The crossing number of $C_m \times C_n$ is as conjectured for $n \geq m(m+1)$

LEV YU. GLEBSKY[1]

Department of Mechanics and Mathematics,
Nizhny Novgorod State University, Russia

and

GELASIO SALAZAR[2]

IICO–UASLP
Av. Karakorum 1470, Lomas 4ta. Seccion
San Luis Potosi SLP, Mexico 78210

29 December 2003

Abstract. It has been long conjectured that the crossing number of $C_m \times C_n$ is $(m-2)n$, for all m, n such that $n \geq m \geq 3$. In this paper it is shown that if $n \geq m(m+1)$ and $m \geq 3$, then this conjecture holds. That is, the crossing number of $C_m \times C_n$ is as conjectured for all but finitely many n, for each m. The proof is largely based on techniques from the theory of arrangements, introduced by Adamsson and further developed by Adamsson and Richter.

2000 Mathematics Subject Classification: 05C10, 05C62, 57M15

Keywords: Crossing number, Cartesian product of cycles, graph drawing

1 Introduction

In 1973, Harary, Kainen, and Schwenk proved that toroidal graphs can have arbitrarily large crossing numbers [7]. In the same paper, they put forward the following conjecture.

Conjecture [HKS–Conjecture] *The crossing number $cr(C_m \times C_n)$ of the Cartesian product $C_m \times C_n$ is $(m-2)n$, for all m, n such that $n \geq m \geq 3$.*

This has been proved for m, n satisfying $n \geq m$, $m \leq 7$ [7, 14, 6, 5, 13, 11, 3, 12, 4, 1]. In this paper we show that the HKS–conjecture holds for all but finitely many n, for each $m \geq 3$.

[1]Current affiliation: IICO-UASLP. Av. Karakorum 1470, Lomas 4ta. Seccion. San Luis Potosi, SLP. Mexico 78210
[2]Corresponding Author. E–mail: `gsalazar@cactus.iico.uaslp.mx`

Theorem 1 *Let m, n be integers such that $n \geq m(m+1), m \geq 3$. Then $cr(C_m \times C_n) = (m-2)n$.*

The Cartesian product $C_m \times C_n$ is a particular case of an (m,n)–graph. A 4–regular graph is an (m,n)*–graph* if it consists of n pairwise disjoint, cyclically ordered m–cycles $R(0), R(1), \ldots, R(n-1)$ (colored red to help comprehension), plus mn (blue) edges, so that each vertex in $R(j)$ is adjacent, via blue edges, to one vertex in $R(j \ominus 1)$ and to one vertex in $R(j \oplus 1)$ (throughout this work addition and subtraction modulo n are denoted $\oplus$ and $\ominus$, respectively).

It is not difficult to exhibit drawings of $C_m \times C_n$ with exactly $(m-2)n$ crossings. Hence $\text{cr}(C_m \times C_n) \leq (m-2)n$. Thus, in order to prove Theorem 1 we need to prove that $\text{cr}(C_m \times C_n) \geq (m-2)n$. This inequality is a consequence of our main result, since $C_m \times C_n$ is an (m,n)–graph.

Main Theorem *Let $\mathcal{E}$ be a drawing of an (m,n)–graph, where m, n satisfy $n \geq m(m+1), m \geq 3$. Then $\mathcal{E}$ has at least $(m-2)n$ crossings.*

The proof of the Main Theorem is largely based on techniques from the theory of arrangements, introduced by Adamsson [1], and further developed by Adamsson and Richter [2].

In a *drawing* of a graph G, (different) vertices are represented by (different) points, and edges are represented by *open arcs* (continuous images of $(0,1)$) in such a way that the end points of the representation of an edge e are precisely the points that represent the vertices incident with e. It is also assumed that no representation of an edge contains a representation of a vertex.

A drawing is *good* if (i) no edge has a self–intersection; (ii) no two adjacent edges intersect; (iii) no two edges intersect each other more than once; (iv) each intersection of edges is a crossing rather than tangential; and (v) no three edges intersect in a common point.

The *crossing number* $\text{cr}(G)$ of G is the minimum number of pairwise intersections of edges in a drawing of G in the plane. A drawing $\mathcal{E}$ of G is *optimal* if its number $\text{cr}(\mathcal{E})$ of pairwise intersections of edges equals $\text{cr}(G)$. It is a routine exercise to show that conditions (i) to (iv) in the previous paragraph hold for every optimal drawing of G, and that every optimal drawing of G can be easily modified to satisfy (v). Thus, in order to calculate the crossing number of G it suffices to consider good drawings of G.

The theory of crossing numbers has recently received a good deal of attention from the point of view of the relationship between the crossing number and the structural properties of a graph. Good examples of work in this direction are the recent papers by Hliněný (see [8, 9]).

The heart of the proof of the Main Theorem is the following.

Theorem 2 *Let $\mathcal{E}$ be a robust drawing of an (m,n)–graph, where m, n satisfy $n \geq m \geq 3$. Then $\mathcal{E}$ has at least $(m-2)n$ crossings.*

Roughly speaking (the formal definition is in Section 2), a drawing of an (m,n)–graph is robust if no red cycle separates two other red cycles (in a sense made precise below) and each red cycle is disjoint from (sufficiently) many other red cycles. As we prove below, if n is large enough compared to m, then the only way to avoid robustness is to have a red cycle with at least m crossings.

Lemma 3 *Let $\mathcal{E}$ be a drawing of an (m,n)–graph, where m, n satisfy $n \geq 2m+2$, $m \geq 3$. Then either $\mathcal{E}$ is robust or there is a red cycle with at least m crossings in $\mathcal{E}$.*

The argument that shows that the Main Theorem follows from Theorem 2 and Lemma 3 can be outlined as follows.

Suppose that m, n satisfy $n \geq m(m+1), m \geq 3$, and let $\mathcal{E}$ be a drawing of an (m,n)–graph. Assume $\mathcal{E}$ is not robust, as otherwise we are done by Theorem 2. Then some red cycle $R(j_1)$ has m

or more crossings in $\mathcal{E}$, by Lemma 3. Repeat the argument with the drawing $\mathcal{E}_1$ (of an $(m, n-1)$–graph) that results by removing $R(j_1)$ from $\mathcal{E}$: either $\mathcal{E}_1$ is robust (in which case we are done, since then $\mathcal{E}_1$ has at least $(m-2)(n-1)$ crossings, and so $\mathcal{E}$ has at least $(m-2)(n-1)+m = (m-2)n+2$ crossings), or some red cycle $R(j_2)$ has at least m crossings in $\mathcal{E}_1$. If we need to repeat this argument $n-(2m+2)$ times (that is, if we have not proved after $n-(2m+2)$ steps that $\mathcal{E}$ has at least $(m-2)n$ crossings), then at each of the $n-(2m+2)$ steps we have found a red cycle with m or more crossings. Thus, after the $(n-(2m+2))$–th step we have proved that $\mathcal{E}$ has at least $m(n-(2m+2)) \geq (m-2)n$ crossings.

The complete, formal proof that the Main Theorem follows from Theorem 2 and Lemma 3 is in Section 7.

Sections 2, 3, 4, 5, and 6 are devoted to the proof of Theorem 2, whose final step is in Section 7. In Section 7 we also prove that Lemma 3 follows easily from the definition of robust drawing.

Section 8 is devoted to a discussion on Adamsson and Richter's pioneer work on arrangements. Section 9 contains some concluding remarks.

2 Overview of the proof of Theorem 2

The strategy of the proof of Theorem 2 is to show that in every robust drawing of an (m, n)–graph, we can associate to each red cycle $R(j)$ a set $\mathcal{I}_j$ of at least $m-2$ crossings, in such a way that no crossing is associated to more than one red cycle.

The overall strategy, and the specification of the crossings that get assigned to each red cycle are quite similar to those of the proof of Theorem 3.13 in [1]. In that statement, Adamsson analyzed crossings in linear (m, n)–arrangements. These structures consist of an ordered collection of $n+2$ closed arcs plus m open arcs that intersect the closed arcs in the given order. Adamsson used his knowledge on drawings of linear (m, n)–arrangements to prove (among other things) that large classes of drawings of (m, n)–graphs have at least $(m-2)n$ crossings.

