamma/720$. Let $\alpha := \gamma/600$, $c := \alpha^2/100$, $r := \alpha^2 n^{1/3}/5$, $s := 1/\alpha^2$, and $t := \sqrt{n/s}$. Note that obviously $r > 5cn^{1/3}$.

Inspired by the construction in [5], G will be the disjoint union of two graphs G_1 and G_2 plus some isolated vertices. Let G_1 be $n/2r$ copies of the complete graph K_r. Clearly $|V(G_1)| \leqslant n/2$. Now let G_2 be obtained from a complete graph K_t by subdividing each edge $s-1$ times, i.e. replacing each edge by a path with s edges (these length s paths are the *branches*). It is easy to check that $|V(G_2)| \leqslant n/2$. Furthermore,

$$\alpha^4 n^2 = t^4 > \mathrm{cr}(G_2) > \frac{t^4}{100} = \frac{n^2}{100s^2} = \frac{\alpha^4 n^2}{100}, \tag{8}$$

where the inequalities $t^4 > \mathrm{cr}(G_2) > t^4/100$ are easily derived bounds for the crossing number of the complete graph on t vertices.

Now let w be a random weight assignment on the edges of G. Let $E_{<\alpha}$ denote the set of edges of G that receive a weight smaller than α under w. Let us say that a branch is *weak* if at least one of its edges is in $E_{<\alpha}$; otherwise the branch is *strong*.

The probability that any fixed branch is strong is

$$(1-\alpha)^s \approx e^{-\alpha s} = e^{-1/\alpha}.$$

Using Chernoff's bound, w.h.p. at most $t^2 e^{-1/\alpha}$ branches are strong. That is, w.h.p. at least $\binom{t}{2} - t^2 e^{-1/\alpha} \approx t^2(1/2 - e^{-1/\alpha})$ branches are weak.

Now consider the drawing of G_2 in which the t vertices of degree $t-1$ are in convex position, and the edges are the straight segments joining them. This drawing of G_2 has $\binom{t}{4} \approx t^4/24$ crossings (this is by no means a crossing-minimal drawing of G_2, but it is enough for our purposes). Moreover, by adjusting the drawing of each branch if needed, we may ensure that each branch is crossed in exactly one edge, namely the edge with smallest weight. It follows that the number of crossings involving two strong branches (and thus, in particular, the number of crossings of weight $\geqslant \alpha$) is w.h.p. at most $(t^2 e^{-1/\alpha})^2$, and so w.h.p.

$$\begin{aligned}\mathrm{cr}(G_2, w) &< t^4 e^{-2/\alpha} + \alpha \cdot t^4(1/24 - e^{-2/\alpha}) < t^4(\alpha/24 + e^{-2/\alpha})\\ &< 100\mathrm{cr}(G_2)(\alpha/24 + e^{-2/\alpha}) \leqslant 5\alpha \cdot \mathrm{cr}(G_2), \end{aligned}\tag{9}$$

where for this last inequality we used that $e^{-1200/\gamma} = e^{-2/\alpha} < \gamma/720 = (5/6)\alpha$.

We finally move on to G. First we note that

$$|E(G)| = |E(G_1)| + |E(G_2)| \geqslant |E(G_2)| = (n/2r)r(r-1)/2 > nr/5 > cn^{4/3}.$$

Using (8), we obtain

$$\mathrm{cr}(G) = \mathrm{cr}(G_1) + \mathrm{cr}(G_2) > \mathrm{cr}(G_2) > \alpha^4 n^2/100. \tag{10}$$

From the other side, using (8) and (9) and the trivial bound $\mathrm{cr}(K_r) \leqslant r^4$, we get

$$\mathrm{cr}(G, w) \leqslant \mathrm{cr}(G_1) + \mathrm{cr}(G_2, w) \leqslant (n/2r)r^4 + 5\alpha^5 n^2 < 6\alpha^5 n^2, \tag{11}$$

where for the last inequality we used the (easily checked) inequality $(n/2r)r^4 < \alpha^5 n^2$.

