On the additivity of crossing numbers of graphs

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Abstract. We describe the relationship between the crossing number of a graph G with a 2-edge-cut C and the crossing numbers of the components of $G - C$. Let G be a connected graph with a 2-edge-cut $C := [V_1, V_2]$. Let u_1u_2, v_1v_2 be the edges of C, so that $u_i, v_i \in V_i$ for $i = 1, 2$, and let $G_i := G[V_i]$ and $G'_i := G_i + u_iv_i$. We show that if either G_1 or G_2 is not connected, then $\mathrm{cr}(G) = \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$, and that if they are both connected then $\mathrm{cr}(G) = \mathrm{cr}(G'_1) + \mathrm{cr}(G'_2)$. We use this to show how to decompose crossing–critical graphs with 2-edge-cuts into smaller, 3–edge–connected crossing–critical graphs. We also observe that this settles a question arising from knot theory, raised by Sawollek, by describing exactly under which conditions the crossing number of the connected sum of two graphs equals the sum of the crossing numbers of the individual graphs.

1 Introduction

Crossing number is a widely studied parameter that measures the nonplanarity of a graph. For a long time, most research on crossing numbers focused on the estimation of the crossing number of interesting families of graphs. While this kind of research is still of interest, in recent years a good deal of effort has been put into more general, structural questions on this parameter (see for instance [3, 4, 6, 7, 8]). For an excellent recent survey, see [10].

One of the most natural questions for a graph theoretical parameter is how it behaves under the (some) composition of graphs: if G is obtained from G_1 and G_2 by means of some operation, how does the parameter applied to G relates to the values of the parameter for G_1 and G_2?

The effect of cut vertices and cut edges on crossing numbers is trivial. Indeed, if G_1, G_2 are disjoint graphs, and G is obtained by identifying a vertex of G_1 with a vertex of G_2, then clearly $\mathrm{cr}(G) = \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$. Similarly, if G has a cut edge e, then the crossing number of G equals the sum of the crossing numbers of the components of $G - e$.

The immediately next level asks for the behaviour of crossing number with respect to 2-edge-cuts. Is it always true that the crossing number of a graph G with a 2-edge-cut C equals the sum of the crossing numbers of the components G_1, G_2 of $G - C$? It is not difficult to find examples that show that the answer to this question is no. A closer examination reveals that it seems more natural to consider instead of G_i (for $i = 1, 2$) the graph G'_i that results by adding to G_i an edge joining the vertices incident with the edges in C.

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1.1 Main result: additivity of crossing number

Our main result shows that $\mathrm{cr}(G)$ is either equal to $\mathrm{cr}(G_1) + \mathrm{cr}(G_2)$ or to $\mathrm{cr}(G'_1) + \mathrm{cr}(G'_2)$. The exact answer depends only on the connectivity of G_1 and G_2.

Theorem 1 (Additivity of crossing number). *Let G be a connected graph, and $C := [V_1, V_2]$ a 2-edge-cut of G. Let u_1u_2, v_1v_2 be the edges of C, so that $u_i, v_i \in V_i$ for $i = 1, 2$. Let $G_i := G[V_i]$ and $G'_i := G_i + u_iv_i$, for $i = 1, 2$. Then:*

(i) If either G_1 or G_2 is not connected, then $\mathrm{cr}(G) = \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$.

(ii) If both G_1 and G_2 are connected, then $\mathrm{cr}(G) = \mathrm{cr}(G'_1) + \mathrm{cr}(G'_2)$.

We prove this statement in Section 2.

We note that (ii) in Theorem 1 follows from Lemmas 1 and 2 in [1] if G_i contains two edge–disjoint u_i–v_i paths, for $i = 1$ and 2. In [1], Bokal explores the additivity of crossing numbers under certain connectivity conditions, and applies his results in an amazingly short and elegant proof of an old conjecture by Jendrol' and Ščerbová [5]. In [2], Bokal uses these results to settle an open question on the existence of crossing–critical graphs with prescribed average degree and crossing number.