The main steps in the proof of Theorem 2 are:

(a) Define the set $\mathcal{I}_j$ of crossings associated to each red cycle $R(j)$ in a robust drawing of an (m, n)–graph;

(b) show that no crossing is associated to more than one red cycle, that is, $j \neq k$ implies $\mathcal{I}_j \cap \mathcal{I}_k = \emptyset$; and

(c) show that at least $m-2$ crossings are associated to each $R(j)$, that is, $|\mathcal{I}_j| \geq m-2$.

These three steps are carried out in Sections 4, 5, and 6, respectively.

A quick look at these sections will reveal that each of these steps involves several nontrivial technical details. Nevertheless, the general ideas involved in the construction and analysis of $\mathcal{I}_j$ are not difficult to explain. We devote this section to explore these ideas in some detail.

Before moving on any further, we wish to settle two potential sources of confusion that may arise when analyzing drawings of graphs. The first one has to do with using the same name for graph–theoretical objects (vertices, edges, paths) and for the subsets in the plane that represent them. The second one arises since we are using two different types of intersections: we are interested in crossings (that is, intersections of representations of edges) in drawings of (m, n)–graphs, and, on the other hand, we often need to analyze common structures (that is, graph–theoretical intersections) of two subgraphs of an (m, n)–graph. In view of the following remarks, none of these issues should be a source of confusion throughout this work.

Remark *Throughout this paper, G is a fixed (m, n)–graph, and $\mathcal{D}$ is a fixed good drawing of G.*

Remark *We often make no distinction between a graph–theoretical object (such as a vertex, or a path, or a cycle) and the subset of* $\mathbb{R}^2$ *that represents it. Throughout this work, we have taken special care to ensure that no confusion arises from this practice.*

Remark *Suppose that* H, K *are subgraphs of* G*, such that no edge is in both* H *and* K*. We denote by* $H \sqcap K$ *the set of pairwise intersections (necessarily crossings, since* $\mathcal{D}$ *is good) of edges in* $\mathcal{D}$ *that involve one edge in* H *and one edge in* K*.*

We now go back to the discussion on the construction and analysis of $\mathcal{I}_j$. First we note that $\mathcal{I}_j$ is constructed under the assumption that $\mathcal{D}$ is robust.

2.1 Robustness

Robustness encompasses two properties of drawings of (m, n)–graphs. The first one (being red–nonseparating) guarantees that no red cycle separates, in a sense made precise below, the other red cycles. The second property guarantees that each red cycle is disjoint from sufficiently many other red cycles.

We now give the formal definitions of these properties.

A *closed arc* is a continuous image of S^1. A collection $\mathcal{C} = \{C(0), \ldots, C(r)\}$ of closed arcs is *strongly nonseparating* if for each $C(j)$ there is a unique component $\mathbf{N}_j$ (the *nice* component) of $\mathbb{R}^2 \setminus C(j)$ that intersects each arc in $\mathcal{C}$ different from $C(j)$. A drawing of an (m, n)–graph is *red–nonseparating* if $\{R(0), \ldots, R(n-1)\}$ is strongly nonseparating.

For the rest of this subsection, we assume that $\mathcal{D}$ is red–nonseparating.

We now specify the additional property that the drawing $\mathcal{D}$ must satisfy in order to be robust.

Let $j \in Z_n = \{0, \ldots, n-1\}$ be fixed. If $R(j) \sqcap R(k) = \emptyset$ for some red cycle $R(k)$, then let

$$\bar{b}(j) = \min\{\beta \in Z_n \setminus \{0\} \mid R(j \ominus \beta) \sqcap R(j) = \emptyset\}.$$

Thus, $R(j \ominus \bar{b}(j))$ is the closest predecessor (in the cyclic order of Z_n) of $R(j)$ that does not cross $R(j)$.

Suppose that $\bar{b}(j)$ is defined for every $j \in Z_n$. For each v in the vertex set $V(R(j))$ of $R(j)$, let

$$b(v) = \min\{\beta \in \{1, \ldots, \bar{b}(j)\} \mid v \in \mathbf{N}_{j \ominus \beta}\}.$$

Thus, $R(j \ominus b(v))$ is the closest predecessor (in the cyclic order of Z_n) of $R(j)$ whose nice component contains v. Note that $b(v)$ is well–defined, since v is in $\mathbf{N}_{j \ominus \bar{b}(j)}$.

Let $v \in V(R(j))$, and suppose that there is an $\alpha \in Z_n \setminus \{0\}$ such that neither $R(j \ominus b(v))$ nor $R(j)$ crosses $R(j \oplus \alpha)$. Then define

$$a(v) = \min\{\alpha \in Z_n \setminus \{0\} \bigm| R(j \ominus b(v)) \sqcap R(j \oplus \alpha) = R(j) \sqcap R(j \oplus \alpha) = \emptyset\}.$$

Let

$$\mathcal{B}_j = \{b(v) \mid v \in V(R(j))\}.$$

Thus, $\mathcal{B}_j$ is a nonempty subset of $\{1, \ldots, \bar{b}(j)\}$.

The (red–nonseparating, by assumption) drawing $\mathcal{D}$ is *robust* if the following hold:

(i) $\bar{b}(j)$ is defined for each $j \in Z_n$ (thus, $b(v)$ is defined for every vertex v in G); and

(ii) for every vertex v in G, $a(v)$ is defined, and $b(v) + a(v) < n/2$.

Since our goal is to prove Theorem 2, from now on we assume that $\mathcal{D}$ is robust.

Remark *For the rest of this paper,* G *is a fixed* (m, n)*–graph, and* $\mathcal{D}$ *is a fixed robust good drawing of* G*.*

2.2 The set $\mathcal{I}_j$ of crossings associated to the red cycle $R(j)$

First we describe an important class of blue subgraphs associated to each vertex in G.

Let $v \in V(R(j))$, and let $i \in Z_n \setminus \{0\}$. Since G is an (m,n)–graph, it follows that there are unique blue paths

$$B_v(i,0) = \big(u(j \ominus i), u(j \ominus (i-1)), \ldots, u(j \ominus 1), u(j)\big), \text{ and}$$

$$B_v(0,i) = \big(u(j), u(j \oplus 1), \ldots, u(j \oplus (i-1)), u(j \oplus i)\big)$$

such that $u(\ell) \in V(R(\ell))$ for each $\ell \in \{j \ominus i, \ldots, j \ominus 1, j, j \oplus 1, \ldots, j \oplus i\}$, and $u(j) = v$. We follow the convention that $B_v(0,0) = v$.

If $i, k \in Z_n$ are such that $i + k \le n$, then define the (blue) subgraph $B_v(i,k)$ as the union of $B_v(i,0)$ and $B_v(0,k)$:

$$B_v(i,k) = B_v(i,0) \cup B_v(0,k).$$

Remark *If $0 < i + k < n$, then $B_v(i,k)$ is a path. If $i + k = n$, then $B_v(i,k)$ is either a path or a cycle. For instance, if $G = C_m \times C_n$, and $i + k = n$, then, for every vertex v in $C_m \times C_n$, the blue subgraph $B_v(i,k)$ is a cycle.*

Let $j \in Z_n$ be fixed. The initial step in the construction of $\mathcal{I}_j$ is to partition $V(R(j))$ into sets $\mathcal{C}_j, \mathcal{T}_j$, according to the following rule: a vertex $v \in V(R(j))$ is in $\mathcal{C}_j$ iff $B_v(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$ crosses $R(j)$. Obviously, $\mathcal{T}_j = V(R(j)) \setminus \mathcal{C}_j$. It is easy to check that, since the drawing $\mathcal{D}$ is robust, it follows that if $v \in \mathcal{T}_j$, then the rotation scheme around v is red–red–blue–blue. This motivates the notation chosen for $\mathcal{T}_j$ and $\mathcal{C}_j$: if $v \in \mathcal{C}_j$, then $B_v(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$ crosses $R(j)$, whereas if $v \in \mathcal{T}_j$, then the blue subgraph $B_v(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$ intersects $R(j)$ tangentially (namely at v). Once $\mathcal{C}_j$ and $\mathcal{T}_j$ have been identified, we are ready to start constructing $\mathcal{I}_j$.

