Approximating the Crossing Number of Toroidal Graphs

Petr Hliněný[1]* and **Gelasio Salazar**[2]**

[1] Faculty of Informatics, Masaryk University
Botanická 68a, 602 00 Brno, Czech Republic
e-mail: hlineny@fi.muni.cz

[2] Instituto de Física, Universidad Autónoma de San Luis Potosí
San Luis Potosí, Mexico, 78000
e-mail: gsalazar@ifisica.uaslp.mx

Abstract. CrossingNumber is one of the most challenging algorithmic problems in topological graph theory, with applications to graph drawing and VLSI layout. No polynomial time constant approximation algorithm is known for this NP-complete problem. We prove that a natural approach to planar drawing of toroidal graphs (used already by Pach and Tóth in [20]) gives a polynomial time constant approximation algorithm for the crossing number of toroidal graphs with bounded degree. In this proof we present a new "grid" theorem on toroidal graphs.

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1 Introduction

We assume the reader is familiar with the standard terminology of graph theory. In this paper we consider finite graphs, with loops or multiple edges allowed. Some standard topological graph theory terminology is briefly introduced throughout this paper. For other related terminology and theory we refer the reader to Mohar and Thomassen [18]. Here our main interest lies in *toroidal graphs*, that is, graphs that can be *embedded* (meaning drawn without edge crossings) on the torus.

The (planar) *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of edge crossings in a drawing of G in the plane. To resolve ambiguity, we consider drawings of graphs such that no edge passes through another vertex, and that no three edges intersect in a common point which is not a vertex. Then a *crossing* is an intersection point of two edges which is not a vertex.

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Computing crossing numbers has important applications in VLSI design, and, naturally, in the graph drawing area. The algorithmic decision problem of *crossing minimization* is formulated as follows:

CrossingNumber

Input: A (multi)graph G and an integer k.
Question: Is $\mathrm{cr}(G) \leq k$? (Possibly: if so, find the corresponding drawing).

The problem is in NP since one could guess the optimal drawing, replace its crossings with new (degree 4, subdividing) vertices, and verify planarity of the resulting graph. It has been proved by Garey and Johnson [9] that crossing minimization is NP-complete if k is a part of the input. The same assertion has been proved true later by Hliněný [12] both for cubic graphs and for the minor-monotone version (cf. [1]) of crossing number. An important, stubborn open problem is to decide whether the crossing number of graphs with bounded tree-width can be computed in polynomial time.

On the positive side, a (surprising) result from Grohe [11], recently improved by Kawarabayashi and Reed [15], states that CrossingNumber is an FPT problem. Unfortunately, these algorithms are not usable in practice, not even for small values of k. Regarding approximability results, the best general result known to date is a polynomial time algorithm by Even, Guha and Schieber [8], which approximates $\mathrm{cr}(G) + |V(G)|$ up to a factor of $\log^3 |V(G)|$ for graphs G of bounded degree (notice the $+|V(G)|$ term).

Our interest in the crossing number of graphs embedded in a given surface follows a recent major trend in crossing numbers research, which emphasizes the relationship of crossing number to topological graph theory and to structural parameters (see for instance [1–3, 10, 15, 22]). Böröczky, Pach and Tóth [3, 20] prove that the crossing number of a toroidal graph G is at most $c \cdot \Delta(G)V(G)$, with an analogous generalization to any fixed surface. (A refinement of this estimate bounds $\mathrm{cr}(G)$ by a factor of the sum of square degrees of G.) In this direction the assymptotically best possible estimate for graphs G of orientable genus $g = o(|V(G)|)$ is $\mathrm{cr}(G) \leq c \cdot g\Delta(G)V(G)$ given by Djidjev and Vrt'o [6]. An even wider generalization of the problem by Telle and Wood [22] shows that any class $\mathcal{G}$ of bounded-degree graphs excluding a fixed minor H satisfies $\mathrm{cr}(G) \leq c_{H,\Delta} \cdot V(G)$ for every $G \in \mathcal{G}$. Although all these estimates are tight in the sense that there exist graph sequences attaining them assymptotically, they give no good algorithmic approximation for CrossingNumber since many other graphs in these classes also have arbitrarily smaller crossing number.

On the other hand, constant factor approximation algorithms of CrossingNumber are known only for some particular families of graphs, such as [10] for projective graphs of bounded degree with an approximation factor $4.5\Delta(G)^2$; or [13] for almost planar graphs of bounded degree with an approximation factor $\Delta(G)$. (A graph is *almost planar* if deleting one edge leaves it planar.) In this relation one should mention that an older result of Riskin [21] implies that, for almost planar graphs coming from cubic 3-connected planar subgraphs, the crossing number can be determined exactly. Other aspects of the crossing number of almost planar graphs are dealt with in [17]. Now we extend our attention to graphs embeddable on the torus.