1.2 2-edge-cuts in crossing–critical graphs

In analogy with other important graph theoretical parameters, it is of great interest to understand the properties of *crossing–critical* graphs, that is, graphs G such that $\mathrm{cr}(G-e) < \mathrm{cr}(G)$ for every edge e of G. The observations above on the trivial role of cut edges imply that connected crossing–critical graphs have no cut edges.

We will show, using Theorem 1, that every crossing–critical graph G with 2-edge-cuts can be naturally decomposed into smaller, 3-edge–connected crossing–critical graphs, such that the sum of their crossing numbers equals $\mathrm{cr}(G)$. In view of this result, further investigations into the structure of crossing–critical graphs may as well focus on 3-edge-connected graphs.

One might expect that the breakdown of crossing–critical graphs (with 2-edge-cuts) into smaller crossing–critical pieces is dictated by Theorem 1. More precisely, if we let G, G_1, G_2, G'_1, G'_2, be as in Theorem 1, we may wonder whether it is always true that either G_1 or G'_1 is crossing–critical and either G_2 or G'_2 is crossing–critical. Although this is not always the case, the decomposition of G into crossing–critical graphs goes along these lines.

To state the decomposition theorem, first we observe that the set K_G of edges in a 2-edge-connected graph G that are in some 2-edge-cut of G breaks naturally into equivalence classes. Indeed, if we define $\sim$ by the rule that $e, e' \in K_G$ satisfy $e \sim e'$ iff either $e = e'$ or $\{e, e'\}$ is a 2-edge-cut of G, then $\sim$ is an equivalence relation. The induced equivalence classes are the *2-cut-classes of* G. It is easy to check that if $[e]$ is such an equivalence class, then each component H of $G - [e]$ is 2-edge-connected, and that there are exactly two edges in $[e]$ incident with vertices in H.

Theorem 2 (Decomposition of crossing–critical graphs with 2-edge-cuts). *Let G be a connected crossing–critical graph with minimum degree at least* 3. *Then G is 2-edge-connected. Let $[e]$ be a 2-cut-class of G. Let $G_1, G_2, \ldots, G_n$ be the components of $G - [e]$, and for $i = 1, 2, \ldots, n$, let u_i, v_i be the (nonnecessarily different) vertices in G_i incident with edges in $[e]$. Then, for each $i = 1, 2, \ldots, n$, either G_i or $G_i + u_iv_i$ is crossing–critical.*

Remark *The assumption of minimum degree at least* 3 *in Theorem 2 is not restrictive at all, since edge subdivisions affect neither the crossing number of a graph nor its criticality.*

Bruce Richter (personal communication) has observed the following.

Remark *It is straightforward to check, using Theorems 1 and 2, that (in the notation of Theorem 2) for at least one i the graph $G_i + u_i v_i$ is crossing–critical.*

A repeated application of Theorems 1 and 2 yields the following.

Corollary 3. *Let G be a connected crossing-critical graph with minimum degree at least* 3*. Then there is a collection $J_1, J_2, \ldots, J_m$ of 3-edge-connected crossing-critical graphs, each of which is a subdivision of G, and such that* $\mathrm{cr}(G) = \sum_{i=1}^{m} \mathrm{cr}(J_i)$

The following is also an easy consequence of Theorems 1 and 2.

Corollary 4. *For each integer $k \geq 1$, there is an integer $f(k)$ with the following property. Suppose that G is a crossing-critical graph with minimum degree at least* 3 *and* $\mathrm{cr}(G) = k$*. Then G has at most $f(k)$ edges that belong to some 2-edge-cut of G.*

It can be shown that $f(k) = 2k - 2$ works, and, moreover, it is best possible.