For each $v \in \mathcal{C}_j$, we assign to $\mathcal{I}_j$ one crossing (we specify which one in Section 4) between $B_v(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$ and $R(j)$. Let $\mathcal{Y}_j$ denote the collection of crossings in $\mathcal{I}_j$ obtained in this way. Thus, $|\mathcal{Y}_j| = |\mathcal{C}_j|$. In order to complete $\mathcal{I}_j$, we need to specify an additional collection $\mathcal{X}_j$ of $m - 2 - |\mathcal{C}_j| = |\mathcal{T}_j| - 2$ crossings that get assigned to $\mathcal{I}_j$.

In the construction of $\mathcal{X}_j$, a predominant role is played by a particular type of blue path. For each $v \in \mathcal{T}_j$, the *main blue path M_v of v* is $B_v(b(v), a(v))$ (note that since $\mathcal{D}$ is robust, $B_v(b(v), a(v))$ is indeed a path for every v). We also let $\mathcal{M}_j = \{M_v \mid v \in \mathcal{T}_j\}$. The first step in the construction of $\mathcal{X}_j$ is to partition $\mathcal{T}_j$ into subcollections $\mathcal{T}_j(\beta)$ as follows:

$$\mathcal{T}_j(\beta) = \{v \in \mathcal{T}_j \mid b(v) = \beta\}.$$

This partition of $\mathcal{T}_j$ naturally induces a partition of $\mathcal{M}_j$. Indeed, for each $\beta \in \mathcal{B}_j$, let $\mathcal{M}_j(\beta) = \{M_v \mid v \in \mathcal{T}_j(\beta)\}$. Then $\mathcal{M}_j$ is the disjoint union of the collections $\mathcal{M}_j(\beta)$.

Remark *If $u, v \in \mathcal{T}_j(\beta)$, then $a(u) = a(v)$ (this follows immediately from the definition of $a(u)$ and $a(v)$). Thus, if M_u, M_v are in $\mathcal{M}_j(\beta)$, then each of M_u and M_v has an end point in $R(j \ominus \beta)$ and the other end point in $R(j \oplus a(u)) = R(j \oplus a(v))$.*

For each $\beta \in \mathcal{B}_j$, we identify a collection $\mathcal{X}_j(\beta)$ of crossings associated to the paths in $\mathcal{M}_j(\beta)$. These crossings (whose precise nature is described in Section 4) are of three different types: (i) crossings between paths in $\mathcal{M}_j(\beta)$; (ii) crossings involving a path M_v in $\mathcal{M}_j(\beta)$ and an edge in either $R(j \ominus \beta)$ or $R(j \oplus a(v))$; and (iii) crossings between $R(j \ominus \beta)$ and $R(j)$.

Finally, we let $\mathcal{X}_j = \cup_{\beta \in \mathcal{B}_j} \mathcal{X}_j(\beta)$.

As we mentioned above, $\mathcal{I}_j$ is defined as the union of $\mathcal{Y}_j$ and $\mathcal{X}_j$. Thus,

$$\mathcal{I}_j = \mathcal{Y}_j \cup \Big(\bigcup_{\beta \in \mathcal{B}_j} \mathcal{X}_j(\beta) \Big). \tag{1}$$

2.3 No crossing is associated to more than one red cycle

As we pointed out above, a crucial property of the collections $\mathcal{I}_j, j \in Z_n$, is that no crossing is in $\mathcal{I}_j$ and $\mathcal{I}_k$ if $j \neq k$.

Lemma 4 *If $j \neq k$, then $\mathcal{I}_j \cap \mathcal{I}_k = \emptyset$. That is, no crossing in $\mathcal{D}$ is associated to more than one red cycle.*

This essential fact is established in Section 5, as a corollary of the following more general result.

Proposition 5 *Let $j, k \in Z_n$, $\beta \in \mathcal{B}_j$, and $\beta' \in \mathcal{B}_k$. Then:*

(a) If $j \neq k$, then $\mathcal{Y}_j \cap \mathcal{Y}_k = \emptyset$.

(b) $\mathcal{Y}_j \cap \mathcal{X}_k(\beta') = \emptyset$.

(c) If $j \neq k$ or $\beta \neq \beta'$, then $\mathcal{X}_j(\beta) \cap \mathcal{X}_k(\beta') = \emptyset$.

We emphasize that only very elementary combinatorial and topological arguments are required in the proof of Proposition 5 (this proof is in Section 5). Indeed, once $\mathcal{Y}_j$ and $\mathcal{X}_j(\beta)$ have been carefully defined in Section 4, the proof of Proposition 5 reduces to a (somewhat painstaking) exercise of careful bookkeeping.

Proposition 5 also implies the following crucial result.

Proposition 6 *Each of the unions on the right hand side of Eq. (1) is a disjoint union.*

This statement is an immediate corollary of Proposition 5. Moreover, this explains why Proposition 5 is stronger than what is strictly required for the proof of Lemma 4. Indeed, the following two assertions contained in Proposition 5 are not needed to establish Lemma 4: (i) if $j = k$, then $\mathcal{Y}_j \cap \mathcal{X}_k(\beta') = \emptyset$; and (ii) if $j = k$ and $\beta \neq \beta'$, then $\mathcal{X}_j(\beta) \cap \mathcal{X}_k(\beta') = \emptyset$. On the other hand, these statements, together with (a) in Proposition 5, immediately imply Proposition 6.

2.4 At least $m - 2$ crossings are associated to each red cycle

The final step in the proof of Theorem 2 is to show that $|\mathcal{I}_j| \geq m - 2$, for each $j \in Z_n$.

Lemma 7 *For each $j \in Z_n$, $|\mathcal{I}_j| \geq m - 2$. In other words, there are at least $m - 2$ crossings associated to each red cycle.*

As we pointed out in Subsection 2.2, in order to show that $|\mathcal{I}_j| \geq m - 2$ it suffices to show that $|\mathcal{X}_j| \geq |\mathcal{T}_j| - 2 = (\sum_{\beta \in \mathcal{B}_j} |\mathcal{T}_j(\beta)|) - 2$.

Since by Proposition 6 $\mathcal{X}_j$ is the disjoint union $\cup_{\beta \in \mathcal{B}_j} \mathcal{X}_j(\beta)$, it follows that in order to show that $|\mathcal{I}_j| \geq m - 2$ it suffices to prove the following.

Proposition 8 *For each $j \in Z_n$, the following statements hold.*

(a) For each $\beta \in \mathcal{B}_j, \beta \neq \overline{b}(j)$, $|\mathcal{X}_j(\beta)| \geq |\mathcal{T}_j(\beta)|$.

(b) If $\overline{b}(j) \in \mathcal{B}_j$, then $|\mathcal{X}_j(\overline{b}(j))| \geq |\mathcal{T}_j(\overline{b}(j))| - 2$.

This statement is established in Section 6. However, as we now explain, its proof heavily relies on some technical results from Section 3.

As we mentioned above, the crossings in $\mathcal{X}_j(\beta)$ are obtained by analyzing the collection of main paths M_v in $\mathcal{M}_j(\beta)$, and the red cycles $R(j \ominus \beta), R(j)$, and $R(j \oplus a(v))$ (recall that $a(v)$ is the same for each $v \in \mathcal{T}_j(\beta)$). We also recall that for each M_v in $\mathcal{M}_j(\beta)$, the rotation scheme around v is red–red–blue–blue.

In view of this, it is not surprising that in order to find lower bounds for the sizes of the collections $\mathcal{X}_j(\beta)$, we need to analyze crossings in $(s, 1)$–arrangements. Roughly speaking, $(s, 1)$–arrangements are drawings of structures that consist of three closed curves C_0, C_1, C_2 (such that neither C_0 nor C_1 intersects C_2) plus s open arcs $A_0, \ldots, A_{s-1}$ that intersect the curves in the given order.

We devote the next section to establishing the facts about $(s, 1)$–arrangements that are required in the proof of Proposition 8.

3 Analysis of crossings in $(s, 1)$–arrangements

The theory of arrangements was introduced and extensively investigated by Adamsson in his Ph.D. thesis [1], and further developed by Adamsson and Richter [2]. Adamsson and Richter showed the enormous relevance that these structures have in the analysis of drawings of (m, n)–graphs. In [1] and [2], the results from (linear and circular) arrangements are used to prove that $\mathrm{cr}(C_7 \times C_n) = 5n$ (as conjectured), and to prove that large classes of drawings of (m, n)–graphs have at least $(m - 2)n$ crossings, as conjectured.

Arrangements consist of two collections of arcs with certain special properties. Before moving on to the definition of an arrangement, we need to give a precise definition of open and closed arcs.

