The number of generalized balanced lines*

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Abstract

Let S be a set of r red points and $b = r + 2\delta$ blue points in general position in the plane, with $\delta \geq 0$. A line ℓ determined by them is *balanced* if in each open half-plane bounded by ℓ the difference between the number of blue points and red points is δ. We show that every set S as above has at least r balanced lines. The main techniques in the proof are rotations and a generalization, sliding rotations, introduced here.

1 Introduction

Let B and R be, respectively, sets of blue and red points in the plane, and let $S = B \cup R$ be in general position. Let $r = |R|$ and $b = |B| = r + 2\delta$, with $\delta \geq 0$. Furthermore, we are given weights $\omega(p) = +1$ for $p \in B$ and $\omega(q) = -1$ for $q \in R$. Given a halfplane H, its weight is then defined as $\omega(H) = \sum_{s \in S \cap H} \omega(s)$. Here and throughout this paper, halfplanes are open unless otherwise stated.

Definition 1. A line ℓ determined by two points of S is *balanced* if the two halfplanes it defines have weight δ. Observe that this implies that the two points of S spanning ℓ have different colors.

For $\delta = 0$, we obtain the original result, as conjectured by George Baloglou, and proved by Pach and Pinchasi via circular sequences:

Theorem 1 ([3])**.** *Let $|R| = |B| = n$. Every set S as above determines at least n balanced lines. This bound is tight.*

Tightness is shown, e.g., by placing S on a convex $2n$-gon in such a way that R is separated from B by a straight line.

The general result was proved by Sharir and Welzl in an indirect manner, via an equivalence with a very special case of the Generalized Lower Bound Theorem. This motivated them to ask for a more direct and simpler proof.

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Theorem 2 ([4])**.** *Let B and R be, respectively, sets of blue and red points in the plane, and let $S = B \cup R$ be in general position. Let $r = |R|$ and $b = |B| = r + 2\delta$, with $\delta \geq 0$. The number of lines that pass through a point in B and a point in R, and such that the two induced halfplanes have weight δ is at least r. This number is attained if R and B can be separated by a line.*

In this paper we give a simple proof of Theorem 2 using elementary geometric techniques. Therefore, via the results in [4], we provide a geometric proof of the following very special case of the Generalized Lower Bound Theorem:

Let $\mathcal{P}$ be a convex polytope which is the intersection of $d+4$ halfspaces in general position in $\mathbb{R}^d$. Let its edges be oriented according to a generic linear function (edges are directed from smaller to larger value; "generic" means that the function evaluates to distinct values at the vertices of $\mathcal{P}$).

Theorem 3 ([4])**.** *The number of vertices with $\lceil \frac{d}{2} \rceil - 1$ outgoing edges is at most the number of vertices with $\lceil \frac{d}{2} \rceil$ outgoing edges.*

All proofs in this paper can be easily translated to the more general setting of circular sequences (see [2]).

2 Geometric tools

We assume that coordinate axes are chosen in such a way that all points have different abscissa. The tools we use are inspired in the rotational movement introduced by Erdős et al. [1].

Definition 2. Let $P \subseteq S$. A P^k*-rotation* is a family of directed lines P_t^k, where $t \in [0, 2\pi]$ is the angle measured from the vertical axis, defined as follows: P_0^k contains a single point of P, and as t increases, it rotates counterclockwise in such a way that

(i) $|P \cap P_t^k| = 1$ except for a finite number of events, when $|P \cap P_t^k| = 2$; and

(ii) whenever $|P \cap P_t^k| = 1$, there are exactly k points of P to the right of P_t^k.

The common point $P \cap P_t^k = \{p\}$ is called the *pivot*, and it changes precisely when $|P \cap P_t^k| = 2$. Observe that $P_0^k = P_{2\pi}^k$.

Definition 3. Let ℓ^+ and ℓ^- denote, respectively, the halfplanes to the right and to the left of ℓ. Let $\omega(\ell)$ be the weight of ℓ^+. Given a P^k-rotation, we say that $P^k \geq \delta$ if $\omega(P_t^k) \geq \delta$ for every $t \in [0, 2\pi]$, and similarly for the rest of inequalities. A rotation B^k is δ*-preserving* if either $B^k \geq \delta$ or $B^k < \delta$. Symmetrically, R^k is δ*-preserving* if either $R^k \leq \delta$ or $R^k > \delta$.