1.3 Application to a question arising from knot theory

A tantalizingly open conjecture in knot theory states that the crossing number of links is additive under connected sums. In a paper relating the crossing number of graphs and the crossing number of knots, Sawollek [9] observed that "...it is an open problem whether the crossing number is additive with respect to a *connected sum* $G_1 \# G_2$ of two graphs G_1 and G_2, i.e., two edges $e_1 = v_1 v_2 \in G_1$ and $e_2 = w_1 w_2 \in G_2$ that are not loops are replaced by edges $e_1' = v_1 w_1$ and $e_2' = v_2 w_2$".

We note that the connected sum depends on the chosen edges e_1, e_2, and on the pairs of vertices we choose to join (there are two such possibilities). To emphasize this dependence on e_1 and e_2, we write $(G_1 \# G_2)_{e_1,e_2}$, and to specify which pairs of vertices get joined, we regard both $e_1 = v_1 v_2$ are $e_2 = w_1 w_2$ as directed edges, so that the newly formed edges are $v_1 w_1$ and $v_2 w_2$. Thus, under this assumption, $(G_1 \# G_2)_{e_1,e_2}$ is a unique, well–defined graph.

In the following statement we settle Sawollek's question, characterizing under which conditions crossing number behaves additively under connected sums.

Theorem 5. *Let H_1, H_2, and let e_1, e_2 be edges in H_1 and H_2, respectively.*

(i) If either each e_i is a cut edge of H_i, or no e_i is a cut edge of H_i, then $\mathrm{cr}((H_1 \# H_2)_{e_1,e_2}) = \mathrm{cr}(H_1) + \mathrm{cr}(H_2)$.

(ii) Otherwise, we assume without loss of generality that e_1 is a cut edge of H_1 but e_2 is not a cut edge of H_2. In this case, $\mathrm{cr}((H_1 \# H_2)_{e_1,e_2}) = \mathrm{cr}(H_1) + \mathrm{cr}(H_2)$ *if and only if* $\mathrm{cr}(H_2 - e_2) = \mathrm{cr}(H_2)$.

We prove Theorem 5 in Section 4. As we shall see, it is a straightforward consequence of Theorem 1.

2 Proof of Theorem 1

We assume that $u_i \neq v_i$ for $i = 1, 2$, as otherwise u_i is a cut vertex, and the theorem follows.

The proof of (i) is straightforward, since in this case both edges in C are cut edges in G. Thus we assume G_1 and G_2 are connected. Now $\mathrm{cr}(G) \leq \mathrm{cr}(G_1') + \mathrm{cr}(G_2')$ is an easy exercise (a special case of Lemma 1 in [1]), so let us proceed to show the reverse inequality. To help comprehension, we color (both abstractly and when they are drawn) the edges and vertices in G_1 (respectively G_2) red (respectively blue). We color u_1u_2 and v_1v_2 yellow.

Let $\mathcal{D}$ be an optimal drawing of G. Let X_{RR} (respectively X_{BB}) denote the number of red–red (respectively blue–blue) crossings in $\mathcal{D}$, that is, crossings in which both edges involved are red (respectively blue). Let $\mathcal{D}_1$ and $\mathcal{D}_2$ denote the drawings of G_1 and G_2, respectively, induced by $\mathcal{D}$.

A point in $\mathcal{D}_1$ is a *vertex point* if it represents a (red) vertex, and a *crossing point* if it represents a (necessarily red–red) crossing. If we remove from $\mathcal{D}_1$ all vertex points and all crossing points, the components of the resulting set are the *red arcs* of $\mathcal{D}_1$. Thus, each red arc either represents a full red edge e (if e is not crossed in $\mathcal{D}_1$) or a portion of a red edge e (if e is crossed in $\mathcal{D}_1$). An analogous discussion applies to $\mathcal{D}_2$, and *blue arcs* are similarly defined.

Let Γ_1 denote the set of all simple arcs (think of these as *green* arcs, to emphasize they do not come from $\mathcal{D}$) that join u_1 and v_1, and otherwise only intersect $\mathcal{D}_1$ in red arcs (that is, not in vertex or crossing points). For $\gamma \in \Gamma_1$, let $|\gamma|_{\mathcal{D}_1}$ denote the number of intersections of γ with red arcs. Let $d_1 := \min_{\gamma \in \Gamma_1}\{|\gamma|_{\mathcal{D}_1}\}$. Define Γ_2 and d_2 similarly.