An *open arc* γ is the image of a continuous map $f : (0, 1) \to \mathbb{R}^2$ such that: (i) no point in $\mathbb{R}^2$ is the image under f of more than two points in $(0, 1)$, and the set of points in $\mathbb{R}^2$ that are the image of more than one point in $(0, 1)$ is finite; (ii) the unique continuous extension $\overline{f} : [0, 1] \to \mathbb{R}^2$ of f to $[0, 1]$ is such that $\overline{f}(0), \overline{f}(1) \notin f(0, 1)$, and $f(0) \neq f(1)$. The points $\overline{f}(0)$ and $\overline{f}(1)$ are the *end points* of γ (note that the end points of γ are not in γ).

A *closed arc* γ is the image of a continuous map $f : S^1 \to \mathbb{R}^2$ such that no point in $\mathbb{R}^2$ is the image under f of more than two points in S^1, and the set of points in $\mathbb{R}^2$ that are the image of more than one point in S_1 is finite.

Following Adamsson and Richter, an $(s, 1)$–*linear arrangement* (or simply an $(s, 1)$–*arrangement*) is a pair $(\mathcal{C}, \mathcal{A})$, where $\mathcal{C} = \{C_0, C_1, C_2\}$ is a collection of closed arcs and $\mathcal{A} = \{A_1, \ldots, A_{s-1}\}$ is a collection of open arcs with the following properties:

(i) $\mathcal{C}$ is strongly nonseparating;

(ii) $(C_0 \cup C_1) \cap C_2 = \emptyset$;

(iii) each A_i has one end point (the *initial* point $t(i)$ of A_i) in C_0, the other end point (the *final* point $f(i)$ of A_i) in C_2, and a *middle* point $w(i) \in A_i$ in C_1;

(iv) each $t(i)$ is in the same component of $\mathbb{R}^2 \setminus \{C_1\}$ as C_2, and each $w(i)$ is in the same component of $\mathbb{R}^2 \setminus \{C_0\}$ as C_2.

We are interested exclusively in certain $(s, 1)$–arrangements induced from good drawings of (m, n)–graphs. An $(s, 1)$–arrangement is *neat* if it satisfies the following additional conditions:

(v) for each i, the only intersection point between A_i and C_1 is $w(i)$ (it is straightforward to check that (iv) implies that then $w(i)$ is a tangential intersection);

(vi) no point in C_1 is the middle point of two different arcs in $\mathcal{A}$.

A linear arrangement $(\mathcal{C}, \mathcal{A})$ is *k–intersecting* if $|C_0 \cap C_1| = k$.

As we mentioned in the previous section, we are interested in $(s,1)$–arrangements because these structures are naturally induced in $\mathcal{D}$ by each collection $\mathcal{M}_j(\beta)$ of main paths. We now formalize this idea.

Proposition 9 *Let $\mathcal{M}_j(\beta)$ be a collection of main paths, and let $|\mathcal{M}_j(\beta)| = s$. Let $M_v \in \mathcal{M}_j(\beta)$, and define $\alpha = a(v)$ (recall that $a(v)$ is the same for each vertex $v \in \mathcal{T}_j(\beta)$). Construct the collections $\mathcal{C} = \{C_0, C_1, C_2\}, \mathcal{A} = \{A_0, A_1, \ldots, A_{s-1}\}$ of closed and open arcs, respectively, in the following way:*

(i) C_0, C_1, and C_2 are the drawings of $R(j \ominus \beta), R(j)$, and $R(j \oplus \alpha)$ induced by $\mathcal{D}$, respectively;

(ii) the arcs in $\mathcal{A}$ are the drawings of the main paths (without their end points) $M_v \in \mathcal{M}_j(\beta)$ induced by $\mathcal{D}$.

Then $(\mathcal{C}, \mathcal{A})$ is a neat $(s,1)$–arrangement.

We call the neat arrangement in the previous statement the *$(s,1)$–arrangement induced by $\mathcal{M}_j(\beta)$*.

We are interested in certain types of crossings that appear in neat $(s,1)$–arrangements. More specifically, we are interested in (i) crossings that involve one open arc in $\mathcal{A}$ and one closed arc in $\{C_0, C_2\}$ (recall that in a neat $(s,1)$–arrangement no open arc crosses C_1); and (ii) crossings between two different open arcs in $\mathcal{A}$. Moreover, the crossings of the latter type in which we are interested involve very particular subarcs of the open arcs in $\mathcal{A}$.

For each A_i in $\mathcal{A}$, let T_i denote the (unique) open subarc of A_i whose end points are $t(i)$ and $w(i)$, and let F_i denote the (unique) open subarc of A_i whose end points are $w(i)$ and $f(i)$. The subarcs T_i and F_i are the *initial* and *final* subarcs of A_i, respectively. Thus, $A_i = T_i \cup F_i \cup \{w(i)\}$.

The main results on crossings in $(s,1)$–arrangements that we will need later are the following.

Lemma 10 *Let $(\mathcal{C}, \mathcal{A}) = (\{C_0, C_1, C_2\}, \{A_0, \ldots, A_{s-1}\})$ be a 0–intersecting neat $(s,1)$–arrangement. Let x_1 denote the number of crossings of $(\mathcal{C}, \mathcal{A})$ that involve one initial subarc and one final subarc. Let x_2 denote the number of initial arcs that cross C_2. Let x_3 denote the number of final arcs that cross C_0. Then $x_1 + x_2 + x_3 \geq s - 2$.*

Lemma 11 *Let $(\mathcal{C}, \mathcal{A}) = (\{C_0, C_1, C_2\}, \{A_0, \ldots, A_{s-1}\})$ be a k–intersecting $(s,1)$–arrangement, where $k > 0$. Let x_1 denote the number of crossings of $(\mathcal{C}, \mathcal{A})$ that involve one initial subarc and one final subarc. Let x_2 denote the number of initial arcs that cross C_2. Let x_3 denote the number of final arcs that cross C_0. Then $x_1 + x_2 + x_3 + k \geq s$.*

These results follow from (the proofs of) Corollary 3.6 and Theorem 3.11, respectively, in [1].

4 The set $\mathcal{I}_j$ of crossings associated to the red cycle $R(j)$

The aim in this section is to define the set $\mathcal{I}_j$ of crossings associated to each red cycle $R(j)$.

As we explained in Section 2, some of the crossings in $\mathcal{I}_j$ involve one edge in $R(j)$ and one blue edge in a blue subgraph $B_v(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$ (recall that this blue subgraph is either a path or a cycle). Since in some cases $R(j)$ and $B_v(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$ cross more than once, we need to specify which such crossing belongs to $\mathcal{I}_j$. This motivates the following definitions.

Let $s, t \in Z_n$ be such that $0 \le s \le \lfloor n/2 \rfloor, 0 \le t \le \lfloor n/2 \rfloor$, and consider the blue subgraph $B_v(s,t)$. Let $R(j)$ be the red cycle that contains v. Thus, $B_v(s,t)$ has one end vertex in $R(j \ominus s)$ and one end vertex in $R(j \oplus t)$, and so we can traverse $B_v(s,t)$ following its *positive* direction (starting at $R(j \ominus s)$) or following its *negative* direction (starting at $R(j \oplus t)$). An $(B_v(s,t), R(k))$–*crossing* is a crossing between an edge in $B_v(s,t)$ and an edge in $R(k)$ (that is, a crossing in $B_v(s,t) \sqcap R(k)$).

Suppose that $B_v(s,t)$ and $R(\ell)$ have the common vertex v_ℓ. If $B_v(s,t)$ crosses $R(k)$, then the *first* $(B_v(s,t), R(k))$*–crossing from* $R(\ell)$ (respectively *last*) is the first $(B_v(s,t), R(k))$–crossing we find as we traverse $B_v(s,t)$, starting at v_ℓ, following the positive (respectively negative) direction of $B_v(s,t)$.

We are now ready to specify the sets $\mathcal{Y}_j$ and $\mathcal{X}_j(\beta)$ of crossings that are associated to $R(j)$.

Recall that $v \in \mathcal{C}_j$ iff $B_v(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$ crosses $R(j)$. As we pointed out in Section 2, for each $v \in \mathcal{C}_j$ one such crossing is assigned to $\mathcal{Y}_j$. To specify which crossing is assigned to $\mathcal{Y}_j$, we introduce sets $\mathcal{C}_j^+, \mathcal{C}_j^-$, whose union equals $\mathcal{C}_j$.

