Making a graph crossing-critical by multiplying its edges

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December 16, 2011

Abstract

A graph is *crossing-critical* if the removal of any of its edges decreases its crossing number. This work is motivated by the following question: to what extent is crossing-criticality a property that is inherent to the structure of a graph, and to what extent can it be induced on a noncritical graph by multiplying (all or some of) its edges? It is shown that if a nonplanar graph G is obtained by adding an edge to a cubic polyhedral graph, and G is sufficiently connected, then G can be made crossing-critical by a suitable multiplication of edges.

1 Introduction

This work is motivated by the recent breakthrough constructions by DeVos, Mohar and Šámal [3] and Dvořák and Mohar [5], which settled two important crossing numbers questions. The graphs constructed in [3] and [5] use weighted (or "thick") edges. A graph with weighted edges can be naturally transformed into an ordinary graph by substituting weighted edges by multiedges (recall that a *multiedge* is a set of edges with the same pair of endvertices). If one wishes to avoid multigraphs, one can always substitute a weight t edge by a $K_{2,t}$, but still the resulting graph is homeomorphic to a multigraph. Sometimes (as in [3]) one can afford to substitute each weighted edge by a slightly richer structure (such as a graph obtained from $K_{2,t}$ by joining the degree 2 vertices with a path), but sometimes (as in [5]) one is concerned with criticality properties, and so no such superfluous edges may be added. In any case, the use of weighted edges is crucial.

After trying unsuccessfully to come up with graphs with similar crossing number properties as those presented in [3] and [5], while avoiding the use of weighted edges, we were left

*Supported by CONACYT Grant 106432.

with a wide open question: in the realm of crossing numbers, more specifically on crossing-criticality issues, to what extent does it make a difference to allow (equivalently, to forbid) weighted edges (or, for that matter, multiedges)?

Recall that the *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane. An edge e of G is *crossing-critical* if $\mathrm{cr}(G-e) < \mathrm{cr}(G)$. If all edges of G are crossing-critical, then G itself is *crossing-critical*. A crossing-critical graph seems naturally more interesting than a graph with some not crossing-critical edges, since a graph of the latter kind contains a proper subgraph that has all the relevant information from the crossing numbers point of view.

Earlier constructions of infinite families of crossing-critical graphs made essential use of multiple edges [11]. On the other hand, constructions such the ones given by Kochol [9], Hliněný [7], and Bokal [1] deal exclusively with simple graphs.

We ask to what extent crossing-criticality is an inherent structural property of a graph, and to what extent crossing-criticality can be induced by multiplying the edges of a (noncritical) graph. Let G, H be graphs. We say that G *is obtained by multiplying edges of* H if H is a subgraph of G and, for every edge of G, there is an edge of H with the same endvertices.

Question 1. *When can a graph be made crossing-critical by multiplying edges? That is, given a (noncritical) graph H, when does there exist a crossing-critical graph G that is obtained by multiplying edges of H?*

Our universe of interest is, of course, the set of nonplanar graphs, since a planar graph obviously remains planar after multiplying any or all of its edges.

We show that a large, interesting family of nonplanar graphs satisfy the property in Question 1. A nonplanar graph G is *near-planar* if it has an edge e such that $G-e$ is planar. Near-planar graphs constitute a natural family of nonplanar graphs. Any thought to the effect that crossing number problems might become easy when restricted to near-planar graphs is put definitely to rest by the recent proof by Cabello and Mohar that CrossingNumber is NP-Hard for near-planar graphs [2].

Following Geelen et al. [6] (who define internally 3-connectedness for matroids), a graph G is *internally* 3*-connected* if G is simple and 2-connected, and for every separation (G_1, G_2) of G of order two, either $|E(G_1)| \leq 2$ or $|E(G_2)| \leq 2$. Hence, internally 3-connected graphs are those that can be obtained from a 3-connected graph by subdividing its edges, with the condition that no edge can be subdivided more than once.

Our main result is that any adequately connected, near-planar graph G obtained by adding an edge to a cubic polyhedral (i.e., planar and 3-connected) graph, belongs to the class alluded to in Question 1.

Theorem 2. *Let G be a near-planar simple graph, with an edge uv such that $G - uv$ is a cubic polyhedral graph. Suppose that $G - \{u, v\}$ is internally* 3*-connected. Then there exists a crossing-critical graph that is obtained by multiplying edges of G.*

We note that some connectivity assumption is needed in order to guarantee that a nonplanar graph can be made crossing-critical by multiplying edges. To see this, consider a graph G which is the 1-sum of a nonplanar graph G_1 plus a planar graph G_2. Since crossing number is additive on the blocks of a graph, it is easy to see that G cannot be made crossing-critical by multiplying edges.

An important ingredient in the proof of Theorem 2 is the following, somewhat curious statement for which we could not find any reference in the literature. We recall that a *weighted graph* is a pair (G, w), where G is a graph and w (the *weight assignment*) is a map that assigns to each edge e of G a number $w(e)$, the *weight* of e. The *length* of a path in a weighted graph is the sum of the weights of the edges in the path. If u, v are vertices of G, then the *distance* $d_w(u, v)$ *from* u *to* v (*under* w) is the length of a minimum length (also called a *shortest*) uv-path. The weight assignment w is *positive* if $w(e) > 0$ for every edge e of G, and it is *integer* if each $w(e)$ is an integer.