Finally, using (10) and (11) and recalling that $\alpha = \gamma/600$, we obtain

$$\mathrm{cr}(G, w) < 6\alpha^5 n^2 = (600\alpha)(\alpha^4 n^2/100) < \gamma \cdot \mathrm{cr}(G),$$

as required. □

We close this subsection with two constructions that further illustrate the discrepancy between the crossing number of a graph and its expected crossing number.

First we describe a construction that highlights the fact that the crossing number (of a family of graphs) may grow with the number of vertices, and yet the expected crossing number (of all graphs in the family) may be bounded by an absolute constant. For any graph G, let $n(G)$ and $m(G)$ denote the number of vertices and edges of G, respectively, and let $s \cdot G$ the graph that consists of s disjoint copies of G. Let $K_5(t)$ denote the graph obtained by replacing each edge of K_5 with a path of length t (a *branch*). Trivially, for any positive integer s, $n(s \cdot K_5(t)) = s(10(t-1)+5) = 10st - 5s, m(s \cdot K_5(t)) = 10st$, and $\mathrm{cr}(s \cdot K_5(t)) = s$. However, the weighted crossing number of $K_5(t)$ is $\min w(e)w(f)$, where the minimum is taken over all pairs of edges e, f that lie on branches that correspond to nonincident edges. A fairly standard calculation shows that $\mathbb{E}(\mathrm{cr}(s \cdot K_5(t)) \leqslant (s/t^2)\log^2 s$. It is worthwhile to explore the consequences of plugging in various values of s. Probably the most interesting case occurs when $s = n^{2/3}/\log n$, for this shows the following:

Proposition 19. *There exists an infinite family of graphs G with crossing number $n^{2/3}/\log n$ and expected crossing number at most 1.* □

Our final construction pertains a family of graphs that seem more natural than the graphs constructed above. We recall that $C_3 \,\square\, C_n$ denotes the Cartesian product of the cycles of sizes 3 and n (see Figure 1).

Proposition 20. *The Cartesian products $C_3 \,\square\, C_n$ satisfy*

$$\mathrm{cr}(C_3 \,\square\, C_n) = n,$$

and yet

$$\mathbb{E}(\mathrm{cr}(C_3 \,\square\, C_n)) \leqslant 2n^{2/3}\log^{1/3} n + 3.$$

Proof. The vertices of $C_3 \,\square\, C_n$ can be labeled $v_{i,j}$, $0 \leqslant i \leqslant 2$, $0 \leqslant j \leqslant n-1$, so that there is an edge joining $v_{i,j}$ and $v_{i',j'}$ if and only if either (i) $j = j'$ and $|i - i'| = 1$ or (ii) $i = i'$ and $|j - j'| = 1$ (indices are modulo n). For $j = 0, 1, \ldots, n-1$, let $V_j := \{v_{i,j} \mid i \in \{0, 1, 2\}\}$. That is, the V_js are the vertex sets of the 3-cycles. For $j = 0, 1, \ldots, n-1$, let $E(j)$ denote the set of (three) edges with an endpoint in V_j and another endpoint in V_{j+1}.