A routine argument shows that $\mathcal{D}_1$ contains a collection $\rho_1, \rho_2, \ldots, \rho_{d_1}$ of simple closed red curves, such that no red arc belongs to more than one ρ_i, with the following properties: as we traverse any simple arc γ in Γ_1 from u_1 to v_1, then (i) *(blocking property)* we must intersect every ρ_i (in a red arc, since $\gamma \in \Gamma_1$); and (ii) *(nesting property)* if $i \geq 2$, then before we intersect any red arc in ρ_i we must intersect some red arc in ρ_{i-1}. There is in $\mathcal{D}_2$ a collection $\beta_1, \beta_2, \ldots, \beta_{d_2}$ of simple closed blue curves with analogous properties.

For each $i = 1, 2, \ldots, d_1$, let R_i denote the set of crossings in $\mathcal{D}$ that involve a yellow or blue arc and a red arc in ρ_i. For each $j = 1, 2, \ldots, d_2$, let B_j denote the set of crossings in $\mathcal{D}$ that involve a yellow or red arc and a blue arc in β_j. Let $\mathbf{R} := \{R_1, R_2, \ldots, R_{d_1}\}$, and $\mathbf{B} := \{B_1, B_2, \ldots, B_{d_2}\}$.

Let $\mathbf{R_1}$ denote the subset of $\mathbf{R}$ consisting of those R_i's with $|R_i| = 1$, and let $\mathbf{B_1}$ denote the subset of $\mathbf{B}$ consisting of those B_j's with $|B_j| = 1$, Note that the blocking property implies that if $d_1 > 0$ (respectively $d_2 > 0$) then no R_i (respectively no B_j) can be empty. Thus each $R_i \notin \mathbf{R}_1$ (and each $B_j \notin \mathbf{B}_1$) has size at least 2. Inspired on this, we define $\mathbf{R}_{\geq 2} := \mathbf{R} \setminus \mathbf{R}_1$, and $\mathbf{B}_{\geq 2} := \mathbf{B} \setminus \mathbf{B}_1$.

Let x be a crossing in some $R_i \in \mathbf{R}_1$. We claim x cannot be in any B_j. Obviously if x is a red–yellow crossing then it cannot be in any B_j. Thus suppose x is a red–blue crossing. Since $x \in R_i \in \mathbf{R}_1$, x is *the only* red–blue crossing that involves a red arc in ρ_i. The blue arc δ involved in x separates $\mathcal{D}_2$ (removing δ disconnects $\mathcal{D}_2$), since it is the only blue arc that crosses ρ_i, and ρ_i is a closed curve. This disconnecting property of δ implies it cannot belong to any simple closed blue curve, in particular δ cannot belong to any β_j. Thus x cannot be in any B_j, as claimed. An analogous argument shows that no crossing x in $B_j \in \mathbf{B}_1$ can also be in an R_i.

Define $X_{\mathbf{R}_1} := \cup_{R_i \in \mathbf{R}_1} R_i$, and $X_{\mathbf{R}_{\geq 2}}, X_{\mathbf{B}_1}, X_{\mathbf{B}_{\geq 2}}$ analogously. Note that $X_{\mathbf{R}_1} \cap X_{\mathbf{R}_{\geq 2}} =$