Lemma 3. *Let G be a 2-connected loopless graph, and let u, v be distinct vertices of G. Then there is a positive integer weight assignment such that every edge of G belongs to a shortest uv-path.*

The rest of this paper is structured as follows.

We prove the auxiliary Lemma 3 in Section 2. We then proceed to reformulate Theorem 2 in terms of weighted graphs. This simply consists on replacing a multigraph with a weighted simple graph so that the weights of the edges are the multiplicities of the edges in the original multigraph. As in [3] and [5], this reformulation, carried out in Section 3, turns out to greatly simplify the discussion and the proofs. The equivalent form of Theorem 2, namely Theorem 4, is then proved in Section 5, the core of this paper. Finally, we present some concluding remarks and open questions in Section 6.

2 Proof of Lemma 3

We use perfect rubber bands, a technique inspired by the work of Tutte [14].

Make every edge a perfect rubber band and pin vertices u and v on a board. Since G is 2-connected, and with use of Menger's theorem, every other vertex w admits two vertex-disjoints paths, one linking w with u and the other linking w with v. Therefore, every vertex will lie on the segment $[u, v]$ on the board. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we may modify vertices positions so that all edge lengths are rational (since they all are barycentric coordinates, it suffices to have a rational distance between u and v). We may also modify these coordinates locally so that no two vertices lie at the same spot.

Therefore, every edge lies on a shortest path from u to v.

In the end, we may multiply every length by the least common multiple of the denominators, so that we get an integer length function on the edges meeting our requirements. □

3 Reformulating Theorem 2 in terms of weighted graphs

In the context of Theorem 2, let G be a simple graph which we seek to make crossing-critical by multiplying (some or all of) its edges. With this in mind, let $\overline{G}$ be a multigraph (that is, a graph with multiedges allowed) whose underlying simple graph is G. Now consider the (positive integer) weight assignment w on $E(G)$ defined as follows: for each edge uv of G, let $w(uv)$ be the number of edges in $\overline{G}$ whose endpoints are u and v (i.e., the *multiplicity* of uv).

If we extend the definition of crossing number to weighted graphs, with the condition that a crossing between two edges contributes to the total crossing number by the products of their weights, then, from the crossing numbers point of view, clearly (G, w) captures all the relevant information from $\overline{G}$. In particular, $\mathrm{cr}(\overline{G}) = \mathrm{cr}(G, w)$. Moreover, by extending the definition of crossing-criticality to weighted graphs in the obvious way (which we now proceed to do), it will follow that $\overline{G}$ is crossing-critical if and only if (G, w) is crossing-critical.

To this end, let G be a graph and w a positive integral weight assignment on G. An edge e of (G, w) is *crossing-critical* if $\mathrm{cr}(G, w_e) < \mathrm{cr}(G, w)$, where w_e is the weight assignment defined by $w_e(f) = w(f)$ for $f \neq e$ and $w_e(e) = w(e) - 1$. As with ordinary graphs, (G, w) is *crossing-critical* if all its edges are crossing-critical.

Under this definition of crossing-criticality for weighted graphs, it is now obvious that if we start with a multigraph $\overline{G}$ and derive its associated weighted graph (G, w) as above, then $\overline{G}$ is crossing-critical if and only if (G, w) is crossing-critical.

In view of this equivalence (for crossing number purposes) between multigraphs and weighted graphs, it follows that Theorem 2 is equivalent to the following:

Theorem 4 (Equivalent to Theorem 2). *Let G be a near-planar simple graph, with an edge uv such that $G - uv$ is a cubic polyhedral graph. Suppose that $G - \{u, v\}$ is internally 3-connected. Then there exists a positive integer weight assignment w such that (G, w) is crossing-critical.*

For the rest of this paper:

- we let $G_{u,v} := G - \{u, v\}$; and
- we refer to the hypotheses that $G - uv$ is 3-connected and $G_{u,v}$ is internally 3-connected simply as the *connectivity assumptions* on $G - uv$ and $G_{u,v}$, respectively.

4 Some facts on $G_{u,v}$ and its dual $G^*_{u,v}$

Before moving on to the proof of Theorem 5, we establish some facts on the graph $G_{u,v}$.

4.1 Remarks on the vertices incident with u and v

Let u_1, u_2, and u_3 be the vertices of G (other than v) adjacent to u. Analogously, Let v_1, v_2, and v_3 be the vertices of G (other than u) adjacent to v.

We start by noting that $u_1, u_2, u_3, v_1, v_2, v_3$ are all distinct. First of all, if $i \neq j$, then since G is simple it follows that $u_i \neq u_j$. Now suppose that $u_i = v_j$ for some i, j, and consider an embedding of $G-uv$ in the plane. It is easy to see that since $u_i = v_j$, and $G-uv$ is cubic, it follows that uv can be added to the embedding of $G-uv$ without introducing any crossings, resulting in an embedding of G. This contradicts the nonplanarity of G. Thus $u_i \neq v_j$ for all $i, j \in \{1, 2, 3\}$, completing the proof that $u_1, u_2, u_3, v_1, v_2, v_3$ are all distinct.