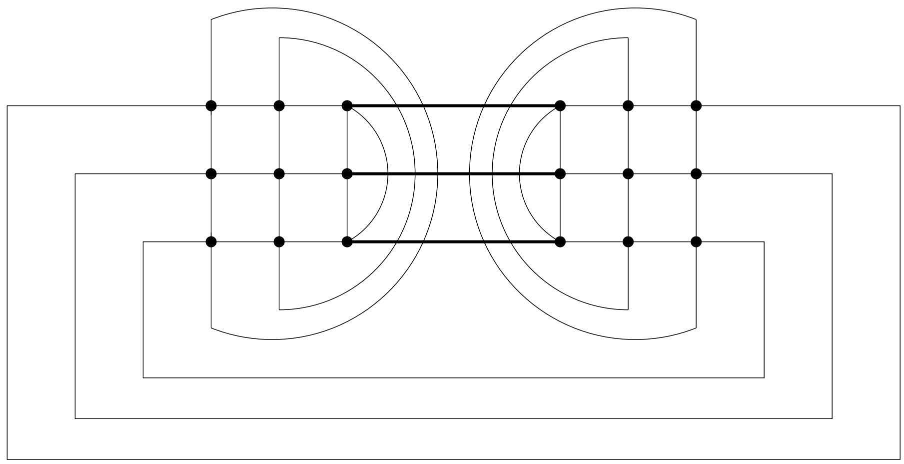

FIGURE 1. A drawing of $C_3 \square C_6$ with 14 crossings, where the thick edges are the edges of one particular $E(j)$. This is easily generalized to obtain, for every even integer $n \geqslant 2$, a (not crossing-minimal) drawing of $C_3 \square C_n$ with $3n-4$ crossings with the following property: there exists a $j \in \{0, 1, 2, \ldots, n-1\}$ such that each crossing involves an edge in $E(j)$.

It is known that $\mathrm{cr}(C_3 \square C_n) = n$ for every $n \geqslant 3$ [23]. In Figure 1 we depict how to produce a (not crossing-minimal) drawing of $C_3 \square C_n$ with $3n-4$ crossings, for every even integer $n \geqslant 2$, with the following property: there is a $j \in \{0, 1, 2 \ldots, n-1\}$ such that every crossing involves an edge in $E(j)$ (the edges in $E(j)$ are the thick edges in Figure 1). Thus,

(A) if the edges in $C_3 \square C_n$ are are weighted, and there exists a j such that the sum of the weights of the edges in $E(j)$ is r, then such a weighted $C_3 \square C_n$ has crossing number at most $r \cdot n$.

For $j = 0, 1, \ldots, n-1$, denote the weights of the edges in $E(j)$ by x_1^j, x_2^j, x_3^j. We have for $t \leqslant 1$ that $\Pr(x_1^j + x_2^j + x_3^j > t) = 1 - t^3/3!$. Using independence,

$$\Pr(\exists j : x_1^j + x_2^j + x_3^j \leqslant t) = 1 - (1 - t^3/6)^n \approx 1 - \exp[-nt^3/6].$$

Choosing $t = 6^{1/3} n^{-1/3} \log^{1/3} n$, this is at least $1 - 1/n$.

Now let $s := \min\{x_1^j + x_2^j + x_3^j \mid j \in \{0, 1, \ldots, n-1\}\}$. Thus $s \leqslant t$ with probability at least $1 - 1/n$. In the complementary scenario (which occurs with probability $< 1/n$), s is obviously at most 3. Using this observation together with (A), it follows that $\mathbb{E}(\mathrm{cr}(C_3 \square C_n)) < \big[(1-1/n)((6)^{1/3} n^{-1/3} \log^{1/3} n) + (1/n)3\big] \cdot n < 2n^{2/3} \log^{1/3} n + 3$. $\square$

7.2. **Concentration of the expected crossing number and the crossing number of randomly sparsened graphs.** Continuing in the theme of expected crossing numbers and its interplay with the decay of crossing numbers, we finally explore the concentration around the crossing number of a randomly sparsened graph, as well as the concentration around the expected crossing number of a graph.