The new contribution of our paper lies in a fine analysis of a natural planar-drawing algorithm for toroidal graphs (analogous to the approach of Pach and Tóth [20]), which is complemented with a matching lower bound on the crossing number. This is summarized next. (We refer to Section 2 for the definition of edge-width, and to Lemma 3.1 for details on the $o(1)$ term appearing there).

Theorem 1.1. *Given a nonplanar toroidal graph G, one can construct in polynomial $O(n\sqrt{n})$ time, where $n = |V(G)| + |E(G)|$, a drawing of G in the plane*

a) with at most $\big(6\Delta(G)^2 + o(1)\big) \cdot \mathrm{cr}(G)$ crossings;

b) with at most $12\Delta(G)^2 \cdot \mathrm{cr}(G)$ crossings if G embeds in the torus with dual edge-width at least $10\lfloor \Delta(G)/2 \rfloor$.

Hence for a fixed maximal degree bound $\Delta(G) \leq \Delta$ we get (b) a polynomial time algorithm which approximates CrossingNumber up to a constant factor $12\Delta^2$ for all graphs which have sufficiently "dense" toroidal embeddings. Notice that, concerning time complexity of our algorithm, we may assume $n = |V(G)|$ if $\Delta(G)$ is bounded, or if G is simple.

Our paper is organized as follows. In Section 2 we describe Drawing Algorithm 2.3 (cf. Theorem 1.1) and some details of its implementation. It uses a natural idea of surgery along a manifold, extensively used in classical topology: "cut and open" a toroidal embedding of a given graph G along a curve intersecting the fewest number of edges, and then redraw the affected edges of G inside the rest of the embedding in the best possible (crossing-wise) way. We prove in Section 3 that this approach gives a good approximation of the correct crossing number of G by exhibiting in G a special minor (a toroidal grid) which itself has crossing number very close to the quantity computed in Algorithm 2.3. This part represents the main new contribution of our paper, not appearing in any of the related previous papers [3, 20, 22, 2]. Theoretical details about finding this grid minor are then given in Section 4.

2 The algorithm

For the coming arguments we have to introduce some common topological terms. A closed curve on a surface is simply called a *loop*. Two loops α, β on a surface Σ are *freely homotopic* if α can be continuously transformed to β on Σ. A closed curve on a surface is *contractible* if it is freely homotopic to a constant curve (it can be continuously deformed to a single point).

Since we are going to work with a toroidal embedding of a given graph, we first resolve the task of finding it. It is widely known how to test planarity efficiently, and a strong generalization of that result by Mohar [16] claims:

Theorem 2.1 (Mohar). *For every surface Σ there is a linear time algorithm which, for a given graph G, either finds an embedding of G on Σ or returns a subgraph of G that is a subdivision of a "minimal obstacle" for Σ.*

In particular, this result provides us a toroidal embedding of the input graph which is known to be toroidal.

The second ingredient in our approach is a well-known concept of measuring "dual density" of a graph embedding. Consider now a graph G embedded on a nonplanar surface Σ (i.e. G is a topological rather than a combinatorial object, and the embedding G itself determines the surface Σ). The *edge-width* $\mathrm{ew}(G)$ of the embedding G is then defined as the length of the shortest cycle in G which is not contractible on Σ.

The edge-width of a given embedding can be efficiently computed by an algorithm of Thomassen [23]. A recent improvement of running time is in [5]:

Theorem 2.2 (Cabello, Mohar). *Given an embedded graph H, one can compute in time $O(n\sqrt{n})$, where $n = |V(H)| + |E(H)|$, the edge-width k of the embedding H, and find a length-k noncontractible cycle in H.*

The basic idea—"cut and open" a toroidal embedding of a given graph G while affecting the fewest number of edges, appears in the core of the proof by Pach and Tóth [20] (Böröczky et al [3]). We adopt it (with a slight modification – using the topological dual instead of a triangulation) in an algorithmical setting. See Fig. 1 for an informal hint to geometric idea of this algorithm.

Algorithm 2.3. Drawing a toroidal graph G in the plane.