4.2 The embeddings of $G - uv$ and $G_{u,v}$

We note that the connectivity assumptions on $G-uv$ and $G_{u,v}$ imply that these two graphs admit unique (up to homeomorphism) embeddings in the plane. This allows us, for the rest of the proof, to regard these as graphs embedded in the plane.

Since $G_{u,v}$ is a subgraph of $G-uv$, it follows that we may assume that the restriction of the embedding of $G-uv$ to $G_{u,v}$ is precisely the embedding of $G_{u,v}$. Conversely, to obtain the embedding of $G-uv$, we may start with the embedding of $G_{u,v}$; then we find the (unique) face F_u incident with u_1, u_2, and u_3, and draw uu_1, uu_2, and uu_3 (and, of course, u) inside F_u; and similarly find the (unique) face F_v incident with v_1, v_2, and v_3, and draw vv_1, vv_2, and vv_3 (and, of course, v) inside F_v. Note that $F_u \neq F_v$, as otherwise the edge uv could be added to the embedding of $G-uv$ without introducing any crossings, resulting in an embedding of G, contradicting its nonplanarity.

4.3 Weight assignments on the dual $G^*_{u,v}$ of $G_{u,v}$

We shall make extensive use of weight assignments on the dual (embedded graph) $G^*_{u,v}$ of $G_{u,v}$. We start by noting that $G^*_{u,v}$ is well-defined (and admits a unique plane embedding) since $G_{u,v}$ admits a unique plane embedding. As with $G-uv$ and $G_{u,v}$, this allows us to unambiguously regard $G^*_{u,v}$ for the rest of the proof as an embedded graph. We shall let $\mathcal{F}$ denote the set of all faces in $G_{u,v}$ (equivalently, the set of all vertices of $G^*_{u,v}$).

A weight assignment λ on $G_{u,v}$ naturally induces a weight assignment λ^* on $G^*_{u,v}$, and vice versa: if e is an edge of $G_{u,v}$ and e^* is its dual edge in $G^*_{u,v}$, then we simply let $\lambda^*(e^*) = \lambda(e)$. Trivially, a weight assignment $\overline{\lambda}$ on the whole graph G also naturally induces a weight assignment λ^* on $G^*_{u,v}$: it suffices to consider the restriction λ of $\overline{\lambda}$ to $G_{u,v}$, and from this we obtain λ^* as we just described.

If λ^* is a weight assignment on $G^*_{u,v}$, then for $F, F' \in \mathcal{F}$ we let $d_{\lambda^*}(F, F')$ denote the length of a shortest FF'-path in $G^*_{u,v}$ under λ^*. We call $d_{\lambda^*}(F, F')$ the *distance* between F and F' under λ^*.

Now since for $i = 1, 2, 3$ the vertex u_i has degree 2 in $G_{u,v}$, it follows that u_i is incident with exactly two faces in $G^*_{u,v}$, one of which is F_u; let F_{u_i} denote the other face.

Thus it makes sense to define the *distance* $d_{\lambda^*}(u_i, F)$ between u_i and any face $F \in \mathcal{F}$ as $\min\{d_{\lambda^*}(F_u, F), d_{\lambda^*}(F_{u_i}, F)\}$. We define F_{v_i} and $d_{\lambda^*}(v_i, F)$ analogously, for $i = 1, 2, 3$.

The connectivity assumption on $G_{u,v}$ ensures that F_{u_1}, F_{u_2} and F_{u_3} are pairwise distinct. Similarly, F_{v_1}, F_{v_2} and F_{v_3} are pairwise distinct. Note that maybe $F_{u_i} = F_v$ for some $i \in \{1, 2, 3\}$, or $F_{v_j} = F_u$ for some $j \in \{1, 2, 3\}$.

Finally, we say that a weight assignment λ^* on $G^*_{u,v}$ is *balanced* if each edge e^* of $G^*_{u,v}$ belongs to a shortest F_uF_v-path in $(G^*_{u,v}, \lambda^*)$.

5 Proof of Theorem 4

First we show (Proposition 5) that if there exists a weight assignment ω on G with certain properties, then (G, ω) is crossing-critical. The existence of a weight assignment with these properties is established in Proposition 6, and so Theorem 4 immediately follows.

Proposition 5. *Suppose that ω is a positive integer weight assignment on G with the following properties:*

(1) The induced weight assignment ω^* on $G^*_{u,v}$ is balanced.

(2) For every pair of edges e, e' of $G_{u,v}$, $\omega(e)\omega(e') > d_{\omega^*}(F_u, F_v) \cdot \omega(uv)$.

(3) $d_{\omega^*}(u_1, F) \cdot \omega(uu_1) + d_{\omega^*}(u_2, F) \cdot \omega(uu_2) + d_{\omega^*}(u_3, F) \cdot \omega(uu_3) \geq d_{\omega^*}(F_u, F) \cdot \omega(uv)$, for every $F \in \mathcal{F}$.

