The convex hull of every optimal pseudolinear drawing of K_n is a triangle

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Abstract

We show that the convex hull of every optimal pseudolinear drawing of K_n is a triangle. This is closely related to the recently proved conjecture that the convex hull of every optimal rectilinear drawing of K_n is a triangle.

1 Introduction

1.1 Our main result

The following statement remained an important, open conjecture for a long time. Recently, a proof was announced by Aichholzer, Orden, and Ramos [2].

Theorem 1 ([2]) *The convex hull of every optimal rectilinear drawing of K_n is a triangle.*

Extending this conjecture to (optimal) nonrectilinear drawings of K_n does not make much sense: there is no distinguished unbounded face if the rectilinear condition is altogether dropped, so a meaningful convex hull cannot even be defined. On the other hand, since the convex hull is well–defined for pseudolinear (which lie in between rectilinear and arbitrary) drawings, it makes sense to ask if a similar property holds for pseudolinear drawings. Our main result is that an analogous statement holds for pseudolinear drawings.

Theorem 2 (Main result) *The convex hull of every optimal pseudolinear drawing of K_n is a triangle.*

1.2 Pseudolinear drawings

Recall that a *pseudoline* in the projective plane $\mathbb{P}^2$ is a simple closed curve whose removal does not disconnect $\mathbb{P}^2$. A collection of pseudolines is a *pseudoline arrangement* if each two pseudolines intersect (necessarily

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cross) in exactly one point. A *generalized configuration* Ω_P *with point set* P consists of a finite set P of points, together with a pseudoline joining each pair, and it is *simple* if there is a single pseudoline for each pair.

Consider a good drawing $\mathcal{D}$ of K_n in the plane $\mathbb{R}^2$ (thus, every edge is represented by a simple curve), contained in a disk with radius $< R$ centered at the origin. Let D denote the disk with radius R, centered at the origin. By identifying antipodal points in the boundary of D (and discarding $\mathbb{R}^2 \setminus D$), we may regard $\mathcal{D}$ as (a new drawing $\mathcal{D}'$, as the host surface has changed) lying in the projective plane. Now if each edge e in $\mathcal{D}'$ can be extended to a pseudoline (an *extension of* e) so that the resulting structure is a simple generalized configuration Ω_P in which P is the set of n vertices, then the original drawing $\mathcal{D}$ is a *pseudolinear drawing of* K_n. The *pseudosegments* are the edges of a pseudolinear drawing; in pseudolinear drawings we use the term "edge" and "pseudosegment" interchangeably. If we start with a pseudolinear drawing of K_n (which, we emphasize, lies in $\mathbb{R}^2$), it is easy to see that we may equivalently stay (all along) in $\mathbb{R}^2$, and for each edge e construct an $\mathbb{R}^2$–*extension* ℓ_e, a set of points homeomorphic to a straight line, which contains e, whose removal disconnects $\mathbb{R}^2$ into two unbounded sets, and so that every pair of $\mathbb{R}^2$–extensions cross at exactly one point.

As we observed above, the convex hull in a pseudolinear drawing of K_n is a well–defined object that naturally generalizes the definition of the convex hull of a rectilinear drawing (the definition actually applies to quite more general objects, namely the *CC*–systems introduced by Knuth; see [7] and [9]). Consider a pseudolinear drawing $\mathcal{D}$ of K_n, and for each edge (pseudosegment) e construct an $\mathbb{R}^2$–extension ℓ_e as described above. An edge in $\mathcal{D}$ is a *convex hull edge* of $\mathcal{D}$ if the $n-2$ points (vertices of K_n) not incident with e lie on the same half–plane of ℓ_e, and the *convex hull* of $\mathcal{D}$ is the collection of all the convex hull edges and their incident vertices. It can be checked that convex hull edges are well–defined, that is, independent of the chosen $\mathbb{R}^2$–extensions.

It is readily verified that no convex hull edge can cross another edge. Therefore Theorem 2 states that the obvious extension of Theorem 1 to pseudolinear drawings is true: the unbounded face in any optimal pseudolinear drawing of K_n is incident with (exactly) 3 vertices and 3 edges.

