Addendum to “Nearly–light cycles in embedded graphs and crossing–critical graphs”

Mario Lomelí* Gelasio Salazar†

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Abstract

The purpose of this discussion is to give a proof of Theorem 6 in “Nearly–light cycles in embedded graphs and crossing–critical graphs” [1].

Since this is clearly not intended as a stand–alone manuscript, but as an addendum to [1], we proceed right away to state and prove Theorem 6 in that paper.

Theorem 1 (Theorem 6 in [1]). *For each $\varepsilon > 0$ and integer $\chi \le 2$ there exist $\ell_0 := \ell_0(\varepsilon, \chi), \Delta_0 := \Delta_0(\varepsilon, \chi)$, $c := c(\varepsilon, \chi)$ with the following property. Let $G = (V, E)$ be a simple connected graph with minimum degree at least 3, embedded in a surface with Euler characteristic χ. Let F denote the set of faces of G. Then G contains at least $\left(\frac{2}{3} - \varepsilon\right)|F| + c$ face boundaries that are (ℓ_0, Δ_0)–nearly–light.*

The *length* $\ell(f)$ of a face f is the size (number of edges) of a boundary walk of f. We recall the convention that an edge that is traversed twice in the boundary walk contributes in two to this length.

*Facultad de Ciencias, UASLP, San Luis Potosí, SLP. Mexico 78210.
†Instituto de Física, UASLP. San Luis Potosí, SLP. Mexico 78210. gsalazar@ifisica.uaslp.mx

First we prove some simple consequences of Euler's formula.

Proposition 2. *Let $G = (V, E)$ be a simple connected graph with minimum degree at least 3, embedded in a surface with Euler characteristic χ. Let F denote the set of faces of the embedding. Let $\Delta_0 > 3$ and $\ell_0 > 3$ be fixed integers. Let $V_{\geq \Delta_0}$ denote the set of vertices that have degree at least Δ_0, and let $F_{\geq \ell_0}$ denote the set of faces with length at least ℓ_0. Then*

$$|V| \leq 2|F| - 2\chi, \tag{1}$$

$$|V_{\geq \Delta_0}| \leq \left(\frac{3}{\Delta_0 - 3}\right)|V| - \frac{6\chi}{\Delta_0 - 3}, \quad \text{and} \tag{2}$$

$$|F_{\geq \ell_0}| \leq \left(\frac{3}{\ell_0 - 3}\right)|F| - \frac{6\chi}{\ell_0 - 3}. \tag{3}$$

Proof. Since G has minimum degree at least 3, it follows that $3|V| \leq 2|E|$. Thus Euler's formula $|V| - |E| + |F| = \chi$ implies (1). Now since G is connected and simple, $3|F| \leq 2|E|$, and so from Euler's formula it follows that $|E| \leq 3|V| - 3\chi$. Now every vertex with degree smaller than Δ_0 has degree at least 3, and so $2|E| = \sum_{v \in V} d(v) \geq \Delta_0 |V_{\geq \Delta_0}| + 3(|V| - |V_{\geq \Delta_0}|) = 3|V| + (\Delta_0 - 3)|V_{\geq \Delta_0}|$. Since $2|E| \leq 6|V| - 6\chi$, it follows that $3|V| + (\Delta_0 - 3)|V_{\geq \Delta_0}| \leq 6|V| - 6\chi$. This shows (2).

An application of $3|V| \leq 2|E|$ yields $|E| \leq 3|F| - 3\chi$. Since each face with length smaller than ℓ_0 has length at least 3, it follows that $2|E| = \sum_f \ell(f) \geq \ell_0 |F_{\geq \ell_0}| + 3(|F| - |F_{\geq \ell_0}|) = 3|F| + (\ell_0 - 3)|F_{\geq \ell_0}|$. Since $2|E| \leq 6|F| - 6\chi$, it follows that $3|F| + (\ell_0 - 3)|F_{\geq \ell_0}| \leq 6|F| - 6\chi$. This shows (3). □

We now combine Euler's formula with some elementary surface topology facts in order to bound the number of faces that can be incident with more than one vertex of large degree.

Proposition 3. *Let $G = (V, E)$ be a simple connected graph with minimum degree at least 3, embedded in a surface with Euler characteristic χ. Let F denote the set of faces of the embedding. Let $\Delta_0 > 3$ be an integer. Let $V_{\geq \Delta_0}$ denote the set of vertices with degree at least Δ_0, and let F^{Δ_0} denote the set of faces that are incident with at least two vertices in $V_{\geq \Delta_0}$. Then*