(4) For each $i = 1, 2, 3$, there is a face $U_i \in \mathcal{F}$ such that $d_{\omega^*}(u_i, U_i) > 0$ and $d_{\omega^*}(u_1, U_i) \cdot \omega(uu_1) + d_{\omega^*}(u_2, U_i) \cdot \omega(uu_2) + d_{\omega^*}(u_3, U_i) \cdot \omega(uu_3) = d_{\omega^*}(F_u, U_i) \cdot \omega(uv)$.

(5) $d_{\omega^*}(v_1, F) \cdot \omega(vv_1) + d_{\omega^*}(v_2, F) \cdot \omega(vv_2) + d_{\omega^*}(v_3, F) \cdot \omega(vv_3) \geq d_{\omega^*}(F_v, F) \cdot \omega(uv)$, for every $F \in \mathcal{F}$.

(6) For each $i = 1, 2, 3$, there is a face $V_i \in \mathcal{F}$ such that $d_{\omega^*}(v_i, V_i) > 0$ and $d_{\omega^*}(v_1, V_i) \cdot \omega(vv_1) + d_{\omega^*}(v_2, V_i) \cdot \omega(vv_2) + d_{\omega^*}(v_3, V_i) \cdot \omega(vv_3) = d_{\omega^*}(F_v, V_i) \cdot \omega(uv)$.

(7) For all $i, j \in \{1, 2, 3\}$, $\omega(uu_i) \cdot \omega(vv_j) < (1/9) \min\{\, \omega(e) \mid e \in E(G_{u,v})\}$.

Then (G, ω) is crossing-critical.

Proof. Throughout the proof, for brevity we let $t := \omega(uv)$.

To help comprehension, we break the proof into several steps.

(A) $\mathrm{cr}(G, \omega) \leq t \cdot d_{\omega^*}(F_u, F_v)$.

Start with the (unique) embedding of $G - uv$, and draw uv following a shortest F_uF_v-path in $(G^*_{u,v}, \omega^*)$. Then the sum of the weights of the edges crossed by uv equals the total weight of the shortest F_uF_v-path, that is, $d_{\omega^*}(F_u, F_v)$ (here we use the elementary, easy to check fact that crossings between adjacent edges can always be avoided; in this case, we may draw

uv so that it crosses no edge adjacent to u or v). Since $\omega(uv) = t$, it follows that such a drawing of (G, ω) has exactly $t \cdot d_{\omega^*}(F_u, F_v)$. crossings.

(B) $\mathrm{cr}(G, \omega) = t \cdot d_{\omega^*}(F_u, F_v)$.

Consider a crossing-minimal drawing $\mathcal{D}$ of (G, ω). An immediate consequence of (2) and (A) is that the drawing of $G_{u,v}$ induced by $\mathcal{D}$ is an embedding (that is, no two edges of $G_{u,v}$ cross each other in $\mathcal{D}$).

Now let F' (respectively, F'') denote the face of $G_{u,v}$ in which u (respectively, v) is drawn in $\mathcal{D}$. Clearly, for $i = 1, 2, 3$ the edge uu_i contributes in at least $\omega(uu_i) \cdot d_{\omega^*}(u_i, F')$ crossings. Analogously, for $i = 1, 2, 3$ the edge vv_i contributes in at least $\omega(vv_i) \cdot d_{\omega^*}(v_i, F'')$ crossings. Thus it follows from (3) and (5) that the edges in $\{uu_1, uu_2, uu_3, vv_1, vv_2, vv_3\}$ contribute in at least $t \cdot d_{\omega^*}(F_u, F') + t \cdot d_{\omega^*}(F_v, F'') = t \cdot (d_{\omega^*}(F_u, F') + d_{\omega^*}(F_v, F''))$ crossings. On the other hand, since the ends u, v of uv are in faces F' and F'', it follows that edge uv contributes in at least $t \cdot d_{\omega^*}(F', F'')$ crossings. We conclude that $\mathcal{D}$ has at least $t \cdot \big(d_{\omega^*}(F_u, F') + d_{\omega^*}(F_v, F'') + d_{\omega^*}(F', F'')\big)$. Elementary triangle inequality arguments show that $d_{\omega^*}(F_u, F') + d_{\omega^*}(F_v, F'') + d_{\omega^*}(F', F'') \geq d_{\omega^*}(F_u, F_v)$, and so $\mathcal{D}$ has at least $t \cdot d_{\omega^*}(F_u, F_v)$ crossings. Thus $\mathrm{cr}(G, \omega) \geq t \cdot d_{\omega^*}(F_u, F_v)$. The reverse inequality is given in (A), and so (B) follows.

(C) *Crossing-criticality of the edges in $G_{u,v}$ and of the edge uv.*

Let e be any edge in $G_{u,v}$. We proceed similarly as in (A). Start with the (unique) embedding of $G - uv$, and draw uv following a shortest F_uF_v-path in $(G^*_{u,v}, \omega^*)$ that includes e^* (the existence of such a path is guaranteed by the balancedness of ω^*). This yields a drawing of (G, ω) with exactly $t \cdot d_{\omega^*}(F_u, F_v)$ crossings, in which e and uv cross each other. Since $\mathrm{cr}(G, \omega) = t \cdot d_{\omega^*}(F_u, F_v)$, it follows that e and uv are both crossed in a crossing-minimal drawing of (G, ω). Therefore both e and uv are crossing-critical in (G, ω).