1.3 Pseudolinear and rectilinear crossing numbers

If $\mathcal{D}$ is a drawing of K_n, then we let $\mathrm{cr}(\mathcal{D})$ denote the number of pairwise crossings of edges in $\mathcal{D}$. The *pseudolinear crossing number* $\widetilde{\mathrm{cr}}(K_n)$ is the minimum of $\mathrm{cr}(\mathcal{D})$ over all pseudolinear drawings $\mathcal{D}$ of K_n. The *rectilinear crossing number* $\overline{\mathrm{cr}}(K_n)$ of K_n is the minimum of $\mathrm{cr}(\mathcal{D})$ over all rectilinear drawings $\mathcal{D}$ of K_n. Since every rectilinear drawing of K_n is also a pseudolinear drawing, $\overline{\mathrm{cr}}(K_n) \geq \widetilde{\mathrm{cr}}(K_n)$.

If a pseudolinear drawing is combinatorially equivalent to a rectilinear drawing, then it is *stretchable*. Since almost all pseudolinear drawings are non–stretchable (see for instance [11]), it is conceivable that $\widetilde{\mathrm{cr}}(K_n) < \overline{\mathrm{cr}}(K_n)$ for some n. We have verified that $\widetilde{\mathrm{cr}}(K_n) = \overline{\mathrm{cr}}(K_n)$ for $n \leq 12$, and in this basis we put forward the following.

Conjecture 3 *For every* n, $\widetilde{\mathrm{cr}}(K_n) = \overline{\mathrm{cr}}(K_n)$.

Settling this conjecture in either direction would be quite interesting by itself: we would know whether or not there is anything to gain, with respect to crossing numbers, by considering non–stretchable pseudolinear drawings of K_n (over rectilinear ones).

2 Background: generalized configurations and allowable sequences

We recall that a *simple allowable sequence on* n *elements* Π is a doubly infinite sequence $(\ldots, \pi_{-1}, \pi_0, \pi_1, \ldots)$ of permutations of an n–element *ground* set (say $\{p_1, p_2, \ldots, p_n\}$), such that (i) any two consecutive permutations differ by exactly one transposition of two elements in adjacent positions; and (ii) after a move in which i and j switch, they do not switch again until every other pair has switched. If a transposition τ swaps elements p_i

and p_j, so that p_i moves from position t to position $t+1$, and p_j moves from position $t+1$ to position t, then we write $\tau = [p_i|p_j]_t$. An allowable sequence $\Pi = (\ldots, \pi_{-1}, \pi_0, \pi_1, \ldots)$ on n elements is equivalently defined by its *transpositions sequence* $T(\Pi) = (\ldots, \tau_{-1}, \tau_0, \tau_1, \ldots)$, where τ_i is the transposition that transforms π_{i-1} into π_i.

It is straightforward to see that a simple allowable sequence on n elements has period $n(n-1)$. We shall be particularly interested in halfperiods of Π, that is, finite subsequences $(\pi_i, \pi_{i+1}, \ldots, \pi_{i+\binom{n}{2}})$. Note that the ending permutation of a halfperiod is the reverse permutation of the starting one.

Simple allowable sequences, introduced by Goodman and Pollack in an influential paper [8], are a fruitful tool to encode any generalized configuration of points: to each generalized configuration of points Ω_P with point set P, one can naturally associate a simple allowable sequence Π_{Ω_P} with ground set P, and, reciprocally, given a simple allowable sequence Π with ground set P one can obtain a generalized configuration of points Ω_P whose associated sequence is $\Pi_{\Omega_P} = \Pi$. The details of this relationship have been lucidly explained in [8] and in subsequent surveys (more recently in [1] or [10], precisely in the context of crossing numbers), so we shall omit them, and refer the interested reader to these sources.

Suppose that $\mathcal{D}$ is a pseudolinear drawing of K_n, with underlying n–point set P. Thus (since $\mathcal{D}$ is pseudolinear) P is the point set of a simple generalized configuration Ω_P. We say that Ω_P is a generalized configuration *associated to* $\mathcal{D}$. Although Ω_P is not unique (as there are infinitely many ways to extend the pseudoedges to form pseudolines), the induced simple allowable sequence Π_{Ω_P} is unique, and thus it is consistent to call $\Pi_{\mathcal{D}} := \Pi_{\Omega_P}$ *the simple allowable sequence* associated to $\mathcal{D}$.