Define $\mathcal{C}_j^+$ and $\mathcal{C}_j^-$ as follows. Let $v \in \mathcal{C}_j$. Then:

(a) $v \in \mathcal{C}_j^+$ iff $B_v(0, \lfloor n/2 \rfloor)$ crosses $R(j)$;

(b) $v \in \mathcal{C}_j^-$ iff $B_v(\lfloor n/2 \rfloor, 0)$ crosses $R(j)$.

Clearly, $\mathcal{C}_j$ is the union of $\mathcal{C}_j^+$ and $\mathcal{C}_j^-$.

One word regarding the introduction of the superscripts $+$ and $-$ in these definitions. Suppose that $v \in \mathcal{C}_j$. Then v is in $\mathcal{C}_j^+$ iff the blue path of length $\lfloor n/2 \rfloor$ starting at v and following the *positive* direction crosses $R(j)$. Similarly, v is in $\mathcal{C}_j^-$ iff the blue path of length $\lfloor n/2 \rfloor$ starting at v and following the *negative* direction crosses $R(j)$.

Now define the collections $\mathcal{Y}_j^+$ and $\mathcal{Y}_j^-$ of crossings specified as follows:

(i) If $v \in \mathcal{C}_j^+$, then let the first $(B_v(0, \lfloor n/2 \rfloor), R(j))$–crossing from $R(j)$ belong to $\mathcal{Y}_j^+$;

(ii) if $v \in \mathcal{C}_j^-$, then let the last $(B_v(\lfloor n/2 \rfloor, 0), R(j))$–crossing from $R(j)$ belong to $\mathcal{Y}_j^-$.

Finally, define $\mathcal{Y}_j = \mathcal{Y}_j^+ \cup \mathcal{Y}_j^-$. Clearly, $|\mathcal{Y}_j| \ge |\mathcal{C}_j|$.

Remark *The union that defines $\mathcal{Y}_j$ is a disjoint union. Indeed, the only blue edge involved in each crossing in $\mathcal{Y}_j^+$ is in $B_v(0, \lfloor n/2 \rfloor)$, and the only blue edge involved in each crossing in $\mathcal{Y}_j^-$ is in $B_v(\lfloor n/2 \rfloor, 0)$, and no blue edge is in both blue subgraphs.*

We now move on to the definition of $\mathcal{X}_j(\beta)$.

For each $j \in Z_n$, and each $\beta \in \mathcal{B}_j$, let $\mathcal{X}_j(\beta)$ denote the set of crossings of the following types:

(i) all the crossings between $R(j \ominus \beta)$ and $R(j)$;

(ii) if $v \in \mathcal{T}_j(\beta)$ and $R(j \ominus \beta) \sqcap B_v(0, a(v)) \ne \emptyset$, the last $(B_v(0, a(v)), R(j \ominus \beta))$–crossing from $R(j \oplus a(v))$;

(iii) if $v \in \mathcal{T}_j(\beta)$ and $R(j \oplus a(v)) \sqcap B_v(\beta, 0) \ne \emptyset$, the first $(B_v(\beta, 0), R(j \oplus a(v)))$–crossing from $R(j \ominus \beta)$;

(iv) if $v, w \in \mathcal{T}_j(\beta)$, $v \ne w$, every crossing between $B_v(\beta, 0)$ and $B_w(0, a(w))$.

We are now ready to complete the definition of the set $\mathcal{I}_j$ of crossings associated to each red cycle $R(j)$:

$$\mathcal{I}_j = \mathcal{Y}_j \cup \Big(\bigcup_{\beta \in \mathcal{B}_j} \mathcal{X}_j(\beta) \Big). \tag{1}$$

In the next section we show that if $j \neq k$, then $\mathcal{I}_j \cap \mathcal{I}_k = \emptyset$.

5 No crossing is associated to more than one red cycle

Our main result in this section is the following.

Lemma 4 *If $j \neq k$, then $\mathcal{I}_j \cap \mathcal{I}_k = \emptyset$. That is, no crossing in $\mathcal{D}$ is associated to more than one red cycle.* ∎

This statement is an immediate consequence of our next result.

Proposition 5 *Let $j, k \in Z_n$, $\beta \in \mathcal{B}_j$, and $\beta' \in \mathcal{B}_k$. Then:*

(a) If $j \neq k$, then $\mathcal{Y}_j \cap \mathcal{Y}_k = \emptyset$.

(b) $\mathcal{Y}_j \cap \mathcal{X}_k(\beta') = \emptyset$.

(c) If $j \neq k$ or $\beta \neq \beta'$, then $\mathcal{X}_j(\beta) \cap \mathcal{X}_k(\beta') = \emptyset$.

Proof of (a). Suppose $j \neq k$. Each crossing in $\mathcal{Y}_j$ (respectively $\mathcal{Y}_k$) is a bichromatic crossing whose red edge involved is in $R(j)$ (respectively $R(k)$). Since $j \neq k$, (a) follows.

Proof of (b). Seeking a contradiction, suppose that for some $j, k \in Z_n$, $\beta' \in \mathcal{B}_k$, some (necessarily bichromatic, by the definition of $\mathcal{Y}_j$) crossing x belongs to both $\mathcal{Y}_j$ and $\mathcal{X}_k(\beta')$. Since x is in $\mathcal{Y}_j$, it follows that there is a vertex u in $R(j)$ such that $B_u(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$ crosses $R(j)$ at x. Thus, in particular, the red edge involved in x is in $R(j)$. Now, since x is in $\mathcal{X}_k(\beta')$, then there is a vertex v in $\mathcal{T}_k(\beta')$ such that x is a $(B_v(\beta', a(v)), R(j))$–crossing (note that $b(v) = \beta'$).

By the definition of $\mathcal{X}_k(\beta')$, a bichromatic crossing in $\mathcal{X}_k(\beta')$ involves an edge in $R(j)$ only if j is either $k \ominus \beta'$ or $k \oplus a(v)$. Thus, either $j = k \ominus \beta'$ or $j = k \oplus a(v)$. We analyze these cases separately.

Let $\bar{l}(x)$ denote the blue edge involved in x.

Case 1. $j = k \ominus \beta'$. By the definition of $\mathcal{X}_k(\beta')$, $\bar{l}(x)$ is in $B_v(0, a(v))$. Moreover, x is the last $(B_v(0, a(v)), R(j))$–crossing from $R(k \oplus a(v))$. Since x is in $\mathcal{Y}_j$, x is in $\mathcal{Y}_j^+$ or in $\mathcal{Y}_j^-$.

Suppose that $x \in \mathcal{Y}_j^-$. Then, x occurs between $R(j)$ and $B_u(\lfloor n/2 \rfloor, 0)$. Thus $\bar{l}(x)$ is in both $B_u(\lfloor n/2 \rfloor, 0)$ and $B_v(0, a(v))$. It is straightforward to check that these blue subgraphs (which are paths, since $\mathcal{D}$ is robust) have edges in common only if $\lfloor n/2 \rfloor + \beta' + a(v) > n$ (recall that $u \in V(R(j))$ and $v \in V(R(k))$). However, this inequality does not hold, since $\mathcal{D}$ is robust.

Suppose now that $x \in \mathcal{Y}_j^+$. Then, x is the first $(B_u(0, \lfloor n/2 \rfloor), R(j))$–crossing from $R(j)$. As we pointed out above, x is also the last $(B_v(0, a(v)), R(j))$–crossing from $R(k \oplus a(v))$. In order for these crossings to be the same, $B_v(0, a(v))$ needs to cross $R(j)$ exactly once. On the other hand, $v \in \mathbf{N}_j$ (by the definition of $\mathcal{T}_k(\beta')$) and $R(k \oplus a(v)) \subseteq \mathbf{N}_j$ (since, by the definition of $a(v)$, $R(k \oplus a(v)) \sqcap R(k \ominus \beta') = \emptyset$, and $j = k \ominus \beta'$). Therefore $|B_v(0, \beta' + a(v)) \sqcap R(j)| \geq 2$, a contradiction.