(D) *Crossing-criticality of the edges $uu_1, uu_2, uu_3, vv_1, vv_2$, and vv_3.*

We prove the criticality of uu_1; the proof of the criticality of the other edges is totally analogous.

Consider the (unique) embedding of $G_{u,v}$. Put u in face U_i (see property (4)) and v in face F_v. Then draw uu_j, for $j = 1, 2, 3$, adding $\omega(uu_j) \cdot d_{\omega^*}(u_j, U_1)$ crossings with the edges in $G_{u,v}$. Since crossings between adjacent edges can always be avoided, it follows that uu_1, uu_2, uu_3 get drawn by adding $\omega(uu_1) \cdot d_{\omega^*}(u_1, U_1) + \omega(uu_2) \cdot d_{\omega^*}(u_2, U_1) + \omega(uu_3) \cdot d_{\omega^*}(u_3, U_1) = t \cdot d_{\omega^*}(F_u, U_1)$ crossings (using (4)). Finally we draw vv_1, vv_2, vv_3 in face F_v. Now this last step may add crossings, but only of the edges vv_1, vv_2, vv_3 with the edges uu_1, uu_2, uu_3. In view of (7), the last step added fewer than $9 \cdot (1/9) \min\{\ \omega(e) \mid e \in E(G_{u,v})\} = \min\{\ \omega(e) \mid e \in E(G_{u,v})\}$ crossings. We finally draw uv; since u is in face U_1 and v is in face F_v, it follows that uv can be drawn by adding $t \cdot d_{\omega^*}(U_1, F_v)$ crossings.

The described drawing $\mathcal{D}$ of G has then fewer than $t \cdot d_{\omega^*}(F_u, U_1) + t \cdot d_{\omega^*}(U_1, F_v) + \min\{\ \omega(e) \mid e \in E(G_{u,v})\} = t \cdot d_{\omega^*}(F_u, F_v) + \min\{\ \omega(e) \mid e \in E(G_{u,v})\} = \mathrm{cr}(G, \omega) +$

$\min\{\ \omega(e) \mid e \in E(G_{u,v})\}$ crossings, where for the first equality we used the balancedness of ω^*, and for the second equality we used (B). Thus $\mathrm{cr}(\mathcal{D}) < \mathrm{cr}(G,\omega) + \min\{\ \omega(e) \mid e \in E(G_{u,v})\}$.

In $\mathcal{D}$, the edge uu_1 contributes in $\omega(uu_1)\cdot d_{\omega^*}(u_1,U_1)$ crossings; note that (4) implies that $\omega(uu_1)\cdot d_{\omega^*}(u_1,U_1) > 0$. Since obviously $d_{\omega^*}(u_1,U_i) \geq \min\{\omega(e) \mid e \in E(G_{u,v})\}$, it follows that uu_1 contributes in at least $\min\{\omega(e) \mid e \in E(G_{u,v})\}$ crossings. Thus, if we remove uu_1 we obtain a drawing of $G - uu_1$ with fewer than $\mathrm{cr}(G,\omega)$ crossings. Therefore uu_1 is critical in (G,ω), as claimed. □

Proposition 6. *There exists a positive integer weight assignment ω on G that satisfies (1)–(7) in Proposition 5*

Proof. We start with a balanced positive integer weight assignment μ^* on $G^*_{u,v}$. The existence of such a μ^* is guaranteed from Lemma 3, which applies since the connectivity assumption on $G_{u,v}$ implies that $G^*_{u,v}$ is also 3-connected. Let μ denote the (positive integer) weight assignment naturally induced on $G_{u,v}$.

Claim I. *There exists a rational point (x_1,x_2,x_3) such that, for every $F \in \mathcal{F}\setminus\{F_u\}$,*

$$d_{\mu^*}(u_1,F)x_1 + d_{\mu^*}(u_2,F)x_2 + d_{\mu^*}(u_3,F)x_3 \geq d_{\mu^*}(F_u,F).$$

Moreover, there exist faces U_1,U_2,U_3 of $G_{u,v}$, such that $d_{\mu^*}(u_i,U_i) > 0$ and, for $i = 1,2,3$,

$$d_{\mu^*}(u_1,U_i)x_1 + d_{\mu^*}(u_2,U_i)x_2 + d_{\mu^*}(u_3,U_i)x_3 = d_{\mu^*}(F_u,U_i).$$

Proof. Let $\mathcal{I}$ be the system of inequalities $\mathcal{I} := \{d_{\mu^*}(u_1,F)x_1 + d_{\mu^*}(u_2,F)x_2 + d_{\mu^*}(u_3,F)x_3 \geq d_{\mu^*}(F_u,F) \mid f \in \mathcal{F}\setminus\{F_u\}\}$, and let Λ denote the set of all triples (x_1,x_2,x_3) that satisfy all inequalities in $\mathcal{I}$. Clearly, if x_1, x_2, and x_3 are all large enough then (x_1,x_2,x_3) is in Λ. Thus Λ is a nonempty convex polyhedron in $\mathbb{R}^3$.