Lemma 4. *In an R^k-rotation, transitions $\delta \rightsquigarrow \delta + 1$ and $\delta + 1 \rightsquigarrow \delta$ in $\omega(R_t^k)$ are always through a balanced line. In a B^k-rotation, transitions $\delta \rightsquigarrow \delta - 1$ and $\delta - 1 \rightsquigarrow \delta$ in $\omega(B_t^k)$ are always through a balanced line.*

Proof. When a red point is found during an R^k-rotation, the weight of the halfplane is preserved because the pivot point changes. Therefore, the change $\delta \rightsquigarrow \delta + 1$ happens when a blue point is found in the head of R_t^k (Figure 1, left), while $\delta + 1 \rightsquigarrow \delta$ happens when a blue point is found in the tail of R_t^k (Figure 1, right). In both cases, the points define a balanced line. For a B^k-rotation, the proof is identical. □

Figure 1: Transitions in an R^k-rotation are always through a balanced line.

Claim 8.1 in [3] has now a more direct proof:

Lemma 5. *If r is odd, there exists a balanced line which is a halving line of S.*

Proof. Let $k = \lfloor \frac{r}{2} \rfloor$ and consider an R^k-rotation. If $R_0^k \leq \delta$, then $R_\pi^k > \delta$, and conversely. Therefore, there exist transitions $\delta \rightsquigarrow \delta+1$ and $\delta+1 \rightsquigarrow \delta$ in $\omega(R_t^k)$ which, from Lemma 4 are always through a balanced line. Observe that both transitions are through the same balanced line, with angles t_0 and $t_0+\pi$. □

Remark 1. Let us observe that Theorem 1.4 in [3], which states that Theorem 1 is true when R and B are separated by a line ℓ, has now an easier proof: if we start R^k-rotations with a line parallel to ℓ, for each k there exist exactly one transition $\delta \rightsquigarrow \delta+1$ and one transition $\delta+1 \rightsquigarrow \delta$ which, from Lemma 4, correspond always to a balanced line. If r is even, there are 2 balanced lines for $k = 0, \ldots, \frac{r}{2}-1$, for a total of r balanced lines, while if r is odd there are 2 balanced lines for $k = 0, \ldots, \lfloor \frac{r}{2} \rfloor - 1$ and 1 balanced line for $k = \lfloor \frac{r}{2} \rfloor$.

Remark 2. Lemmas 4 and 5 conclude the proof of Theorem 1 if no R^k-rotation is δ-preserving or if no B^k-rotation (with $k \geq \delta$) is δ-preserving. Hence, in the following we assume that there exists either at least one R^k-rotation or one B^k-rotation (with $k \geq \delta$) which is δ-preserving.

Lemma 6. *Let $0 \leq j \leq \lfloor \frac{r}{2} \rfloor$. If $R^j > \delta$ then $B^{j+\delta} \geq \delta$, while if $B^{j+\delta} < \delta$ then $R^j \leq \delta$.*

Proof. Consider the line $R_{t_0}^j$. The halfplane $(R_{t_0}^j)^+$ contains j red points and $b > j+\delta$ blue points. Therefore, the line $B_{t_0}^{j+\delta}$ is to the right of $(R_{t_0}^j)^+$ and contains at most j red points. Then, $\omega(B_{t_0}^{j+\delta}) \geq \delta$. The proof of the second statement is analogous. □

The next definition generalizes the concept of P^k-rotation in two different ways: parallel movements are permitted and the number of points to the right of the line can change.

Definition 4. A *P-sliding rotation* consists in moving a directed line ℓ continuously, starting with an ℓ_0 which contains a single point $p_0 \in P$, and composing rotation around a point of P (the pivot) and parallel displacement (in either direction) until the next point of P is found. Furthermore, after a 2π rotation is completed, the line ℓ_0 must be reached again.

This movement is clearly a continuous curve in the space of lines in the plane. For instance, if a line is parameterized as a point in $S^1 \times \mathbb{R}$, a P-sliding rotation describes a (non-strictly) angular-wise monotone curve, with vertical segments corresponding to parallel displacements.

Let Σ be a P-sliding rotation. Let us denote by Σ_t the line with angle t with respect to the vertical axis defined as follows: if there is no parallel displacement at angle t, then Σ_t denotes the corresponding line. Otherwise, it denotes the leftmost line corresponding to angle t.

Definition 5. A P-sliding rotation Σ is *positively oriented* if $\Sigma_{t+\pi}$ is to the left of Σ_t for all $t \in [0, \pi)$.

That $\Sigma \geq \delta$, as well as the rest of inequalities, is defined exactly as in Definition 3. Similarly, a B-sliding rotation Σ is δ-preserving if $\Sigma \geq \delta$, while an R-sliding rotation is δ-preserving if $\Sigma \leq \delta$. The following definition is the crux of the rest of the paper.