1. Given a toroidal graph G, we first test planarity of G. (If G is plane, we are done.) We construct an embedding $\bar{G}$ of G on the torus $\mathcal{S}_1$ using Theorem 2.1.
2. We construct the topological dual G^* for $\bar{G}$ on $\mathcal{S}_1$. We compute k, the edge-width of G^*, and the corresponding length-k cycle C^* in G^* as described in Theorem 2.2.
3. Let γ be the simple loop of $\mathcal{S}_1$ formed by C^*. We transform $\mathcal{S}_1$ into a cylinder $\mathcal{R}$ by "cutting along" γ. The cylinder $\mathcal{R}$ has two boundary curves γ_1 and γ_2 which are the copies of γ. In this way the embedded (dual) graph G^* is naturally transformed into $G^\star$ on $\mathcal{R}$ such that $C_1^\star$ and $C_2^\star$ are the two copies of C^* embedded as γ_1 and γ_2, respectively.
4. Let G^o be the graph resulting from $G^\star$ by contracting each of $C_1^\star$ and $C_2^\star$ into single vertices w_1 and w_2. Note that since G is not planar, it follows that G^o is connected. We then use breadth-first search to compute the shortest path P^o of length ℓ between w_1 and w_2 in G^o. Let δ be the simple curve on $\mathcal{R}$ formed by the embedding of P^o in $G^\star$. Hence δ connects a point x_1 on γ_1 to a point $x_2$on γ_2, and δ intersects ℓ edges of the original embedding $\bar{G}$.
5. Let $F \subseteq E(G)$ be the set of those edges in the embedding $\bar{G}$ which are crossed by γ, and $F' \subseteq E(G)$ be the set of those crossed by δ. Hence $\bar{G} - F$ is actually embedded on $\mathcal{R}$, and we extendthis crossing-free subdrawing (of $G - F$) into a new drawing $\widetilde{G}$of the whole graph G on $\mathcal{R}$ as follows: each edge from F is newly drawn along an appropriate section of γ_1 up to x_1, then along δ (crossing the ℓ edges from F') until reaching x_2, and finally along an appropriate section of γ_2. We output $\widetilde{G}$ as a drawing of G.

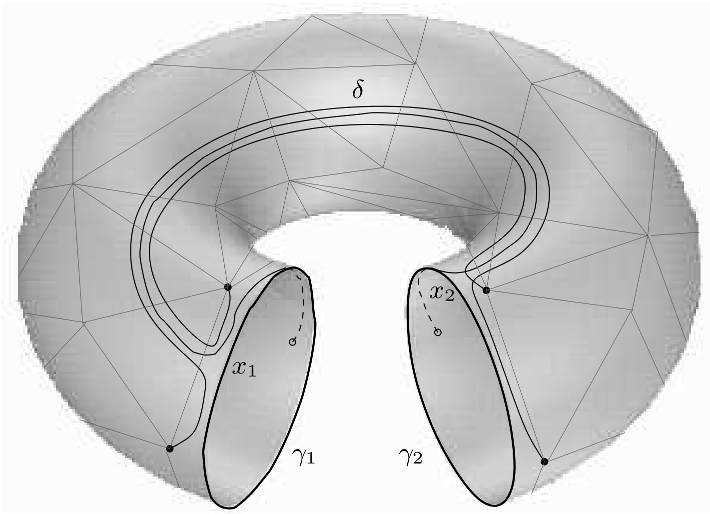

Fig. 1. Cutting a toroidal embedding by γ, and redrawing the affected edges along δ.

Lemma 2.4. *The output graph $\widetilde{G}$ in Algorithm 2.3 is a planar drawing of G with at most $k\ell + \lfloor k^2/4 \rfloor$ edge crossings, where k, ℓ are computed in the algorithm.*

Proof. Since cutting the torus along any noncontractible loop (Step 3) results in a cylinder, the graph $\bar{G} - F$ is cylinder-embedded, and hence plane (F are the edges intersected by γ, that is, dual to $E(C^*)$). Now since G is nonplanar, $k > 0$ and the graph G^o is connected. So in Step 4 we find a dual path P^o and the associated curve δ connecting x_1 with x_2 on the two boundaries γ_1, γ_2 of our cylinder $\mathcal{R}$.

The drawing $\bar{G} - F$ is disjoint from both γ_1, γ_2 in $\mathcal{R}$, and by the definition of F, each $e = u_1u_2 \in F$ has u_i on the face incident with γ_i, $i = 1, 2$. Hence such $e \in F$ can be drawn along γ_i from u_i to x_i without crossings, for $i = 1, 2$, and (in the middle) along δ making ℓ crossings with the edges from F'. See in Fig. 1. Furthermore, two edges $e, e' \in F$ must cross each other in $\widetilde{G}$ if and only if x_1, x_2 (visualized as points back on γ) separate the intersections $e \cap \gamma$ from $e' \cap \gamma$. This makes at most $\lfloor k/2 \rfloor \lceil k/2 \rceil = \lfloor k^2/4 \rfloor$ crossings in addition to the $k\ell$ crossings between F and F'. ∎

Lemma 2.5. *Algorithm 2.3 runs in time $O(n\sqrt{n})$, where $n = |V(H)| + |E(H)|$.*

Proof. We represent an embedded graph by its rotation system (of edges at the vertices). Step 1 runs in linear time with this representation, by Theorem 2.1. Now the dual embedding G^* is easily obtained in linear time, too, and so Step 2 runs in time $O(n\sqrt{n})$, by Theorem 2.2. The transformation into a cylindrical embedding $G^\star$ described in Step 3 is simply done in $O(k)$ time: we duplicate C^* into $C_1^\star, C_2^\star$ and "split" the local rotations of $V(C^*)$ accordingly. In Step 4 we deal with an abstract graph G^o, and the breadth-first search (for P^o) on it also runs in $O(n)$ time. Then in Step 5 we get the embedding $\bar{G} - F$ in linear time as the plane dual of $G^\star$, excluding the $C_1^\star, C_2^\star$-faces. We also identify F' as the edge set dual to $E(P^o)$ computed in Step 4.