Each inequality J in $\mathcal{I}$ naturally defines a plane in $\mathbb{R}^3$, namely the plane obtained by substituting $\geq$ with $=$. We call this the *plane associated to J*.

Since each u_i is incident with face F_{u_i}, it follows that $d_{\mu^*}(u_1,F_{u_1}) = d_{\mu^*}(u_2,F_{u_2}) = d_{\mu^*}(u_3,F_{u_3}) = 0$, Therefore the inequalities corresponding to F_{u_1}, F_{u_2} and F_{u_3}, respectively, define the following system Γ:

$$\begin{array}{ccccccc} & & d_{\mu^*}(u_2,F_{u_1})x_2 & + & d_{\mu^*}(u_3,F_{u_1})x_3 & \geq & d_{\mu^*}(F_u,F_{u_1}) \quad (\Gamma 1)\\ d_{\mu^*}(u_1,F_{u_2})x_1 & & & + & d_{\mu^*}(u_3,F_{u_2})x_3 & \geq & d_{\mu^*}(F_u,F_{u_2}) \quad (\Gamma 2)\\ d_{\mu^*}(u_1,F_{u_3})x_1 & + & d_{\mu^*}(u_2,F_{u_3})x_2 & & & \geq & d_{\mu^*}(F_u,F_{u_3}) \quad (\Gamma 3)\end{array}$$

where all coefficients (since the faces $F_{u_1}, F_{u_2}, F_{u_3}$ are pairwise distinct) and all the right-hand sides are strictly positive integers. We refer to this as the *positive integrality* property of (Γ1), (Γ2), and (Γ3).

This positive integrality property implies that the planes associated to (Γ1) and (Γ2) intersect in a line whose intersection with the positive octant is a (full one-dimensional) segment. In particular, there is a point (a_1, a_2, a_3), all of whose points are positive and rational, and that lies on the intersection of the planes associated to $(\Gamma 1)$ and $(\Gamma 2)$.

Now consider the set of points $r^{\rightarrow} := \{(x_1, a_2, a_3) \mid x_1 \geq 0\}$. Thus $r^{\rightarrow}$ is a ray starting at $(0, a_2, a_3)$ and parallel to the x_1-axis, lying on the plane associated to $(\Gamma 1)$. Now since $(\Gamma 1)$ is the only inequality in $\mathcal{I}$ whose x_1 coefficient is zero, it follows that for every large enough x_1, the point (x_1, a_2, a_3) is in Λ. Now every inequality in $\mathcal{I}$ distinct from $(\Gamma 1)$ intersects $r^{\rightarrow}$ in at most one point. Let a_1' be largest possible such that (a_1', a_2, a_3) is the intersection of $r^{\rightarrow}$ with a plane associated to an inequality I in $\mathcal{I} \setminus \{(\Gamma 1)\}$; since a_1 is the intersection of $(\Gamma 2)$ with $(\Gamma 1)$, it follows that a_1' is well-defined.

Since a_1', a_2, a_3 all arise from the intersection of planes with integer coefficients, it follows that they are all rational numbers. We claim that the rational point (a_1', a_2, a_3) satisfies the requirements in the Claim.

Consider any inequality J in $\mathcal{I}$ distinct from $(\Gamma 1)$, $(\Gamma 2)$, and $(\Gamma 3)$. Thus all the coefficients of J are nonzero, and so the intersection of the plane associated to J with the positive octant is a triangle. Let (j, a_2, a_3) be the intersection of this triangle with the line $\{(x_1, a_2, a_3) \mid x_1 \in \mathbb{R}\}$. Then $j \leq a_1'$, and so the ray $\{(x_1, a_2, a_3) \mid x_1 \in \mathbb{R}, x_1 \geq j\}$ is contained in the *feasible* region of J (that is, the region of $\mathbb{R}^3$ that consists of those points for which J is satisfied). In particular, (a_1', a_2, a_3) is in the feasible region of J. A similar argument shows that also in the case in which J is either $(\Gamma 2)$ or $(\Gamma 3)$, then (a_1', a_2, a_3) is in the feasible region of J. This proves the first part of Claim I.

Now we recall that (a_1', a_2, a_3) lies on the plane associated to $(\Gamma 1)$, that is, $d_{\mu^*}(u_2, F_{u_1})a_2 + d_{\mu^*}(u_3, F_{u_1})a_3 = d_{\mu^*}(F_u, F_{u_1})$. Since $d_{\mu^*}(u_1, F_{u_1}) = 0$, we have $d_{\mu^*}(u_1, F_{u_1})a_1' + d_{\mu^*}(u_2, F_{u_1})a_2 + d_{\mu^*}(u_3, F_{u_1})a_3 = d_{\mu^*}(F_u, F_{u_1})$. Noting that $d_{\mu^*}(u_2, F_{u_1}) > 0$ and $d_{\mu^*}(u_e, F_{u_1}) > 0$, the second part of Claim I follows for $i = 2$ and 3 by setting $U_2 = U_3 = F_{u_1}$. Now let U_1 be the face in $\mathcal{G}$ associated to inequality I. Thus $d_{\mu^*}(u_1, U_1)a_1' + d_{\mu^*}(u_2, U_1)a_2 + d_{\mu^*}(u_3, U_1)a_3 = d_{\mu^*}(F_u, U_1)$. We recall that $(\Gamma 1)$ is the only inequality in $\mathcal{I}$ whose x_1 coefficient is 0; since inequality I is distinct from $(\Gamma 1)$ it follows that $d_{\mu^*}(u_1, U_1) > 0$. Thus the second part of Claim I follows for $i = 1$ for this choice of U_1. □