3 Allowable sequences and convex hulls: proof of Theorem 2

The encoding scheme from generalized configurations of points to simple allowable sequences [8] makes it particularly easy to identify the convex hull of a pseudolinear drawing of K_n, as follows.

Proposition 4 *Let $\mathcal{D}$ be a pseudolinear drawing of K_n, and let P denote the underlying n–point set. Let Π_0 be any halfperiod of the associated simple allowable sequence. Then a point p in P is in the convex hull of $\mathcal{D}$ iff it occupies either position 1 or position n in a permutation of Π_0.*

In view of this, in order to establish Theorem 2 it suffices to show that if $\mathcal{D}$ is optimal among pseudolinear drawings (that is, $\widetilde{\mathrm{cr}}(\mathcal{D}) = \widetilde{\mathrm{cr}}(K_n)$), then at most 3 elements in P ever occupy position 1 or position n in some permutation in Π_0 (any halfperiod of $\Pi_{\mathcal{D}}$). In order to prove such a result, we need a useful characterization of which simple allowable sequences are induced from optimal pseudolinear drawings of K_n.

Such a characterization can be obtained from results in [1] and [10] that give the crossing number in a pseudolinear drawing of K_n in terms of properties of its associated simple allowable sequence. In order to present this result, we need to define one local and one global function. Let $\tau = [p_i|p_j]_t$ be a transposition in the transpositions sequence of a simple allowable sequence Π. The *impact* $f(\tau)$ of τ is defined as follows:

$$f(\tau) = f([a|b]_t) = \left(\frac{n-2}{2} - (t-1)\right)^2. \tag{1}$$

Now if Π_0 is a halfperiod of a simple allowable sequence, then its *weight* $F(\Pi_0)$ is

$$F(\Pi_0) = \sum_{\tau} f(\tau), \tag{2}$$

where the summation is over all the τ_i's in the transpositions sequence of Π_0. That is, the weight of Π_0 is simply the sum of the impacts of all the transpositions in its transpositions sequence.

The relevance of the weight of a halfperiod of a simple allowable sequence induced by a pseudolinear drawing of K_n comes from the following result.

Theorem 5 ([1],[10]) *Let $\mathcal{D}$ be a pseudolinear drawing of K_n, and let Π be a halfperiod of its associated simple allowable sequence. Then*

$$\widetilde{\mathrm{cr}}(\mathcal{D}) = 3\binom{n}{4} - F(\Pi_0).$$

Our last required result, which is proved in Section 4, gives us a crucial piece of information on halfperiods that maximize F.

Proposition 6 *Let Π_0 be a halfperiod of a simple allowable sequence on n elements. Suppose that Π_0 maximizes F over all halfperiods of simple allowable sequences on n elements. Then there are (exactly) 3 elements that occupy either position 1 or position n in a permutation of Π_0.*

Proof of Theorem 2.

Since every simple allowable sequence can be induced from a pseudolinear drawing of K_n, it follows from Theorem 5 that a pseudolinear drawing of K_n is optimal iff any halfperiod of its associated simple allowable sequence maximizes F over all possible halfperiods of simple allowable sequences. Propositions 6 and 4 complete the proof. ∎

4 Proof of Proposition 6

Throughout this proof, $\Pi_0 = (\pi_0, \pi_1, \pi_2, \ldots, \pi_{\binom{n}{2}})$ is a halfperiod of a simple allowable sequence that minimizes F. Unless otherwise stated, all transpositions and permutations hereby mentioned occur are associated to Π_0.

Let us label the points so that the initial permutation is $a_1 a_2 \ldots a_n$.

Claim A *Let i satisfy $\lceil n/2 \rceil \le i < n$. Let τ_s be the transposition that moves a_n to position i. Suppose that a_ℓ is to the right of a_n in π_s. Then, after τ_s occurs, the first transposition that involves a_ℓ moves a_ℓ to the left, and the other element involved in the transposition is to the left of a_n in π_s.*

Proof. Seeking a contradiction, let i be smallest possible so that the statement is false. Label $b_1, b_2, \ldots, b_{n-i}$ the last $n-i$ elements in π_s, in the order in which they appear in π_s. Note that $\tau_s = [b_1|a_n]_i$.