Definition 6. Let $\mathcal{S}$ be the set of all positively oriented, δ-preserving B-sliding rotations and R-sliding rotations. The *waist* of a P-sliding rotation $\Sigma \in \mathcal{S}$ is

$$\min_{t\in[0,\pi]} |P \cap \Sigma_t^- \cap \Sigma_{t+\pi}^-|.$$

We denote by Γ the sliding rotation of $\mathcal{S}$ with the smallest waist.

Note that the set $\mathcal{S}$ is non-empty because we have assumed that there exist δ-preserving B^k- or R^k-rotations, which are a particular type of sliding rotations. Furthermore, the waist takes only a finite number of values, so it has a minimum. If the minimum is not unique, we can pick any of the sliding rotations achieving it.

3 Main result

Assume that Γ is a δ-preserving R-sliding rotation (i.e. $\Gamma \leq \delta$). In this case, we will manage to prove that there exist at least r balanced lines. For the case of Γ being a δ-preserving B-sliding rotation, the same arguments would show that there exist at least b balanced lines.

Lemma 7. *Let Γ_0 and Γ_π be the lines achieving the waist of Γ, let $\overline{\Gamma}_0^+$ be the closed halfplane to the right of Γ_0 and let $F = R \cap \overline{\Gamma}_0^+$. For every $k \in \{0, \dots, |F| - 1\}$, during an F^k-rotation a balanced line is found. Similarly, let $H = R \cap \overline{\Gamma}_\pi^+$. For every $k \in \{0, \dots, |H| - 1\}$, during an H^k-rotation a balanced line is found.*

Proof. Figure 2 illustrates the situation. On the one hand, F_0^k is to the right of Γ_0 and, since Γ is positively oriented, F_π^k is to the left of Γ_π. This implies that there is a $t_1 \in [0, \pi]$ such that $F_{t_1}^k = \Gamma_{t_1}$ and therefore $\omega(F_{t_1}^k) \leq \delta$. On the other hand, F_0^k is to the left of Γ_π and F_π^k is to the right of Γ_0, therefore, there exists a $t_2 \in [0, \pi]$ such that $F_{t_2}^k$ and $\Gamma_{t_2+\pi}$ are the same line with opposite directions. Since $\omega(\Gamma_{t_2+\pi}) \leq \delta$, then $\omega(F_{t_2}^k) \geq \delta$. If $\omega(\Gamma_{t_2+\pi}) = \delta$ and the line contains a blue point, then it is a balanced line found in a transition $\delta \rightsquigarrow \delta + 1$. Otherwise, $\omega(F_{t_2}^k) > \delta$ and hence a transition $\delta \rightsquigarrow \delta + 1$ has occurred for a $t \in (t_1, t_2)$.

Now, observe that $R \setminus F \subset {\Gamma_0}^-$. Hence, in the F^k-rotation for $t \in [0, \pi]$, all the points in $R \setminus F$ are found by the head of the line. This implies that a change $\delta \rightsquigarrow \delta + 1$ in the weight of the right halfplane can only occur when a blue point is found in the head of the ray (as in Figure 1, left), hence defining a balanced line. The proof for H is identical. $\square$

Before moving on, let us point out that the $|F| + |H|$ balanced lines given by Lemma 7 are different, because they have exactly k points of F, respectively H, to the right. Let now C_t^Γ be the *central region* defined by the sliding rotation Γ at instant t, defined as $C_t^\Gamma = \Gamma_t^- \cap \Gamma_{t+\pi}^-$. Observe that, for the corresponding t, the transitions $\delta \rightsquigarrow \delta + 1$ in the proof of Lemma 7 correspond to balanced lines inside or in the boundary of the central region.

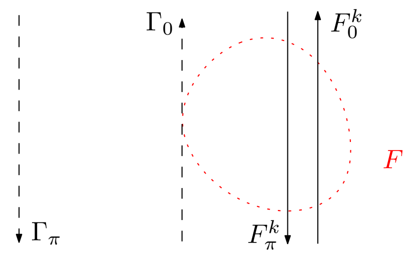

Figure 2: Illustration of the proof of Lemma 7.

Lemma 8. *Let $G = R \setminus (F \cup H)$. For $k \in \{0, \dots, \lceil |G|/2 \rceil - 1\}$, every G^k-rotation has at least two transitions between δ and $\delta+1$, which correspond to lines inside or in the boundary of the central region., i.e., for the corresponding t, $G_t^k \in C_t^\Gamma$.*

Proof. Let us consider first the case when r is odd and $k = \lfloor |G|/2 \rfloor$. G_0^k and G_π^k are the same line with opposite directions. Therefore, if $\omega(G_0^k) \leq \delta$ then $\omega(G_\pi^k) > \delta$ and there must be at least two transitions as stated. These transitions correspond to lines in the central region because Γ is positively oriented.