$X_{\mathbf{B}_1} \cap X_{\mathbf{B}_{\geq 2}} = \emptyset$. The observations in the previous paragraph imply that $X_{\mathbf{R}_1} \cap (X_{\mathbf{B}_1} \cup X_{\mathbf{B}_{\geq 2}}) = X_{\mathbf{B}_1} \cap (X_{\mathbf{R}_1} \cup X_{\mathbf{R}_{\geq 2}}) = \emptyset$. Therefore the total number of crossings in $\mathcal{D}$ that are neither red–red nor blue–blue is at least $|X_{\mathbf{R}_1} \cup X_{\mathbf{B}_1} \cup X_{\mathbf{R}_{\geq 2}} \cup X_{\mathbf{B}_{\geq 2}}| = |X_{\mathbf{R}_1}| + |X_{\mathbf{B}_1}| + |X_{\mathbf{R}_{\geq 2}}| + |X_{\mathbf{B}_{\geq 2}}| - |X_{\mathbf{R}_{\geq 2}} \cap X_{\mathbf{B}_{\geq 2}}|$, which is at least $|X_{\mathbf{R}_1}| + |X_{\mathbf{B}_1}| + (|X_{\mathbf{R}_{\geq 2}}| + |X_{\mathbf{B}_{\geq 2}}|)/2$. Since $|X_{\mathbf{R}_1}| = |\mathbf{R}_1|$, $|X_{\mathbf{B}_1}| = |\mathbf{B}_1|$, $|X_{\mathbf{R}_{\geq 2}}| \geq 2|\mathbf{R}_{\geq 2}|$, $|X_{\mathbf{B}_{\geq 2}}| \geq 2|\mathbf{B}_{\geq 2}|$, and $|\mathbf{R}_1| + |\mathbf{R}_{\geq 2}| = |\mathbf{R}| = d_1$ and $|\mathbf{B}_1| + |\mathbf{B}_{\geq 2}| = |\mathbf{B}| = d_2$, then the total number of crossings that are neither red–red nor blue–blue is at least $d_1 + d_2$. Thus the total number of crossings in $\mathcal{D}$, and consequently $\mathrm{cr}(G)$, is at least $X_{RR} + X_{BB} + d_1 + d_2$. On the other hand, $\mathcal{D}_1$ draws G_1 with X_{RR} crossings, and over $\mathcal{D}_1$ the edge u_1v_1 can be drawn with an additional d_1 crossings. Thus G_1' can be drawn with at most $X_{RR} + d_1$ crossings, that is, $X_{RR} + d_1 \geq \mathrm{cr}(G_1')$. A similar argument shows that $X_{BB} + d_2 \geq \mathrm{cr}(G_2')$. Therefore $\mathrm{cr}(G) \geq \mathrm{cr}(G_1') + \mathrm{cr}(G_2')$. □

3 Proof of Theorem 2

We note that no graph with a cut edge is crossing–critical. Thus G is 2-edge-connected.

For each i, let $\overline{u}_i$ (respectively $\overline{v}_i$) denote the vertex in $G - G_i$ adjacent to u_i (respectively v_i), and let $K_i := \{u_i\overline{u}_i, v_i\overline{v}_i\}$. Thus K_i is the 2-edge-cut that separates G_i from $G - G_i$.

Let e be an edge in G_i. We recall that G_i is 2-edge-connected, and so $G_i - e$ is connected. Thus (ii) in Theorem 1 can be applied to the 2-edge-cut K_i on $G - e$ (and as well on G). Therefore $\mathrm{cr}(G - e) = \mathrm{cr}((G_i - e) + u_iv_i) + \mathrm{cr}((G - G_i) + \overline{u}_i\overline{v}_i)$, and $\mathrm{cr}(G) = \mathrm{cr}(G_i + u_iv_i) + \mathrm{cr}((G - G_i) + \overline{u}_i\overline{v}_i)$. On the other hand, the criticality of G implies that $\mathrm{cr}(G - e) < \mathrm{cr}(G)$. Therefore $\mathrm{cr}((G_i + u_iv_i) - e) = \mathrm{cr}((G_i - e) + u_iv_i) < \mathrm{cr}(G_i + u_iv_i)$. Let $G_i' := G_i + u_iv_i$. Thus $\mathrm{cr}(G_i' - e) < \mathrm{cr}(G_i')$ (that is, e is a critical edge in G_i'). Now if $\mathrm{cr}(G_i' - u_iv_i) < \mathrm{cr}(G_i')$, then every edge is critical in G_i', that is, G_i' is crossing–critical. Since in this case we are done, we assume that $\mathrm{cr}(G_i' - u_iv_i) = \mathrm{cr}(G_i')$, that is, $\mathrm{cr}(G_i) = \mathrm{cr}(G_i')$.