We claim that the first transposition τ_t after τ_s that involves an element in $\{b_1, b_2, \ldots, b_{n-i}\}$ must be the transposition swapping elements b_1 and b_2. Recall that Claim A holds if we substitute i by $i-1$. This implies, in particular, that the first element in $\{b_2, \ldots, b_{n-i}\}$ that gets involved in a transposition after τ_s must be b_2, and that the other element involved in the transposition is to the left of b_2 in π_s. Now the first transposition after τ_s that involves b_1 cannot involve an element to the left of b_1 in π_s, as otherwise (it is easy to check) Claim A would then also hold for i. Thus τ_t must involve b_1 and b_2, that is, $\tau_t = [b_1|b_2]_{i+1}$. Again using the assumption that Claim A holds for $i-1$, it follows that the last transposition τ_r before τ_s that involves an element in $b_1, b_2, \ldots, b_{n-i}$ is precisely the transposition that swaps b_2 and a_n, that is, $\tau_r = [b_2|a_n]_{i+1}$.

Thus, the following transpositions occur in the given order: $\tau_r = [b_2|a_n]_{i+1}$, $\tau_s = [b_1|a_n]_i$, and $\tau_t = [b_1|b_2]_{i+1}$. Moreover, the only transposition between τ_r and τ_t that involves an element in position $i+1$ or further right is precisely τ_s. This last observation implies that if we modify the transpositions sequence by delaying τ_r (if necessary) and letting it occur immediately before τ_s, and then accelerating τ_t (if necessary) and letting it occur immediately after τ_s, and leaving the transposition sequence otherwise unchanged, the resulting transpositions sequence will still correspond to a (valid) halfperiod $\widetilde{\Pi}_0$ of a simple allowable sequence. More precisely, if we let $\tau_i' = \tau_i$ for $1 \le i < r$, $\tau_i' = \tau_{i+1}$ for $r \le i \le s-2$, $\tau_{s-1}' = [b_1|b_2]_i$, $\tau_s' = [b_1|a_n]_{i+1}$, $\tau_{s+1}' = [b_2|a_n]_i$, $\tau_i' = \tau_{i-1}$ for $s+2 \le i \le t$, and $\tau_i' = \tau_i$ for $i > t$, then $\tau_0', \tau_1', \ldots, \tau_{\binom{n}{2}}'$ is the transpositions sequence of a simple allowable sequence $\overline{\Pi}_0$. Clearly, $\sum_{\tau_i \notin \{\tau_r, \tau_s, \tau_t\}} f(\tau_i) = \sum_{\tau_i' \notin \{\tau_{s-1}', \tau_s', \tau_{s+1}'\}} f(\tau_i')$, Moreover, $f(\tau_r) = f(\tau_s')$ and $f(\tau_s) = f(\tau_{s-1}')$, and so $\sum_{\tau_i \ne \tau_t} f(\tau_i) = \sum_{\tau_i' \ne \tau_{s+1}'} f(\tau_i')$. However, $f(\tau_t) = (\frac{n-2}{2} - ((i+1)-1))^2 < (\frac{n-2}{2} - (i-1))^2 = f(\tau_{s+1}')$ (note that here we are using that $i \ge \lceil n/2 \rceil$). Therefore $F(\Pi_0) =$

$\sum_{\tau_i} f(\tau_i) < \sum_{\tau_i'} f(\tau_i') = F(\overline{\Pi}_0)$, contradicting the assumption that assumption that Π_0 maximizes F over all halfperiods of simple allowable sequences of size n. ∎

Claim B *Either a_1 moves a_n from position n or a_n moves a_1 from position* 1.

Proof of Claim B. We suppose that a_1 reaches position $\lceil n/2 \rceil$ before a_n reaches position $\lfloor n/2 \rfloor + 1$ (it is readily checked that these cannot occur simultaneously), and show that in this case a_1 moves a_n out of position n. The other possibility, that a_n reaches position $\lfloor n/2 \rfloor + 1$ before a_1 reaches position $\lceil n/2 \rceil$ (in which case the conclusion is that a_n moves a_1 from position 1), is dealt with in a totally analogous manner.