Finally, we visualize each crossing in the final planar drawing $\widetilde{G}$ as a ("dummy") degree 4 vertex in an associated planar graph G'. Knowing C^* and P^o, and their dually associated edge sets F and F', the construction of G' is computationally achieved in time $O(k\ell+k^2)$ (see Lemma 2.4), which is also $O(n)$ by further Lemma 4.1 and Theorem 3.2. ∎

3 Lower crossing bound

Let G be a graph with maximum degree $\Delta(G) = \Delta$. For simplification we consider G already *embedded in the torus*. As we have observed in Lemma 2.4, Algorithm 2.3 yields a drawing of G in the plane with at most $k\ell+k^2/4$ crossings. Our basic claim, in order to prove Theorem 1.1 that this drawing is a good estimate for $\mathrm{cr}(G)$, is that the computed quantity $k\ell$ is within a constant factor of $\mathrm{cr}(G)$ (more precisely, a factor that depends only on Δ for large enough k). Equivalently, there is a suitable function $f(\Delta) > 0$ such that $\mathrm{cr}(G) \geq (f(\Delta) - o_k(1)) \cdot k\ell$. The goal of this section is to prove this claim.

Assuming a cycle C and a path P with both ends on C, simultaneously embedded in an orientable surface Σ, we say that P is *C-separated* if P is internally disjoint from C, and the first and the last edges of P appear on opposite sides of the loop C in Σ. To give formal mathematical meaning of the variables k, ℓ in Algorithm 2.3, we let $\mathrm{ew}^*(G)$ denote the *dual edge-width* (the edge-width of the topological dual of G), and we let $\mathcal{L}(G)$ denote the set of *orthogonal widths*, that is, the set of all integers ℓ possessing the following property: there is a noncontractible cycle C^* of length $\mathrm{ew}^*(G)$ in the topological dual G^*, such that ℓ is the length of the shortest path P^* in G^* with both ends in $V(C^*)$ which is C^*-separated. Note that P^* may be a cycle, and so with a slight abuse of terminology we do allow the ends of P^* to be the same. Clearly, Algorithm 2.3 computes $k = \mathrm{ew}^*(G)$ and $\ell \in \mathcal{L}(G)$.

Lemma 3.1. *Let G be a graph embedded in the torus with maximum degree Δ, $k = \mathrm{ew}^*(G)$ and $\ell = \max \mathcal{L}(G)$. Then*

$$\mathrm{cr}(G) \;\geq\; \left(\frac{1}{16\lfloor \Delta/2 \rfloor^2} - o_k(1)\right) \cdot k\ell \;\geq\; \left(\frac{1}{4\Delta^2} - o_k(1)\right) \cdot k\ell\,,$$

where $o_k(1) \to 0$ as $k \to \infty$ with fixed Δ.

Before we move on with the proof, we recall that the $p \times q$ *toroidal grid* is the Cartesian product $C_p \times C_q$ of cycles of lengths p and q. This 4–regular graph embeds naturally in the torus with the edge-width $\min\{p, q\}$.

Proof. The main new ingredient for the proof is the following statement, which guarantees the existence of a large toroidal grid minor contained in G.

Theorem 3.2. *Let G be a graph embedded in the torus, $k = \mathrm{ew}^*(G)$ and $\ell \in \mathcal{L}(G)$. Then G contains a minor isomorphic to the toroidal grid of size*

$$\left\lceil \frac{1}{2}\left(\frac{k}{\lfloor \Delta/2 \rfloor} - 1\right)\right\rceil \;\times\; \left\lceil \frac{\ell}{\lfloor \Delta/2 \rfloor}\right\rceil .$$

Assuming this result for the moment (we devote the next section to its proof), we finish the proof of Lemma 3.1.

First we recall that if H is a minor of G, and H has maximum degree at most 4, then $\mathrm{cr}(G) \geq \frac{1}{4}\,\mathrm{cr}(H)$ [19]. It is known that the crossing number of the toroidal grid of size $q \times p$, where $p \geq q \geq 3$, is at least $\frac{1}{2}(q-2)p$ [14]. Combining these facts with Theorem 3.2, and using $\ell \geq k/2$ (from Lemma 4.1), we obtain

$$\mathrm{cr}(G) \geq \frac{1}{4} \cdot \frac{1}{2}(q-2)p \geq \frac{1}{8} \cdot \frac{k}{2\lfloor \Delta/2 \rfloor} \cdot \frac{\ell}{\lfloor \Delta/2 \rfloor} - O(\ell) \geq \left(\frac{1}{16\lfloor \Delta/2 \rfloor^2} - o_k(1) \right) \cdot k\ell \,. \qquad \blacksquare$$