$$|F^{\Delta_0}| \leq \frac{|F|}{3} + 4|V_{\geq \Delta_0}| - 4\chi.$$

Proof. We first define an associated graph $H = (V_H, E_H)$, and an embedding Υ of H, as follows. Let $V_H := V_{\geq \Delta_0}$. Now E_H is defined in terms of the faces in F^{Δ_0} in the following way. Let f be a face in F^{Δ_0}, and let W_f be a boundary walk of f. Let $v_1, v_2, \ldots, v_{r-1}, v_r = v_1$ be the vertices in W_f that are in $V_{\geq \Delta_0}$, in the cyclic order in which they appear in W_f. Suppose first that $r \geq 4$. Then, for each $i = 1, 2, \ldots, r-1$, we define an edge (v_i, v_{i+1}) and let it belong to E_H, and we call f the *host* face of each (v_i, v_{i+1}). Now if $r = 3$ (the only other possibility, since $f \in F^{\Delta_0}$), then we let the edge (v_1, v_2) belong to E_H (again, f is the *host* face of (v_1, v_2)). Note that H may have parallel edges. It is a trivial observation that $|F^{\Delta_0}| \leq |E_H|$.

It is easy to use the current embedding Π of G to define an embedding of H: it suffices to let the vertices in $V_H = V_{\geq \Delta_0}$ be embedded as under Π, and clearly every edge in E_H can be embedded inside its host face, in such a way that the result is an embedding Υ of H.

A *1–gon* (respectively, *2–gon*) is a contractible face in Υ whose boundary walk has exactly one edge (respectively, two edges). A (≤ 2)–gon is either an 1–gon or a 2–gon. Now the number of edges that H may have without having any (≤ 2)–gon in Υ is (by a standard Euler characteristic argument) at most the number of edges in a triangulation on $|V_H| = |V_{\geq \Delta_0}|$ vertices, that is, $3|V_{\geq \Delta_0}| - 3\chi$. It follows that Υ has at least $|E_H| - (3|V_{\geq \Delta_0}| - 3\chi)$ (≤ 2)–gons.

Let d be a 2–gon, and let u, v be the two vertices in $V_{\geq \Delta_0}$ in the boundary walk of d. We say that d is *good* if it does not contain an edge of G (embedded under Π) joining u and v. A *good* (≤ 2)–gon is either a 1–gon or a good 2–gon. We observe that at most $3|V_{\geq \Delta_0}| - 3\chi$ 2–gons are not good. Indeed, this follows since no edge (embedded under Π) is contained in more than one 2–gon, and there are (using the same Euler characteristic argument as above) at most $3|V_{\geq \Delta_0}| - 3\chi$ edges of G that have both endpoints in $V_{\geq \Delta_0}$. Thus it follows that Υ has at least $|E_H| - (3|V_{\geq \Delta_0}| - 3\chi) - (3|V_{\geq \Delta_0}| - 3\chi) = |E_H| - 6|V_{\geq \Delta_0}| + 6\chi$ good (≤ 2)–gons.

The crucial observation is that, since G is simple, it follows that each good (≤ 2)–gon contains at least two faces of F that are not in F^{Δ_0}. Thus there are at least $2(|E_H| - 6|V_{\geq \Delta_0}| + 6\chi)$ faces in $F \setminus F^{\Delta_0}$. That is, $|F^{\Delta_0}| \leq |F| - 2(|E_H| - 6|V_{\geq \Delta_0}| + 6\chi)$. Since $|F^{\Delta_0}| \leq |E_H|$, the claimed inequality follows. □

Proof of Theorem 1.

Let G be a simple connected graph with minimum degree at least 3, embedded in a surface with Euler characteristic χ. Let $\varepsilon > 0$ and $\chi \leq 2$ be given. Let $\Delta_0 > 3$ and $\ell_0 > 3$ be (for the time being any) integers.

Combining Proposition 3 with (1) and (2) in Proposition 2, we obtain

$$|F^{\Delta_0}| \leq \left(\frac{1}{3} + \frac{24}{\Delta_0 - 3}\right)|F| - \left(4 + \frac{48}{\Delta_0 - 3}\right)\chi. \tag{4}$$

Let F_{ℓ_0,Δ_0} denote the set of faces that are (ℓ_0, Δ_0)–nearly–light. Thus $F_{\ell_0,\Delta_0} \supseteq F \setminus (F^{\Delta_0} \cup F_{\geq \ell_0})$, and so $|F_{\ell_0,\Delta_0}| \geq |F| - |F^{\Delta_0}| - |F_{\geq \ell_0}|$. Then (3) in Proposition 2 and (4) imply that

$$|F_{\ell_0,\Delta_0}| \geq \left(\frac{2}{3} - \frac{24}{\Delta_0 - 3} - \frac{3}{\ell_0 - 3}\right)|F| + \left(\frac{6\chi}{\ell_0 - 3} - \frac{48}{\Delta_0 - 3} - 4\right)\chi. \tag{5}$$

To conclude the proof, we note that by making ℓ_0 and Δ_0 sufficiently large, we can make the coefficient of $|F|$ in this equation to be at least $2/3 - \varepsilon$. □

References

[1] M. Lomelí and G. Salazar, Nearly–light cycles in embedded graphs and crossing–critical graphs. Manuscript (2005).