Case 2. $j = k \oplus a(v)$. By the definition of $\mathcal{X}_k(\beta')$, $\bar{l}(x)$ is in $B_v(\beta', 0)$. Moreover, x is the first $(B_v(\beta', 0), R(j))$–crossing from $R(k \ominus \beta')$. Since x is in $\mathcal{Y}_j$, it is in $\mathcal{Y}_j^+$ or $\mathcal{Y}_j^-$.

Suppose that $x \in \mathcal{Y}_j^+$. Then, x occurs between $R(j)$ and $B_u(0, \lfloor n/2 \rfloor)$. Thus $\bar{l}(x)$ is in both $B_u(0, \lfloor n/2 \rfloor)$ and $B_v(\beta', 0)$. It is straightforward to check that these blue paths have edges in common only if $\lfloor n/2 \rfloor + \beta' + a(v) > n$. However, this inequality does not hold, since $\mathcal{D}$ is robust.

Suppose now that $x \in \mathcal{Y}_j^-$. Then, x is the last $(B_u(\lfloor n/2 \rfloor, 0), R(j))$–crossing from $R(j)$. As we pointed out above, x is also the first $(B_v(\beta', 0), R(j))$–crossing from $R(k \ominus \beta')$. In order for these crossings to be the same, $B_v(\beta', 0)$ needs to cross $R(j)$ exactly once. On the other hand, since both $R(k \ominus \beta')$ and $R(k)$ are contained in $\mathbf{N}_j$ (this is true since none of them crosses $R(j)$), it follows that $|(B_v(\beta', 0) \sqcap R(j)| \geq 2$, a contradiction.

Proof of (c). We derive a contradiction from the assumption that the following hold: (i) either $j \neq k$ or $\beta \neq \beta'$; and (ii) there is a crossing x in both $\mathcal{X}_j(\beta)$ and $\mathcal{X}_k(\beta')$.

It follows from the very definitions of $\mathcal{X}_j(\beta)$ and $\mathcal{X}_k(\beta')$ that if $j = k$ and $\beta \neq \beta'$, then no crossing can belong to both $\mathcal{X}_j(\beta)$ and $\mathcal{X}_k(\beta')$. Thus we assume, without any loss of generality, that (i) $k \ominus j \leq n/2$; and (ii) $k \neq j$.

Suppose that both edges involved in x are red. Then, $x \in R(j \ominus \beta) \sqcap R(j)$ and $x \in R(k \ominus \beta') \sqcap R(k)$. This clearly cannot happen, since there are at least three different cycles in $\{R(j \ominus \beta), R(j), R(k \ominus \beta'), R(k)\}$ (this follows since $\mathcal{D}$ is robust). Therefore x involves either one blue edge and one red edge or two blue edges. We analyze these cases separately.

Before moving on to this case analysis, we need to make the following crucial observation.

Claim *Suppose that $w \in \mathcal{T}_k(\beta')$, and let z denote the vertex in $R(j)$ that is also in $B_w(k \ominus j, 0)$. Suppose further that $k \ominus j \leq n/2$. If $z \in \mathcal{T}_j$, then $\beta' \leq k \ominus j$.*

Assuming this statement for the moment, we complete the proof of (c).

Case 1. *x involves one blue edge $\bar{l}(x)$ and one red edge $\bar{r}(x)$.* Since x is in $\mathcal{X}_j(\beta)$, there is a vertex $v \in \mathcal{T}_j(\beta)$ such that either (i) x is the last $(B_v(0, a(v)), R(j \ominus \beta))$–crossing from $R(j \oplus a(v))$, or (ii) x is the first $(B_v(\beta, 0), R(j \oplus a(v)))$–crossing from $R(j \ominus \beta)$. since x is in $\mathcal{X}_k(\beta')$, there is a vertex $u \in \mathcal{T}_k(\beta')$ such that either (a) x is the last $(B_u(0, a(u)), R(k \ominus \beta'))$–crossing from $R(k \oplus a(u))$, or (b) x is the first $(B_u(\beta', 0), R(k \oplus a(u)))$–crossing from $R(k \ominus \beta')$.

Hence, either (I) Statements (i) and (a) hold; (II) Statements (i) and (b) hold; (III) Statements (ii) and (a) hold; (IV) Statements (ii) and (b) hold. We analyze these four possibilities separately.

It is straightforward to check that, since $\bar{l}(x)$ is in both $B_v(\beta, a(v))$ and $B_u(\beta', a(u))$, the Claim above holds with $u = w$ and $v = z$. Thus, $\beta' \leq k \ominus j$.

- **Subcase (I)** *Statements (i) and (a) hold.* By (i), $\bar{r}(x)$ is in $R(j \ominus \beta)$, and by (a), $\bar{r}(x)$ is in $R(k \ominus \beta')$. Thus, $j \ominus \beta = k \ominus \beta'$. But this is impossible, since $(k \ominus j) \oplus \beta = \beta' \leq k \ominus j \leq n/2$, and $\beta < n/2$.

- **Subcase (II)** *Statements (i) and (b) hold.* By (i), $\bar{r}(x)$ is in $R(j \ominus \beta)$, and by (b), $\bar{r}(x)$ is in $R(k \oplus a(u))$. Thus, $j \ominus \beta = k \oplus a(u)$. It follows from the definitions of $\mathcal{T}_j(\beta)$ and $\mathcal{T}_k(\beta')$ that $\bar{l}(x)$ is in both $B_v(0, a(v))$ and $B_u(\beta', 0)$. However, these paths have edges in common only if $\beta + \beta' + a(v) + a(u) > n$ (recall that $u \in V(R(j))$ and $v \in V(R(k))$), contradicting the assumption that $\mathcal{D}$ is robust.

- **Subcase (III)** *Statements (ii) and (a) hold.* By (ii), $\bar{r}(x)$ is in $R(j \oplus a(v))$, and by (a), $\bar{r}(x)$ is in $R(k \ominus \beta')$. Thus, $j \oplus a(v) = k \ominus \beta'$. It follows from the definitions of $\mathcal{T}_j(\beta)$ and $\mathcal{T}_k(\beta')$ that $\bar{l}(x)$ is in both $B_v(\beta, 0)$ and $B_u(0, a(u))$. However, these paths have edges in common only if $\beta + \beta' + a(v) + a(u) > n$, contradicting the assumption that $\mathcal{D}$ is robust.

Subcase (IV) *Statements (ii) and (b) hold.* By (ii), $\overline{r}(x)$ is in $R(j \oplus a(v))$, and by (b), $\overline{r}(x)$ is in $R(k \oplus a(u))$. Thus, $j \oplus a(v) = k \oplus a(u)$. It follows from the definitions of $\mathcal{T}_j(\beta)$ and $\mathcal{T}_k(\beta')$ that $\overline{l}(x)$ is in both $B_v(\beta, 0)$ and $B_u(\beta', 0)$. However, these paths have no edges in common, since $\beta' \leq k \ominus j \leq n/2$ and $\beta < n/2$.

Case 2. *x involves two blue edges.* By the definitions of $\mathcal{X}_j(\beta)$ and $\mathcal{X}_k(\beta')$, there exist vertices $v, v_0 \in \mathcal{T}_j(\beta)$ and $u, u_0 \in \mathcal{T}_k(\beta')$ such that (i) x occurs between an edge e in $B_v(\beta, 0)$ and an edge e_0 in $B_{v_0}(0, a(v_0))$; and (ii) x occurs between an edge f in $B_u(\beta', 0)$ and an edge f_0 in $B_{u_0}(0, a(u_0))$.

As in Case 1, it is straightforward to check that $\beta' \leq k \ominus j$.

Since $\beta' \leq k \ominus j \leq n/2$ and $\beta < n/2$, it follows that if $w \in \mathcal{T}_k(\beta')$, then $e \notin B_w(\beta', 0)$. Thus, $e = f_0$ is in both $B_{u_0}(0, a(u_0))$ and $B_v(\beta, 0)$. A similar argument shows that $f = e_0$ is in both $B_{v_0}(0, a(v_0))$ and $B_u(\beta', 0)$.