To derive Theorem 1.1 a) from Lemma 2.4 and this estimate of Lemma 3.1, we note that, using Lemma 4.1,

$$k\ell + k^2/4 \leq k\ell + k\ell/2 \leq \mathrm{cr}(G) \cdot \frac{3}{2} \left(\frac{1}{4\Delta^2} - o_k(1) \right)^{-1} \leq \mathrm{cr}(G) \cdot \left(6\Delta^2 + o_k(1) \right) \,.$$

The same argument proves also part b) of Theorem 1.1, with a constant factor $\frac{3}{2}\,8\Delta^2 = 12\Delta^2$, if we adapt Lemma 3.1 without asymptotic terms:

Corollary 3.3. *Let G be a graph embedded in the torus, $k = \mathrm{ew}^*(G)$ and $\ell = \max \mathcal{L}(G)$. If $k \geq 10\lfloor \Delta/2 \rfloor$, then $\mathrm{cr}(G) \geq \frac{1}{8\Delta^2} \cdot k\ell$.*

Proof. We just slightly modify the above proof of Lemma 3.1:

$$\mathrm{cr}(G) \geq \frac{1}{8} \left(\frac{k}{2\lfloor \Delta/2 \rfloor} - \frac{1}{2} - 2 \right) \cdot \frac{\ell}{\lfloor \Delta/2 \rfloor} \geq \frac{1}{8} \cdot \frac{k}{4\lfloor \Delta/2 \rfloor} \cdot \frac{\ell}{\lfloor \Delta/2 \rfloor} \geq \frac{1}{8\Delta^2} \cdot k\ell \,. \qquad \blacksquare$$

4 Finding a grid minor

For readers' convenience, we use throughout the coming arguments the same notation as introduced in Algorithm 2.3 and used in Theorem 3.2.

Thus, let G be a graph embedded in the torus $\mathcal{S}_1$ with maximum degree Δ, and G^* be its topological dual. (Although its embedding may not be unique, the following arguments can use *any embedding* G in $\mathcal{S}_1$ to derive the conclusions.) Set $k = \mathrm{ew}^*(G)$ and choose *any* orthogonal width (see in Section 3) $\ell \in \mathcal{L}(G)$. Consequently select any appropriate C^*, a length-k noncontractible cycle in G^* such that the shortest C^*-separated path P^* in G^* has length ℓ. Denote by γ the simple loop in $\mathcal{S}_1$ determined by C^*, and by δ the curve determined by P^*.

Lemma 4.1. *If $k = \mathrm{ew}^*(G)$ and $\ell \in \mathcal{L}(G)$, then $\ell \geq k/2$.*

Proof. Seeking a contradiction, we suppose that $\ell < k/2$. The ends of P^* on C^* determine two subpaths of C^* (both with the same ends as P^*), and one of them, say Q^*, has length at most $k/2$. Then $Q^* \cup P^*$ is noncontractible and its length is at most $\ell + k/2 < k/2 + k/2 = \mathrm{ew}^*(G)$, a contradiction. $\blacksquare$

In order to finish the arguments of Section 3, we have to provide a proof of Theorem 3.2, that is, find a sufficiently large toroidal grid minor in the graph G relatively to the parameters k, ℓ. For that we have to carefully examine the structure and "density" of a toroidal embedding of G.

Remark. A beautiful result by de Graaf and Schrijver [7] precisely relates the size of the largest guaranteed grid minor in a toroidal graph to the "face-width" of its toroidal embedding. It is, unfortunately, not directly usable in our context since the lower bound on the size of a toroidal grid (see Theorem 3.2) implied by [7] would be of order $k \times k$, and not of $k \times \ell$ as we need.

On the other hand, our proof of Theorem 3.2 could be viewed as a graph-theoretical alternative to [7] (de Graaf and Schrijver's proof relies on results on the geometry of numbers), giving a slightly worse estimate in the case of $k = 2\ell$, but significantly stronger for $\ell \gg k$.

Lemma 4.2. *Let G, γ and ℓ be as above. Then the embedded graph G contains at least $\frac{\ell}{\lfloor \Delta/2 \rfloor}$ pairwise disjoint cycles, all freely homotopic to γ.*

Proof. Let F be the set of those edges of G intersected by γ on $\mathcal{S}_1$. We "cut and open" $\mathcal{S}_1$ along γ into the cylinder $\mathcal{R}$ with boundary loops γ_1 and γ_2. Then the (planar) subgraph $H = \bar{G} - F$ is embedded in $\mathcal{R}$. We denote by δ' a curve on $\mathcal{R}$ connecting a point of γ_1 to a point of γ_2, such that δ' has the fewest possible points in common with the embedding H. We claim that δ' intersects H in $p \geq \frac{\ell}{\lfloor \Delta/2 \rfloor}$ points, which can clearly be assumed to be vertices of H. Indeed, if $p < \frac{\ell}{\lfloor \Delta/2 \rfloor}$, then the union of all faces incident with the p vertices intersected by δ' would contain a dual path Q^*, which had length at most $p \cdot \lfloor \Delta/2 \rfloor < \frac{\ell}{\lfloor \Delta/2 \rfloor} \cdot \lfloor \Delta/2 \rfloor = \ell$, and (considering in G^*) Q^* would be C^*-separated. That contradicts the above definition of ℓ.