The proof of the following statement is totally analogous:

Claim II. *There exists a rational point (y_1, y_2, y_3) such that, for every $F \in \mathcal{F} \setminus \{F_v\}$,*

$$d_{\mu^*}(v_1, F)y_1 + d_{\mu^*}(v_2, F)y_2 + d_{\mu^*}(v_3, F)y_3 \geq d_{\mu^*}(F_v, F).$$

Moreover, there exist faces V_1, V_2, V_3 of $G_{u,v}$, such that $d_{\mu^*}(v_i, V_i) > 0$ and, for $i = 1, 2, 3$,

$$d_{\mu^*}(v_1, V_i)y_1 + d_{\mu^*}(v_2, V_i)y_2 + d_{\mu^*}(v_3, V_i)y_3 = d_{\mu^*}(F_u, V_i). \quad \square$$

Let $(p_1/q_1, p_2/q_2, p_3/q_3)$ be a point as in Claim I, and let $(a_1/b_1, a_2/b_2, a_3/b_3)$ be a point as in Claim II, where all p_is, q_is, a_is, and b_is are integers. Let $M := q_1q_2q_3b_1b_2b_3$, and let $r_1 := p_1q_2q_3b_1b_2b_3, r_2 := p_2q_1q_3b_1b_2b_3$, $r_3 := p_3q_1q_2b_1b_2b_3$, $s_1 := a_1b_2b_3q_1q_2q_3, s_2 := a_2b_1b_3q_1q_2q_3$, and $s_3 := a_3b_1b_2q_1q_2q_3$.

Then (r_1, r_2, r_3) is a positive integer solution to the set of inequalities $\{d_{\mu^*}(u_1, F)r_1 + d_{\mu^*}(u_2, F)r_2 + d_{\mu^*}(u_3, F)r_3 \geq M \cdot d_{\mu^*}(F_u, F) : F \in \mathcal{F} \setminus \{F_u\}\}$, and for each $i = 1, 2, 3$, we have $d_{\mu^*}(u_1, U_i)r_1 + d_{\mu^*}(u_2, U_i)r_2 + d_{\mu^*}(u_3, U_i)r_3 = M \cdot d_{\mu^*}(F_u, U_i)$.

Similarly, (s_1, s_2, s_3) is a positive integer solution to the set of inequalities $\{d_{\mu^*}(v_1, F)s_1 + d_{\mu^*}(v_2, F)s_2 + d_{\mu^*}(v_3, F)s_3 \geq M \cdot d_{\mu^*}(F_v, F) : F \in \mathcal{F} \setminus \{F_v\}\}$, and for each $i = 1, 2, 3$, we have $d_{\mu^*}(v_1, V_i)s_1 + d_{\mu^*}(v_2, V_i)s_2 + d_{\mu^*}(v_3, V_i)s_3 = M \cdot d_{\mu^*}(F_v, V_i)$.

Finally, let c be any integer greater than $M \cdot d_{\mu^*}(F_u, F_v)/(\min\{\mu(e) \mid e \in E(G_{u,v})\})^2$ and also greater than $9r_i s_j / \min\{\mu(e) \mid e \in E(G_{u,v})\}$, for all $i, j \in \{1, 2, 3\}$.

Define the weight assignment ω on G as follows:

- $\omega(uv) = M$;
- $\omega(uu_i) = r_i$ and $\omega(vv_i) = s_i$ for $i = 1, 2, 3$;
- $\omega(e) = c \cdot \mu(e)$, for all edges e in $G_{u,v}$.

We claim that ω (and its induced weight assignment ω^* on $G^*_{u,v}$) satisfies (1)–(7) in Proposition 5.

To see that ω^* satisfies (1), it suffices to note that ω^* inherits the balancedness (when restricted to $G^*_{u,v}$) from μ^*.

Now let e, e' be edges of $G_{u,v}$. Then $\omega(e)\omega(e') = c^2 \cdot \mu(e)\mu(e') \geq c^2 \cdot (\min\{\mu(f) \mid f \in E(G_{u,v})\ \})^2 > c \cdot M \cdot d_{\mu^*}(F_u, F_v) = \omega(uv)(c \cdot d_{\mu^*}(F_u, F_v)) = \omega(uv) \cdot d_{\omega^*}(F_u, F_v)$. This proves (2).