Since G_i is a subgraph of G_i' it follows that $\mathrm{cr}(G_i - e) \leq \mathrm{cr}(G_i' - e)$. Since $\mathrm{cr}(G_i' - e) < \mathrm{cr}(G_i') = \mathrm{cr}(G_i)$, this implies that $\mathrm{cr}(G_i - e) < \mathrm{cr}(G_i)$. Since e is an arbitrary edge of G_i, it follows that G_i is crossing–critical. □

4 Proof of Theorem 5

We assume both H_1 and H_2 are connected, since the crossing number of a graph equals the sum of the crossings numbers of its components. Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$, so that $(H_1\#H_2)_{e_1,e_2} = ((H_1 - e_1) \cup (H_2 - e_2)) + u_1u_2 + v_1v_2$. For brevity, we will omit the reference to e_1 and e_2 in $(H_1\#H_2)_{e_1,e_2}$, and simply write $H_1\#H_2$.

The case in which e_i is a cut edge of H_i for both $i = 1$ and 2 follows easily using the fact that for every graph J and a cut edge e of J, $\mathrm{cr}(J)$ equals the sum of the crossing numbers of the components of $J - e$, noting that in this case $H_1\#H_2$ is disconnected, and each of u_1u_2 and v_1v_2 is a cut edge in a component of $H_1\#H_2$.

Suppose now that no e_i is a cut edge of its corresponding H_i. Apply Theorem 1 with $G = H_1\#H_2$, $G_i = H_i - u_iv_i$, and $C = \{u_1u_2, v_1v_2\}$. Each $H_i - u_iv_i$ is connected, so (ii) applies, and then $\mathrm{cr}(H_1\#H_2) = \mathrm{cr}(H_1) + \mathrm{cr}(H_2)$, as claimed.

Finally suppose that e_1 is a cut edge of H_1, but e_2 is not a cut edge of H_2. Let H_1^u, H_1^v denote the components of $H_1 - e_1$ containing u_1 and v_1, respectively. Clearly, u_1u_2 and

v_1v_2 are cut edges in $H_1\#H_2$, and again from the effect of cut edges on crossing number it follows that $\mathrm{cr}(H_1\#H_2) = \mathrm{cr}(H_1^u) + \mathrm{cr}(H_1^v) + \mathrm{cr}(H_2 - e_2)$. Since e_1 is a cut edge of H_1, then $\mathrm{cr}(H_1) = \mathrm{cr}(H_1 - e_1) = \mathrm{cr}(H_1^u) + \mathrm{cr}(H_1^v)$. Thus $\mathrm{cr}(H_1\#H_2) = \mathrm{cr}(H_1) + \mathrm{cr}(H_2 - e_2)$, and so $\mathrm{cr}(H_1\#H_2) = \mathrm{cr}(H_1) + \mathrm{cr}(H_2)$ if and only if $\mathrm{cr}(H_2 - e_2) = \mathrm{cr}(H_2)$. □

5 Concluding Remark

In the present context of understanding the effect of small edge cuts on crossing numbers, we believe the next natural problem to address is the following. Let G be a graph, and $C := [V_1, V_2]$ a 3-edge-cut of G. Let G_1' (respectively G_2') denote the graph that results by contracting V_2 (respectively V_1) to a single vertex. A routine argument shows that $\mathrm{cr}(G) \leq \mathrm{cr}(G_1') + \mathrm{cr}(G_2')$. Give necessary and sufficient conditions under which $\mathrm{cr}(G) \geq \mathrm{cr}(G_1') + \mathrm{cr}(G_2')$. In [1], Bokal gives a sufficient condition. How much can it be weakened?

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