We now "cut and open" the cylinder $\mathcal{R}$ along δ', getting a rectangle with sides $\gamma_1, \delta'_1, \gamma_2, \delta'_2$ in this cyclic order. By duplicating all p vertices of H on δ', we get an embedded graph H'. Let w_i and $w_{i,2}$ for $i = 1, \ldots, p$ denote these duplicated vertices. We note that there is no vertex cut X of size at most $p - 1$ in H' separating $\{w_1, \ldots, w_p\}$ from $\{w_{1,2}, \ldots, w_{p,2}\}$, since that would give a curve ε from γ_1 to γ_2 intersecting H in $|X| < p$ points, which is a contradiction to our choice of δ'. (One may notice that a similar argument appears, for instance, also in [20, 10, 3].) Hence we get p pairwise disjoint paths from $\{w_1, \ldots, w_p\}$ to $\{w_{1,2}, \ldots, w_{p,2}\}$ by Menger's theorem in H'. Moreover, by a planarity argument we immediately see that each of these paths connects w_i to the corresponding $w_{i,2}$ for $i = 1, \ldots, p$. Thus they form the desired collection of p pairwise disjoint cycles in H, and they are also disjoint from γ and freely homotopic to it. ∎

The following statement could be proved using similar means as Lemma 4.2, but this time with many more complications, and so we prefer an indirect approach using [4].

Lemma 4.3. *Let G, γ and ℓ be as above. Then the embedded graph G contains at least $\frac{1}{2}\left(\frac{k}{\lfloor \Delta/2 \rfloor} - 1\right)$ pairwise disjoint pairwise freely homotopic cycles which are not homotopic to an iteration of γ.*

Proof. Let λ be any simple noncontractible loop in $\mathcal{S}_1$ intersecting γ in one point. We define a loop $\lambda_{i,j}$ in $\mathcal{S}_1$ as the composition of j iterations of λ and i iterations of γ. (Classical topology says that any loop on $\mathcal{S}_1$ is freely homotopic to one of $\lambda_{i,j}$. We neglect the orientation of the loops here.) Using the terminology of [4], we define a "complete set of loops" on $\mathcal{S}_1$ as all those which are freely homotopic to $\lambda_{i,0}$ for any i. By an analogous argument as in the proof of Lemma 4.2 we notice that the face-width of G is $r \geq \frac{k}{\lfloor \Delta/2 \rfloor}$. Thus [4, Corollary 7.1] gives the required $\frac{1}{2}(r-1)$ pairwise disjoint cycles, all freely homotopic to $\lambda_{s,t}$ for some s and $t \neq 0$. ∎

Proof of Theorem 3.2. We denote by $C_1, C_2, \ldots, C_p$ (the "C-cycles") the pairwise disjoint cycles in our graph G from Lemma 4.2 and by $D_1, D_2, \ldots, D_q$ (the "D-cycles") the cycles from Lemma 4.3, where $p = \left\lceil \frac{\ell}{\lfloor \Delta/2 \rfloor} \right\rceil$ and $q = \left\lceil \frac{1}{2} \left(\frac{k}{\lfloor \Delta/2 \rfloor} - 1 \right) \right\rceil$. Notice that $p \geq q$ by Lemma 4.1, so we may assume $p \geq 3$ since otherwise the statement is trivial. To simplify notation, we use *cyclic indexing* of the C-cycles modulo p and of the D-cycles modulo q. We also let $C_+ := C_1 \cup C_2 \cup \ldots \cup C_p$ and $D_+ := D_1 \cup D_2 \cup \ldots \cup D_q$.

Remark. It may appear that we already have the desired grid as a minor in $C_+ \cup D_+$, since every D_j, $j \in \{1, \ldots, q\}$, has to intersect each C_i, $i \in \{1, \ldots, p\}$, in some vertex of G. This is because the homotopy types of C_i and D_j on $\mathcal{S}_1$ are distinct. The cycles C_i and D_j, however, could have many "zigzag" intersections, and besides, D_j may "wind" many times in the direction orthogonal to C_i. These problems will be dealt with in the coming proof.

First, we can assume that among all possible choices of the collection $C_1, \ldots, C_p$, we have gotten one which minimizes $|E(C_+) \setminus E(D_+)|$. An *$F$-ear* is a path having both ends in a subgraph F, but otherwise disjoint from F. Then the following is true for our choice:

Claim 4.4. No C_+-ear contained in D_+ has both ends on the same cycle C_i.

Indeed, if a C_+-ear $P \subset D_+$ with both ends on some C_i contradicted our claim, we could rectify the cycle C_i by following P in the appropriate section, thus decreasing the value of $|E(C_+) \setminus E(D_+)|$.