For the rest of cases, observe that, by construction, $G_0^k \in C_0^\Gamma$. According to the value of $\omega(G_0^k)$, we distinguish two cases:

- $\omega(G_0^k) \leq \delta$. If there exist some values for which $G_t^k = \Gamma_t$, let t_1 and t_2 be, respectively, the minimum and maximum of them. If there is no such value, take $t_1 = t_2 = 2\pi$. If G^k takes the value $\delta+1$ in the interval $(0, t_1)$ it must have transitions $\delta \rightsquigarrow \delta+1$ and $\delta+1 \rightsquigarrow \delta$, and the same is true for $(t_2, 2\pi)$. Finally, observe that G^k must take the value $\delta+1$ at least once, because in other case the sliding rotation obtained by concatenating G^k in $(0, t_1)$, Γ in (t_1, t_2) and G^k in $(t_2, 2\pi)$ would be a δ-preserving sliding rotation of waist smaller than the waist of Γ.

- $\omega(G_0^k) > \delta$. If there exist some values for which $G_t^k = \Gamma_t$, let t_1 and t_2 be, respectively, the minimum and maximum of them. G_t^k takes the value δ in the intervals $(0, t_1)$ and $(t_2, 2\pi)$ and therefore the lemma follows. In other case, if G_t^k takes the value δ in the central region, it must have also transition $\delta \rightsquigarrow \delta+1$. Finally, if $\omega(G_t^k) > \delta$ for all $t \in [0, 2\pi]$ we could construct a sliding rotation Σ contradicting the choice of Γ: for each t, consider as Σ_t the parallel to G_t^k which passes through the first blue point to the right of G_t^k. It is easy to see that $\Sigma_t \geq \delta$, because between Γ_t and G_t^k there are always at least two blue points. ∎

The following lemma, which already appeared as Claim 6.4 in [3], will be enough to conclude the proof of Theorem 1.

Lemma 9. *Transitions $\delta \rightsquigarrow \delta+1$ and $\delta+1 \rightsquigarrow \delta$ in a G^k-rotation are always either a balanced line or a $\delta+1 \rightsquigarrow \delta$ transition in an F^j-rotation, $j \in \{0, \dots, |F|-1\}$ or an H^j-rotation, $j \in \{0, \dots, |H|-1\}$.*

Proof. On the one hand, a balanced line is achieved if there is such a transition because a blue point is found. See Figure 1. On the other hand, if the point inducing the transition is $r \in R$, then necessarily $r \in R \setminus G$ (since the G^k-rotation changes pivot whenever a point of G is found). Figure 3 illustrates that a $\delta+1 \rightsquigarrow \delta$ transition appears for an F^j-rotation with pivot g, both if $f \in F$ is found in the tail (left picture) or if $f \in F$ is found in the head (right picture). Note that in the right picture the weight of both halfplanes is $\delta+1$. The case

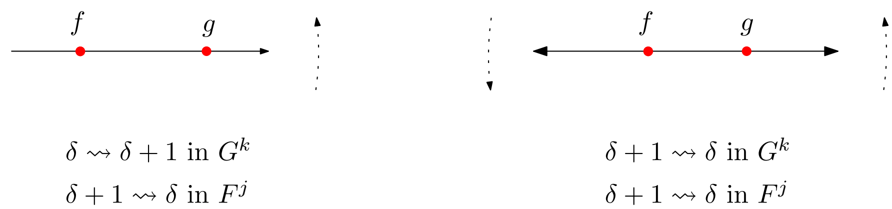

Figure 3: Transitions when a point $f \in F \subset R$ found in a G^k-rotation induces a $\delta+1 \rightsquigarrow \delta$ transition in an F^j-rotation.

in which the point found is $h \in H$ works similarly. □

The following simple observations show that the number of balanced lines is at least r which, together with Remark 1, finishes the proof of Theorem 2:

i) Lemma 7 gives $|F| + |H|$ different balanced lines.

ii) Lemmas 8 and 9 give $|G|$ lines which are, either a balanced line, or a $\delta+1 \rightsquigarrow \delta$ transition at the central region for an F^j- or H^j-rotation.

iii) Each transition in ii) forces a new $\delta \rightsquigarrow \delta+1$ transition at the central region for an F^j- or H^j-rotation which correspond, as in the proof of Lemma 7, to a new balanced line.

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