Hence, $B_{u_0}(0, a(u_0))$ has an edge in common with $B_v(\beta, 0)$, and $B_{v_0}(0, a(v_0))$ has an edge in common with $B_u(\beta', 0)$. It is straightforward to check that this implies $\beta + a(u_0) + \beta' + a(v_0) > n$, contradicting the assumption that $\mathcal{D}$ is robust. ∎

Proof of Claim $\beta' \leq k \ominus j$**.**

Suppose that $\beta' > k \ominus j$. The following statements imply that $B_z(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$ is contained in $\mathbf{N}_j$:

(i) $B_z(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor) \sqcap R(j) = \emptyset$ (since $z \in \mathcal{T}_j$);

(ii) $R(j \ominus b(j)) \subseteq \mathbf{N}_j$ (this follows from the definitions of $b(j)$ and $\mathbf{N}_j$);

(iii) $B_z(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$ has a vertex in common with $R(j \ominus b(j))$ (since $\mathcal{D}$ is robust).

On the other hand, the assumption $\beta' > k \ominus j$ implies $w \notin \mathbf{N}_j$, by the definition of $\mathcal{T}_k(\beta')$. This is a contradiction, since the inequality $k \ominus j < \beta' \leq \lfloor n/2 \rfloor$ implies that w is in $B_z(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$, and $B_z(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor) \subseteq \mathbf{N}_j$. ∎

We close this section by observing the following consequence of Proposition 5.

Proposition 6 *Each of the unions on the right hand side of Eq. (1) is a disjoint union.* ∎

This statement will play a central role in the proof that $|\mathcal{I}_j| \geq m - 2$.

6 At least $m - 2$ crossings are associated to each red cycle

The purpose of this section is to prove the following.

Lemma 7 *For each $j \in Z_n$, $|\mathcal{I}_j| \geq m - 2$. In other words, there are at least $m - 2$ crossings associated to each red cycle.*

Proof. Suppose that $\bar{b}(j) \in \mathcal{B}_j$. Then, by Proposition 6, $|\mathcal{I}_j| = |\mathcal{Y}_j| + \left(\sum_{\beta \in \mathcal{B}_j, \beta \neq \bar{b}(j)} |\mathcal{X}_j(\beta)|\right) + |\mathcal{X}_j(\bar{b}(j))|$. Applying (a) and (b) in Proposition 8 below (also recall that $|\mathcal{Y}_j| \geq |\mathcal{C}_j|$), $|\mathcal{I}_j| \geq |\mathcal{C}_j| + \left(\sum_{\beta \in \mathcal{B}_j, \beta \neq \bar{b}(j)} |\mathcal{T}_j(\beta)|\right) + |\mathcal{T}_j(\bar{b}(j))| - 2$. Since $V(R(j))$ is the disjoint union of $\mathcal{C}_j$ and the sets $\mathcal{T}_j(\beta)$ (for all $\beta \in \mathcal{B}_j$), and $|V(R(j))| = m$, it follows that $|\mathcal{I}_j| \geq m - 2$, as required.

Now suppose $\bar{b}(j) \notin \mathcal{B}_j$. Then, by Proposition 6, $|\mathcal{I}_j| = |\mathcal{Y}_j| + \left(\sum_{\beta \in \mathcal{B}_j, \beta \neq \bar{b}(j)} |\mathcal{X}_j(\beta)|\right)$. Applying (a) in Proposition 8 ((b) is not required, since $b(j) \notin \mathcal{B}_j$), and the inequality $|\mathcal{Y}_j| \geq |\mathcal{C}_j|$, $|\mathcal{I}_j| \geq |\mathcal{C}_j| + \left(\sum_{\beta \in \mathcal{B}_j, \beta \neq \bar{b}(j)} |\mathcal{T}_j(\beta)|\right)$. Since $V(R(j))$ is the disjoint union of $\mathcal{C}_j$ and the sets $\mathcal{T}_j(\beta)$ (for all $\beta \in \mathcal{B}_j$), and $|V(R(j))| = m$, we obtain $|\mathcal{I}_j| \geq m$. ∎

Proposition 8 *For each $j \in Z_n$, the following statements hold.*

(a) For each $\beta \in \mathcal{B}_j, \beta \neq \bar{b}(j)$, $|\mathcal{X}_j(\beta)| \geq |\mathcal{T}_j(\beta)|$.

(b) If $\bar{b}(j) \in \mathcal{B}_j$, then $|\mathcal{X}_j(\bar{b}(j))| \geq |\mathcal{T}_j(\bar{b}(j))| - 2$.

Proof of (a). This follows from Proposition 9 and Lemma 11 (in Section 3), and the definition of $\mathcal{X}_j(\beta)$, since the $(|\mathcal{T}_j(\beta)|, 1)$–arrangement induced by $\mathcal{M}_j(\beta)$ is a neat $|R(j \ominus \beta) \sqcap R(j)|$–intersecting arrangement. ∎

Proof of (b). This follows from Proposition 9 and Lemma 10 (in Section 3), and the definition of $\mathcal{X}_j(\beta)$, since the $(|\mathcal{T}_j(\beta)|, 1)$–arrangement induced by $\mathcal{M}_j(\beta)$ is a neat 0–intersecting arrangement for $\beta = \bar{b}(j)$. ∎

7 Proofs of Theorem 2, Lemma 3, and the Main Theorem

We are now ready to prove Theorem 2 and the Main Theorem.

Proof of Theorem 2. This follows from the definition of $\mathcal{I}_j$, Lemma 4, and Lemma 7. ∎

Now for the proof of the Main Theorem.

First we show that if n is sufficiently large compared to m, then every drawing of an (m, n)–graph either is robust or has a red cycle with at least m crossings.

Lemma 3 *Let $\mathcal{E}$ be a drawing of an (m, n)–graph, where m, n satisfy $n \geq 2m + 2$, $m \geq 3$. Then either $\mathcal{E}$ is robust or there is a red cycle with at least m crossings in $\mathcal{E}$.*

Proof. We assume that $\mathcal{E}$ is not robust, and show that then there is a red cycle with at least m crossings in $\mathcal{E}$.

If every red cycle different from $R(j)$ crosses $R(j)$ (at least twice, by the Jordan Curve Theorem), then $R(j)$ has at least $2(n-1) > m$ crossings, as required. Thus there is some red cycle $R(k)$ that does not cross $R(j)$.

Now we show that if $\mathcal{E}$ is not red–nonseparating, then there is a red cycle with m or more crossings.

Suppose that $\mathcal{E}$ is not red–nonseparating. Then there is a red cycle $R(j)$ such that either (i) no component of $\mathbb{R}^2 \setminus R(j)$ intersects every red cycle different from $R(j)$; or (ii) two (or more) components of $\mathbb{R}^2 \setminus R(j)$ intersect every red cycle different from $R(j)$. It is easy to check that, since $R(k)$ does not cross $R(j)$, then (ii) cannot hold. Suppose now that (i) holds. Let Ω denote the component of $\mathbb{R}^2 \setminus R(j)$ that contains $R(k)$. By assumption, some red cycle $R(\ell)$ does not intersect Ω. It is easy to check that, since the graph under consideration is an (m, n)–graph, then $R(j)$ is crossed by at least m blue edges. Thus $R(j)$ has at least m crossings, as required.

Thus it suffices to show that if $\mathcal{E}$ is red–nonseparating and not robust, then there is a red cycle with m or more crossings.

Suppose then that $\mathcal{E}$ is red–nonseparating and not robust. Then there is a vertex v in a red cycle $R(j)$ such that either (i) $\bar{b}(j)$ is not defined; or (ii) $\bar{b}(j)$ (and consequently $b(v)$) is defined, but $a(v)$ is not defined; or (iii) $\bar{b}(j)$ (and consequently $b(v)$) and $a(v)$ are defined, but $b(v) + a(v) > n/2$.

If $\bar{b}(j)$ is not defined, then $R(j)$ intersects every red cycle different from $R(j)$, and so $R(j)$ has at least $2(n-1) > m$ crossings, as required. Similarly, if $b(j)$ (and thus $b(v)$) is defined but $a(v)$ is not defined, then every red cycle $R(k)$ not in $\{R(j), R(j \ominus b(v))\}$ crosses either $R(j)$ or $R(j \ominus b(v))$. In this case, either $R(j)$ or $R(j \ominus b(v))$ crosses at least $(n-2)/2$ other red curves, and so it has at least $n - 2 > m$ crossings, as required. Thus we may assume that (iii) holds.