We further assume that the cycles $C_1, C_2, \ldots, C_p$ appear in this cyclic order around the torus; precisely, that for none $2 < i < i' \leq p$ the cycles C_1 and C_i share a face in the toroidal (sub)embedding of $C_1 \cup C_2 \cup C_i \cup C_{i'}$. A *quasicycle* is a graph-homomorphic image of a cycle without degree-1 vertices, implicitly retaining its cyclic ordering of vertices. Consider an arbitrary quasicycle D'_j in G homotopic to D_1 (say, initially $D'_j = D_j$). We say that D'_j is *C_+-ear good* if (cf. Claim 4.4) no C_+-ear of D'_j has both ends on the same C_i.

With respect to the chosen quasicycle D'_j, we define an *intersection sequence* $a(j,i)$, $i = 1, \ldots, s_j$, of integers such that D'_j intersects all the C-cycles in the cyclic order $C_1 = C_{a(j,1)}, C_{a(j,2)}, \ldots, C_{a(j,s_j)}$, choosing appropriately s_j and the same orientation as with $C_1, \ldots, C_p$. We denote by $Q_{j,t}$, $t = 1, 2, \ldots, s_j$, the path of D'_j (possibly a single vertex) forming the corresponding intersection with the cycle $C_{a(j,t)}$, and by $T_{j,t}$ the path of D'_j between $Q_{j,t}$ and $Q_{j,t+1}$. Clearly,

$a(j,t+1) \neq a(j,t)$ if D'_j is C_+-ear good, and hence $|a(j,t+1)-a(j,t)| \in \{1,p-1\}$ for $t = 1,2,\ldots,s_j$.

A collection of C_+-ear good quasicycles $D'_1, D'_2, \ldots, D'_q$ in G is *quasigood* if it satisfies the property that whenever D'_n intersects D'_m in a path P (counting also the case of a self-intersection with $m = n$), the following hold up to symmetry between n and m: $P \subseteq Q_{n,x}$ for an appropriate index x of the intersection sequence of D'_n for which $a(n,x-1) = a(n,x+1)$ and $a(n,x) - a(n,x-1) \in \{1, 1-p\}$, and the adjacent paths $T_{n,x-1}$,$Q_{n,x}$,$T_{n,x}$ of D'_n stay locally on one side of the drawing of D'_m in $\mathcal{S}_1$. (Informally, this means that if D'_n intersects D'_m in P, then D'_n makes a $C_{a(n,x-1)}$-ear with P "touching" D'_m from the left side.) For further refernce we say that D'_n is locally on the *left side* of the intersection P.

Among all choices of a quasigood collection $D'_1, D'_2, \ldots, D'_q$ in G, we select one minimizing $s_1 + \ldots + s_q$ where s_j is the above length of the intersection sequence for D'_j.

Claim 4.5. For all $1 \leq j \leq q$ the intersection sequence of D'_j satisfies $a(j,t-1) \neq a(j,t+1)$ for any $1 < t \leq s_j$. Consequently, $D'_1, D'_2, \ldots, D'_q$ is a collection of pairwise disjoint proper cycles in G.

The idea of a proof of this claim is simple—if $a(j,t-1) = a(j,t+1)$, then we could rectify D'_j by following $C_{a(j,t-1)}$ instead of $T_{j,t-1} \cup Q_{j,t} \cup T_{j,t}$; decreasing s_j by 2. We make this formally precise now.

Let $\mathcal{R}_i$ denote the (sub)cylinder of $\mathcal{S}_1$ between the boundaries C_i and C_{i+1}. Notice that if $a(j,t-1) = a(j,t+1)$ happens for $a(j,t) - a(j,t-1) \in \{-1, p-1\}$, then neccessarily for some other index t' it holds $a(j,t'-1) = a(j,t'+1)$ and $a(j,t') - a(j,t'-1) \in \{1, 1-p\}$. So suppose for a contradiction that $a(j,t-1) = a(j,t+1) = i$ and $a(j,t) = i+1$. Then the path $P = T_{j,t-1} \cup Q_{j,t} \cup T_{j,t}$ is drawn in $\mathcal{R}_i$ with both ends on C_i and "touching" C_{i+1}. We denote by $R_0 \subset \mathcal{R}_i$ the open region bounded by P and C_i, and by P' the section of the boundary of R_0 not belonging to D'_j.

Assuming that R_0 is minimal possible over all choices of j for which $a(j,t-1) = a(j,t+1)$, we show that no D'_m, $m \in \{1,\ldots,q\}$ enters R_0: If some D'_m intersected R_0, then D'_m could not enter R_0 across P by the "stay on one side" property of a quasigood collection. Hence D'_m should enter and leave R_0 across $P' \subseteq C_i$, but not touch $Q_{j,t} \subseteq C_{i+1}$ by minimality of R_0. So D'_m would make a C_+-ear with both ends on C_i, contradicting the assumption that D'_m is C_+-ear good.