Denote $R = R(G, p)$ the random graph obtained from G by randomly and independently removing edges, each with probability p. Using a standard martingale concentration inequality we show that $\mathrm{cr}(R)$ is concentrated around its mean. Let $E(G) = \{e_1, \ldots, e_m\}$, and consider the random variable $\mathrm{cr}(R)$ as a Doob's martingale, where the edges are exposed one by one. The length of the martingale is $|E(G)|$. Removing or adding an edge changes the crossing number by at most $|E(G)|$. Thus, by the Azuma-Hoeffding's inequality, for every $\lambda > 0$ we have

$$\Pr[|\mathbb{E}(\mathrm{cr}(R)) - \mathrm{cr}(R)| > \lambda] \leqslant \exp\left[\frac{-\lambda^2}{2|E(G)|^3}\right]. \tag{12}$$

Let $\beta(n)$ be any function tending to infinity. Inequality (12) shows concentration with radius $\lambda = \beta(n)|E(G)|^{3/2}$:

$$\Pr[|\mathbb{E}(\mathrm{cr}(R)) - \mathrm{cr}(R)| > \beta(n)|E(G)|^{3/2}] \leqslant \exp\left[\frac{-\beta(n)^2}{2}\right]. \tag{13}$$

Similary, we can get concentration around the expected crossing number. Assign to each edge a random variable taking values from $[0, 1]$ (which could be different for each edge), which provides to each of them a random weight. Formally, it could be a function $w : E(G) \to \mathcal{F}$, where $\mathcal{F}$ is a collection of random variables taking values from $[0, 1]$. Then $\mathbb{E}(cr(G, w))$ is the expected crossing number for a given w, and $\mathrm{cr}(G, w)$ is a random variable, which is the crossing number of a weighted graph G. As with the random graph R above, resampling the weight of one edge changes the weighted crossing number by at most $|E(G)|$, and so we obtain:

$$\Pr[|\mathbb{E}(\mathrm{cr}(G)) - \mathrm{cr}(G, w)| > \beta(n)|E(G)|^{3/2}] \leqslant \exp\left[\frac{-\beta(n)^2}{2}\right]. \tag{14}$$

These inequalities are meaningful only when G is dense enough, i.e. $|E(G)| \geqslant n^{5/4}$. Note that we could have obtained sharper concentration results for sparse graphs, under the assumption that removing any edge makes the crossing number drop by $o(|E(G)|)$.

8. Concluding remarks

Lemma 8 falls into the realm of light subgraphs. We recall that the *weight* of a subgraph H of a graph G is the sum of the degrees (in G) of its vertices. For a class $\mathscr{G}$ of graphs, define $w(H, \mathscr{G})$ as the smallest integer w

such that each graph $G \in \mathscr{G}$ which contains a subgraph isomorphic to H has a subgraph isomorphic to H of weight at most w. If $w(H, \mathscr{G})$ is finite then H is *light* in $\mathscr{G}$.

Fabrici and Jendrol' [8] proved that paths (and no other connected graphs) are light in the class of 3-connected planar graphs. Fabrici et al. [9] proved that this remains true even if the minimum degree is at least 4, and Mohar [16] extended this to 4-connected planar graphs.

Although some cycles are light in certain families of planar graphs (see for instance [11, 12, 15, 17]), it is easy to see that cycles are not light on the class of planar graphs (consider, for instance, a wheel W_n with n large: each cycle in W_n is either very long or incident with a large degree vertex). However, as Richter and Thomassen illustrated in [22], for some applications one does not need the full lightness condition. A cycle C in a graph is (ℓ, Δ)-*nearly light* if it has length less than ℓ and at most one of its vertices has degree Δ or greater. Richter and Thomassen proved that every planar graph has a $(6, 11)$-nearly light cycle. This was later refined in [14], where it was shown that if the graphs under consideration are sufficiently large, then there is a $\Delta > 0$ such that a linear proportion of the face boundaries are $(6, \Delta)$-nearly light.

The concept of (ℓ, Δ)-earrings extends the idea of nearly light cycles: we allow both vertices u, v incident with some edge e to have arbitrarily large degree, and ask for the existence of *two* cycles that contain e, have bounded length, and (other than u and v) bounded degree. The following immediate corollary (since every 3-connected graph is obviously irreducible) of Lemma 11 guarantees the existence of many pairwise edge-disjoint earrings in 3-connected planar graphs.