By the definitions of $\bar{b}(j)$ and $b(v)$, $R(j)$ crosses every red cycle in $\{R(j \ominus (b(v)-1)), \ldots, R(j \ominus 1)\}$. By the definition of $a(v)$, each red cycle in $\{R(j \oplus 1), \ldots, R(j \oplus (a(v)-1))\}$ is crossed by either $R(j)$

or $R(j \ominus b(v))$. Thus, the set $\mathcal{R}$ of red cycles crossed by (at least) one of $R(j)$ and $R(j \ominus b(v))$ has size at least $b(v)+a(v)-2$. If neither $R(j)$ nor $R(j \ominus b(v))$ has m or more crossings, then $|\mathcal{R}| \leq m-1$ (recall that red cycles that cross each other do so at least twice). Thus $b(v) + a(v) - 2 \leq m - 1$. Since by assumption $b(v) + a(v) > n/2$, it follows that $n/2 < m + 1$, that is, $n < 2m + 2$, a contradiction. Therefore either $R(j)$ or $R(j \ominus b(v))$ has m or more crossings, as required. ∎

Proof of Main Theorem. First we note that if $n \geq m(m + 1)$, then $\min\{(m - 2)n, m(n - (2m + 2))\} = (m - 2)n$. Therefore it suffices to show that if $n \geq 2m + 2$, then every drawing of an (m, n)–graph has at least $\min\{(m - 2)n, m(n - (2m + 2))\}$ crossings. We prove this by induction on n.

The base case is $n = 2m + 2$, for which there is nothing to prove. Suppose that the statement holds for $n = k - 1 \geq 2m + 2$, and consider a drawing $\mathcal{E}$ of an (m, k)–graph. If $\mathcal{E}$ is robust, then we are done, since by Theorem 2 $\mathcal{E}$ has at least $(m - 2)k$ crossings. Thus we assume $\mathcal{E}$ is not robust. Since $k > 2m + 2$, it follows from Lemma 3 that there is a red cycle $R(j)$ with m or more crossings in $\mathcal{E}$. The drawing $\mathcal{E}'$ that results by removing $R(j)$ from $\mathcal{E}$ has, by the induction hypothesis, at least $\min\{(m - 2)(k - 1), m(k - 1 - (2m + 2))\}$ crossings, and so $\mathcal{E}$ has at least $\min\{(m - 2)(k - 1) + m, m(k - 1 - (2m + 2)) + m\} = \min\{(mk - 2(k - 1), m(k - (2m + 2))\}$ crossings. Since $mk - 2(k - 1) > (m - 2)k$, then $\mathcal{E}$ has at least $\min\{(m - 2)k, m(k - (2m + 2))\}$ crossings, as required. ∎

8 On the Adamsson and Richter work on arrangements

As it transpires from our discussions in Sections 2 and 3, the present work is heavily influenced by techniques and results from the theory of arrangements, introduced by Adamsson [1] and further developed by Adamsson and Richter [2].

Our aim in this section is to discuss Adamsson and Richter's work in more detail. We also explain the difference between their and our assignment of crossings to red cycles.

The fundamental structures analyzed in [1] are (linear) (m, n)–arrangements. An (m, n)–arrangement consists of a collection $\mathcal{C}$ of $n+2$ ordered (red) closed curves $S = C_0, C_1, C_2, \ldots, C_n, T = C_{n+1}$ plus a collection $\mathcal{P}$ of m (blue) open arcs, each of which intersects the red curves in the given order. Each of these (forced by definition) intersections is regarded as a *vertex*, and additional intersections are counted as *crossings* (which they are indeed, since the curves in $\mathcal{C} \cup \mathcal{P}$ can be assumed to be in general position). It can be assumed with no loss of generality that each closed curve in $\mathcal{C} \setminus \{S\}$ is contained in the unbounded component of $\mathbb{R}^2 \setminus \{S\}$ and each closed curve in $\mathcal{C} \setminus \{T\}$ is contained in the unbounded component of $\mathbb{R}^2 \setminus \{T\}$.

A crucial result in [1] is that each (m, n)–arrangement has at least $(m - 2)n$ crossings. This is proved by assigning to each red closed curve in $\{C_1, \ldots, C_n\}$ a collection of at least $m-2$ crossings. It is natural to ask what is the difference between this assignment of crossings and the assignment we describe in Section 4.

In order to discuss the difference between these assignments, one must point out that the objects under study are different at this stage. Indeed, our objects of study are drawings of (m, n)–graphs, and in the assignment in [1], the objects under consideration are (m, n)–arrangements (which appear naturally as induced drawings in certain drawings of (m, n)–graphs).

This distinction has an impact in the difference between our assignment and the assignment in [1]. In [1], each of the curves $C_1, \ldots, C_n$ gets assigned $m - 2$ crossings, and this assignment is based on the assumption that all the curves $C_1, \ldots, C_n$ have a common predecessor $S = C_0$ and a common successor $T = C_{n+1}$ with the properties mentioned above. In the present paper, we work with each red curve $R(j)$, find a collection of predecessors and (respective) successors, and *for each predecessor–successor pair we describe an assignment of crossings quite similar to the one used in [1].* An important distinction is that we do not require the predecessors to be disjoint from the red curve being analyzed.

Once Adamsson and Richter prove that (m, n)–arrangements have at least $(m-2)n$ crossings, they move on to apply this result to *circular* (m, n)–arrangements. A circular (m, n)–arrangement is the underlying drawing of an (m, n)–graph. Adamsson and Richter prove that if a circular (m, n)–arrangement satisfies certain additional conditions (namely, if it is *partitioned*), then their knowledge on linear (p, q)–arrangements can be applied to show that the circular arrangement under consideration has at least $(m-2)n$ crossings.

The exact definition of when a circular arrangement is partitioned is somewhat technical. However, to give the reader an idea of the generality and power of this concept, we mention that every drawing of an (m, n)–graph in which the union of the closed curves $R(j)$ has at least three components is a partitioned (m, n)–circular arrangement. This implies the following quite general result.

Theorem [Adamsson and Richter] *Let $\mathcal{E}$ be a drawing of an (m, n)–graph. Suppose that the union of the induced drawings of the red cycles $R(j)$ has at least three components. Then $\mathcal{E}$ has at least $(m-2)n$ crossings.*

In Chapter 6 of his Ph.D. thesis, Adamsson wrote: "This work on arrangements has pushed the knowledge of crossing numbers further than it was before, but it has the potential to lead to even more impressive results, and maybe to provide the final proof of the HKS–conjecture". In the present work we have extended these techniques to show that the HKS–Conjecture holds for all but finitely many values of n, for each m. We hope that a combination of the ideas involved in the powerful concept of partitioned circular arrangements (a concept we did not invoke at all in this paper) with the refinement of the techniques of linear (m, n)–arrangements presented in this work will lead to a final proof of the HKS–Conjecture.

9 Concluding Remarks

In the proof of the Main Theorem we used as a base case of the induction the (obviously true) fact that for every $(m, 2m+2)$–graph G, $\text{cr}(G) \geq 0$. It is natural to ask if Theorem 1 is substantially improved if instead we use a nontrivial bound for $\text{cr}(C_m \times C_{2m+2})$. The best general lower bound known for the crossing number of $C_m \times C_n$ (for $n \geq m \geq 3$) is $\text{cr}(C_m \times C_n) \geq (1/2)(m-2)n$ [10]. Using this bound, we obtain the following slightly improved version of Theorem 1.

Theorem 1 [Improved version]. *Let m, n be integers such that $n \geq (m+1)(m+2)/2$, $m \geq 3$. Then $cr(C_m \times C_n) = (m-2)n$.*

While Theorem 1 (and its improved version above) settles the HKS–Conjecture for all but finitely many values of n, for each m, the HKS–Conjecture remains open for $n < (m+1)(m+2)/2$, $m \geq 8$. For values of n sufficiently close to m (more precisely, for m, n such that $m \geq 8$, $m \leq n \leq 5(m-1)/4$), it is known that $\text{cr}(C_m \times C_n) \geq (5/7)mn$ [15]. For n between $5(m-1)/4$ and $(m+1)(m+2)/2$, the best general lower bound known is $cr(C_m \times C_n) \geq (m-2)n/2$ [10].

Acknowledgements

We thank Bruce Richter for discussions. This work was done while the first author visited the Instituto de Investigacion en Comunicacion Optica of the Universidad Autonoma de San Luis Potosi (IICO–UASLP). He thanks the staff of IICO–UASLP for their hospitality. The second author is supported by FAI–UASLP and CONACYT through grant J32168E.

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