Now we form D^o_j as the symmetric difference of D'_j with the boundary of R_0 (hence D^o_j follows P'). To argue that $D'_1, \ldots, D^o_j, \ldots, D'_q$ is a quasigood collection again, it suffices to verify all possible new intersections of D^o_j along P'. So suppose there is D'_n such that its intersection $Q_{n,x}$ with C_i contains some internal vertex of P'. Since D'_n is disjoint from (open) R_0, it will "stay on one side" of D^o_j. If $Q_{n,x}$ intersects D'_j, then D'_n must be locally on the left side of this intersection, and so it is also on the left side of the intersection with new D^o_j according to the above definition. If, on the other hand, $Q_{n,x}$ is disjoint from D'_j, then the adjacent paths $T_{n,x-1}$ and $T_{n,x}$ have to connect to C_{i-1} by Claim 4.4, and so we have $a(n,x) = i$ and $a(n,x-1) = a(n,x+1) = i-1$ as required by the definition for D'_n on the left side. Hence Claim 4.5 is proved.

Claim 4.6. There is a collection of pairwise disjoint cycles $D''_1, D''_2, \ldots, D''_q$ in G where $D''_j \subset D'_j \cup C_j$, $j = 1,2,\ldots,q$, such that the cyclic intersection sequence of each D''_j is $a(j,1) = 1, a(j,2) = 2, \ldots, a(j,p) = p$ of length p.

By Claim 4.5 the intersection sequence of each D'_j has a "nice" form $a(j,1) = 1, a(j,2) = 2, \ldots, a(j,p) = p,\ a(j,p+1) = 1, \ldots$ Our task is (unless already true) to "shortcut" each D'_j such that it "winds only once" in the direction orthogonal to the loop γ. First notice that, for all $i = 1,\ldots,p$, every C_i-ear of each D'_j is C_i-separated (cf. Section 3) by Claim 4.5. We implicitly *orient* every C_i-ear so that it intersects C_{i+1} before C_{i+2}. If we take any C_1-ear $T_1 \subset D'_1$ with start x_1 and end y_1 on C_1, and any one $W_1 \subset C_1$ of the two paths between x_1, y_1, then the cycle $D''_1 = T_1 \cup W_1$ has the desired intersection sequence.

Secondly, notice that since D''_1 is not homotopic to D'_1, every D'_j has to intersect D''_1 in W_1. We may assume that the cycles $D'_2, \ldots, D'_q$ have this ordering of their first intersections with W_1 from x_1. Now for $j = 2,3,\ldots,q$ we do: let $Q_{j,x}$ be the intersection of D'_j with W_1 closest to x_1, and let $T'_j \subset D'_j$ be the unique C_1-ear starting at $Q_{j,x}$. Then let $T_j \subset D'_j$ be the unique C_j-ear starting inside T'_j (and hence not intersecting W_1), and $W_j \subset C_j$ be the path between the ends of T_j disjoint from T_1. We set $D''_j = T_j \cup W_j$. It is straightforward to verify that $D''_1, \ldots, D''_q$ is a collection of pairwise disjoint cycles in G.

Finally, with Claim 4.6 at hand it is easy to finish the whole theorem: contractions of all the paths of $D''_j \cap C_i$, $1 \le i \le p$, $1 \le j \le q$, into single vertices, create a subdivision of the $q \times p$ toroidal grid in G. ∎

5 Conclusions

We observe that the apparent "weakness" of our approximation (Theorem 3.2) in requiring large dual edge-width of $\bar{G}$ with respect to Δ is unavoidable. Indeed, a toroidal embedded graph of dual edge-width $k = 2$ may easily be planar. By multiplying edges of such a graph and some local modification one can get (multi)graphs of crossing number one but arbitrarily large dual edge-width on the torus, at the expense of growing Δ.

It is natural to ask whether our results can be extended to higher genus surfaces. The upper bound techniques, as worked out in [3] or [6], seem to provide a road map for such an extension: Specifically, for G embedded on the orientable surface $\mathcal{S}_g$, we can iterate g-times the "cut and open" construction from Algorithm 2.3. Denoting by k_i the dual edge-width and by ℓ_i the associated orthogonal width obtained at steps $i = 1,2,\ldots,g$, we straightforwardly conclude with a planar drawing of G of at most $O\big(g^2 \cdot \max\{k_i\ell_i : i = 1,\ldots,g\}\big)$ crossings. On the other hand, a nontrivial lower bound of order $\Omega(k_g\ell_g/\Delta^2)$ is easy to obtain using Theorem 3.2 at the last iteration. Unfortunately, this bound generally falls way short of matching the upper bound within a constant factor, even with fixed g and Δ. We have not yet been able to find a remedy for this problem.

Finally, we remark that our "grid" Theorem 3.2 itself seems to be of some interest in structural topological graph theory.

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