Let $m+1$ be the position of a_1 immediately after it swaps with a_n. Thus, the transposition between a_1 and a_n is $[a_1|a_n]_m = \tau_q$ for some q. Since a_1 only moves right, and a_n only moves left, it follows that a_1 is in position $m \geq \lceil n/2 \rceil$ just before this permutation, that is, in π_{q-1}.

To prove the statement, for the rest of the proof we assume that $m < n-1$, and derive a contradiction.

Let b denote the element in position $m+2$ in π_{q-1} (and still there in π_q). Now b is to the right of a_n already in π_{q-1}. An application of Claim A with $i = m+1$ (that is, when a_n first moved into position $m+1$) yields that b could not have arrived to position $m+2$ (in π_{q-1}) by transposing with an element other than a_n. Thus b and a_n swap when b is in position $m+1$ (and a_n is in position $m+2$). Thus this transposition is $[b|a_n]_{m+1} = \tau_p$ for some $p < q$.

We note again that a_1 never moves left. Applying Claim A (again with $i = m+1$), we obtain that the transposition τ_r with $r > q$ smallest possible that involves an element in position $m+1$ or further right is the transposition that swaps a_1 and b. That is, $\tau_r = [a_1|b]_{m+1}$.

Thus, the following transpositions occur in the given order: $\tau_p = [b|a_n]_{m+1}$, $\tau_q = [a_1|a_n]_m$, and $\tau_r = [a_1|b]_{m+1}$. Moreover, τ_q is the only transposition between τ_p and τ_r that involves an element in position $m+1$ or further right (this follows again from Claim A). This observation implies that if we modify the transpositions sequence by delaying τ_p (if necessary) and letting it occur immediately before τ_q, and then accelerating τ_r (if necessary) and letting it occur immediately after τ_q, and leaving the transposition sequence otherwise unchanged, the resulting transpositions sequence will still induce a (valid) simple allowable sequence $\widetilde{\Pi}_0$. More precisely, if we let $\tau_i' = \tau_i$ for $1 \leq i < p$, $\tau_i' = \tau_{i+1}$ for $p \leq i \leq q-2$, $\tau_{q-1}' = [a_1|b]_m$, $\tau_q' = [a_1|a_n]_{m+1}$, $\tau_{q+1}' = [b|a_n]_m$, $\tau_i' = \tau_{i-1}$ for $q+2 \leq i \leq r$, and $\tau_i' = \tau_i$ for $i > r$, then $\tau_0', \tau_1', \ldots, \tau_{\binom{n}{2}}'$ is the transpositions sequence of a simple allowable sequence $\overline{\Pi}_0$. Clearly, $\sum_{\tau_i \notin \{\tau_p, \tau_q, \tau_r\}} f(\tau_i) = \sum_{\tau_i' \notin \{\tau_{q-1}', \tau_q', \tau_{q+1}'\}} f(\tau_i')$, Moreover, $f(\tau_p) = f(\tau_q')$ and $f(\tau_q) = f(\tau_{q-1}')$, and so $\sum_{\tau_i \neq \tau_r} f(\tau_i) = \sum_{\tau_i' \neq \tau_{q+1}'} f(\tau_i')$. However, $f(\tau_r) = (\frac{n-2}{2} - ((m+1)-1))^2 < (\frac{n-2}{2} - (m-1))^2 = f(\tau_{q+1}')$. Therefore $F(\Pi_0) = \sum_{\tau_i} f(\tau_i) < \sum_{\tau_i'} f(\tau_i') = F(\overline{\Pi}_0)$ (here we are using that $m \geq \lceil n/2 \rceil$), contradicting the assumption that Π_0 maximizes F over all halfperiods of simple allowable sequences of size n. ∎

Conclusion of proof of Proposition 6.

By Claim B, either a_1 moves a_n from position n or a_n moves a_1 from position 1. Suppose the former case holds. Let x be the element that moves a_1 from position 1. Immediately after a_1 and x transpose, x is in position 1, and a_n is in position n. Thus another application of Claim B (with the suitable relabeling) implies that either x moves a_n out of position n or a_n moves x out of position 1. The former case is impossible, since $a_1 \neq x$ is the element that moves a_n out of position n. Thus a_n moves x out of position 1. Therefore, the only elements that ever occupy position 1 are a_1, x, and a_n, and the only elements that ever occupy position n are a_1 and a_n. ∎

A slightly different proof is given in [12].

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