For (3) and (4), recall that (r_1, r_2, r_3) is a positive integer solution to the set of inequalities $\{d_{\mu^*}(u_1, F)r_1 + d_{\mu^*}(u_2, F)r_2 + d_{\mu^*}(u_3, F)r_3 \geq M \cdot d_{\mu^*}(F_u, F) : F \in \mathcal{F} \setminus \{F_u\}\}$, and for each $i = 1, 2, 3$, we have $d_{\mu^*}(u_1, U_i)r_1 + d_{\mu^*}(u_2, U_i)r_2 + d_{\mu^*}(u_3, U_i)r_3 = M \cdot d_{\mu^*}(F_u, U_i)$. The definition of ω (and its induced ω^*) then immediately imply (3) and (4) (we are using that for any faces $F, F' \in \mathcal{F} \setminus \{F_u\}$, $d_{\omega^*}(F, F') = c \cdot d_{\mu^*}(F, F')$). The proof that (5) and (6) hold is totally analogous.

Finally, we recall that we defined c so that $c > 9r_i s_j / \min\{\mu(e) \mid e \in E(G_{u,v})\}$ for all $i, j \in \{1, 2, 3\}$. By the definition of ω, this is equivalent to $c \cdot \min\{\mu(e) \mid e \in E(G_{u,v})\} > 9\omega(uu_i)\omega(vv_j)$, that is, $\min\{\omega(e) \mid e \in E(G_{u,v})\} > 9\omega(uu_i)\omega(vv_j)$, which is in turn obviously equivalent to (7). □

6 Concluding Remarks and Open Questions

Let $\mathcal{G}$ be the class of graphs that can be made crossing-critical by a suitable multiplication of edges. In this work we have proved that a large family of graphs is contained in $\mathcal{G}$ (note that the cubic condition is only used around vertices $u, v, u_1, u_2, u_3, v_1, v_2$ and v_3; other vertices can have arbitrary degrees). Which other graphs belong to $\mathcal{G}$? Is there any hope of fully characterizing $\mathcal{G}$?

It is not difficult to prove that we can restrict our attention to simple graphs: if $\overline{G}$ is a graph with multiple edges and G is a maximal simple graph contained in G, then $\overline{G}$ is in $\mathcal{G}$ if and only if G is in $\mathcal{G}$.

Jesús Leaños has observed that the graph $K_{3,3}^{+}$ obtained by adding to $K_{3,3}$ an edge (between vertices in the same chromatic class) is not in $\mathcal{G}$. Following Širáň [12, 13], an edge e in a graph G is a *Kuratowski edge* if there is a subgraph H of G that contains e and is homeomorphic to a Kuratowski graph (that is, $K_{3,3}$ or K_5). It is trivial to see that the added edge in Leaños's example is not a Kuratowski edge of $K_{3,3}^{+}$. This observation naturally gives rise to the following.

Conjecture 7. *If G is a graph all whose edges are Kuratowski edges, then G can be made crossing-critical by a suitable multiplication of its edges.*

We remark that the converse of this statement is not true: Širáň [12] gave examples of graphs that contain crossing-critical edges that are not Kuratowski edges.

The only positive result we have in this direction is that Kuratowski edges can be made individually crossing-critical:

Proposition 8. *If e is a Kuratowski edge of a graph G, then e can be made crossing-critical by a suitable multiplication of the edges of G.*

Proof. Let H be a subgraph of G, homeomorphic to a Kuratowski graph, such that e is in H. Let f be another edge of H such that there is a drawing $\mathcal{D}_H$ of H with exactly one crossing, which involves e and f. Extend $\mathcal{D}_H$ to a drawing $\mathcal{D}$ of G. Let p be the number of crossings in $\mathcal{D}$. If $p = 1$ then e is already critical in G, so there is nothing to prove. Thus we may assume that $p \geq 2$. Add $p^2 - 1$ parallel edges to each of e and f, add $p^4 - 1$ parallel edges to all edges in $H \setminus \{e, f\}$, and do not add any parallel edge to the other edges of G. Let G' denote the resulting graph.

We claim that $\mathrm{cr}(G') \leq p^5$. To see this, consider the drawing $\mathcal{D}'$ of G' naturally induced by $\mathcal{D}$. It is easy to check that each crossing from $\mathcal{D}$ yields at most p^4 crossings in $\mathcal{D}'$ (here we use that e and f are the only edges in H that cross each other in $\mathcal{D}$). Thus $\mathcal{D}'$ has at most $p \cdot p^4 = p^5$ crossings, and so $\mathrm{cr}(G') \leq p^5$, as claimed.

On the other hand, it is clear that a drawing of G' in which e and f do not cross each other has at least p^6 crossings. Since $p^6 > p^5 \geq \mathrm{cr}(G')$, it follows that no such drawing can be optimal. Therefore e and f cross each other in every optimal drawing of G'. This immediately implies that e is critical in G'. $\square$

The immediate next step towards Conjecture 7 seems already difficult enough so as to prompt us to state it:

Conjecture 9. *Suppose that e, f are Kuratowski edges of a graph G. Then there exists a graph H, obtained by multiplying edges of G, such that both e and f are crossing-critical in H.*

Acknowledgements

We thank Jesús Leaños for helpful discussions.

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