Lemma 21. *If $G = (V, E)$ is a 3-connected planar graph, then G has at least $10^{-10}|E|$ pairwise edge-disjoint $(5000, 500)$-earrings.*

We remark that the linear dependence on $|E|$ in Lemma 21 is clearly best possible, since there cannot be more pairwise edge-disjoint earrings than edges in a graph.

Finally, it is natural to ask if the 3-connectedness requirement can be weakened. The construction illustrated in Figure 2 answers this in the negative.

It might be argued that the graphs constructed in the proof of Theorem 18 are somewhat artificial, since many edges are subdivided a large number of times. However, these graphs can be turned into 3-connected graphs, with equivalent properties, as follows. Consider the graph G_2 in the proof of Theorem 18, and some fixed drawing of G_2 (for instance, as in the proof of Theorem 18, draw the degree $t-1$ vertices on a circumference, and the branches as the straight edges joining them). Let $u_1, u_2, \ldots, u_t$ be the *nodes* (degree $t-1$ vertices) of G_2. Thus each branch with endpoints u_i, u_j can be written as $u_i = u_{i,j}^0, u_{i,j}^1, \ldots, u_{i,j}^{s-1}, u_{i,j}^s = u_j$ (the same branch, traversing the

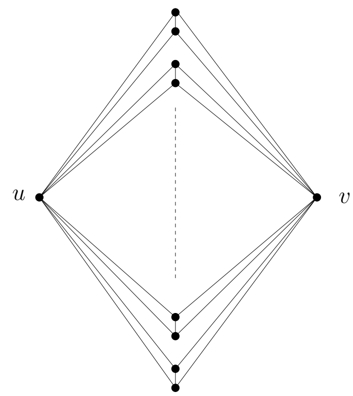

FIGURE 2. The graph H_n obtained by identifying n copies of $K_4 - e$ on their degree 2 vertices u, v. This family of 2-connected graphs shows that the 3-connectedness condition in Lemma 21 cannot be weakened: for each pair of integers ℓ, Δ there is an $n_0 := n_0(\ell, \Delta)$ such that for all $n \geqslant n_0$, H_n does not contain any (ℓ, Δ)-earring.

vertices in the reverse order, reads $u_j = u_{j,i}^0, u_{j,i}^1, \ldots, u_{j,i}^{s-1}, u_{j,i}^s = u_i$, so that $u_{i,j}^k = u_{j,i}^{s-k}$ for $k = 0, 1, \ldots, s$). Now for each branch $u_{i,j}^0, u_{i,j}^1, \ldots, u_{i,j}^{s-1}, u_{i,j}^s$, add the edges $u_{i,j}^k$ and $u_{i,j}^{k+2}$, for $k = 0, 1, \ldots, s-2$. The augmented graph is already 2-connected, but each pair of nodes (that is, degree $t-1$ vertices) is a 2-vertex-cut, so we need to strenghten the connectivity around each node. Consider the node u_1, and suppose for simplicity that the edges $u_1 u_{1,2}^1, u_1 u_{1,3}^1, \ldots, u_1 u_{1,t}^1$ leave u_1 in the given (say clockwise) cyclic order. Then, for each $j = 2, 3, \ldots, s$, it is possible to draw an edge from one of $u_{1,j}^1$ and $u_{1,j}^2$ to one of $u_{1,j+1}^1$ and $u_{1,j+1}^2$ without introducing any crossings (indices are read modulo s). By performing this procedure around each node, we obtain a 3-connected graph that also witnesses Theorem 18. The proof is analogous to the proof of Theorem 18; the only difference is that instead of requiring a weak edge of a branch (say between u_i and u_j), we need weak triplets of edges of the form $(u_{i,j}^\ell, u_{i,j}^{\ell+1}), (u_{i,j}^{\ell-1}, u_{i,j}^{\ell+1}), (u_{i,j}^\ell, u_{i,j}^{\ell+2})$, where $3 \leqslant \ell \leqslant s-3$; we omit the details.

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