On Gibbs measures and lumped Markov chains [1]

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BIRS, October 2, 2007

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[1] Joint work with Jean–René Chazottes, École Polytechnique, Palaiseau

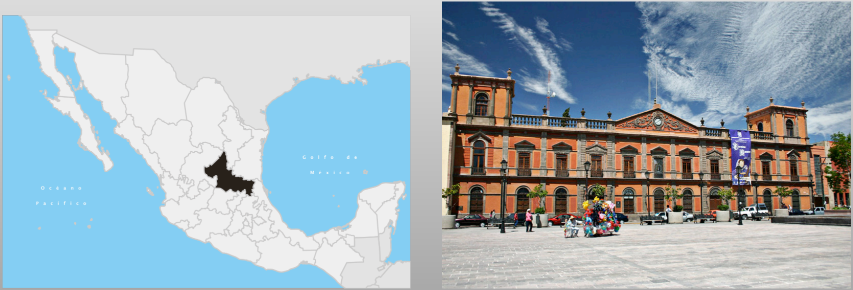

Figure: San Luis Potosí and its University

The Plan

Preliminaries

Gibbs measures

HMC as projections

Ansatz for the potential

Perron-Frobenius theorems

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Let $\{X_n\}_{n\in\mathbb{N}}$ be a stationary process in the finite set (alphabet) A. Assume that there exists a "good function" $\psi : A^{\mathbb{N}} \to \mathbb{R}$ and $K \geqslant 1$ such that for any $n \in \mathbb{N}$ and $a \in A^{\mathbb{N}}$ we have

$$\exp(-K) \leqslant \frac{\mathbb{P}(X_0^n = a_0^n)}{\exp\left(\sum_{j=0}^n \psi(a_j^\infty)\right)} \leqslant \exp(K).$$

Then this process defines a Bowen–Gibbs measure (BGM) $\mu_\psi : \mathcal{B}(A^{\mathbb{N}}) \to [0,1]$ by $\mu_\psi[a_0^n] := \mathbb{P}(X_0^n = a_0^n)$.

A “good function” $\psi : A^{\mathbb{N}} \to \mathbb{R}$ is one such that

$$\mathrm{var}_n(\psi) := \max\{|\psi(a) - \psi(b)| : \; a_0^n = b_0^n\} \leqslant C\theta^n$$

for some $C > 0$ and $0 \leqslant \theta < 1$.

Such function is Hölder continuous with respect to the distance $d : A^{\mathbb{N}} \times A^{\mathbb{N}} \to \mathbb{R}^+$ such that

$$d(a, b) = \exp(-\max\{n \in \mathbb{N} : \; a_0^n = b_0^n\}).$$

A specified subshift $S \subset A^{\mathbb{N}}$ is a closed (with respect to d) and shift–invariant set, for which there exists $\ell \in \mathbb{N}$ such that for each $a, b \in S$ and $m, n \in \mathbb{N}$ there exists $c \in S$ with

$$c_0^m = a_0^m \text{ and } c_{m+\ell+1}^{m+\ell+n+1} = b_0^n.$$

Example. A primitive matrix $M : A \times A \to \{0, 1\}$ $(M^\ell > 0)$ defines the topological Markov chain (TMC)

$$A_M := \{a \in A^{\mathbb{N}} : \ M(a_i, a_{i+1}) = 1 \ \forall \ i \in \mathbb{N}\} \subset A^{\mathbb{N}}.$$

It is a specified subshift.

Theorem (Bowen, Ruelle, and others)

Let $S \subset A^{\mathbb{N}}$ be a specified subshift, and $\psi : S \to \mathbb{R}$ a Hölder continuous function. Then there exists a unique σ–invariant probability measure μ_ψ for which there are constants $P_\psi \in \mathbb{R}$ and $K_\psi \geqslant 1$, such that

$$\exp(-K_\psi) \leqslant \frac{\mu_\psi[a_o^n]}{\exp\left(\sum_{j=0}^{n} \psi(\sigma^j a) - (n+1)P_\psi\right)} \leqslant \exp(K_\psi)$$

for each $a \in S$ and $n \in \mathbb{N}$.

Example. A stationary Markov chain $(X_n)_{n\in\mathbb{N}}$ in A defines a BGM with potential

$$\psi(a) = \log \mathbb{P}(X_0 = a_0 |\ X_1 = a_1).$$

In this case

$$\exp(-K) \leqslant \frac{\mathbb{P}(X_0^n = a_0^n)}{\exp\left(\sum_{j=0}^n \psi(\sigma^j a)\right)} \leqslant \exp(K)$$

with

$$K = \log\left(\max_{a_0^1}\left(\frac{\mathbb{P}(X_0 = a_0)\mathbb{P}(X_1 = a_1)}{\mathbb{P}(X_0^1 = a_0^1)}, \frac{\mathbb{P}(X_0^1 = a_0^1)}{\mathbb{P}(X_0 = a_0)\mathbb{P}(X_1 = a_1)}\right)\right)$$

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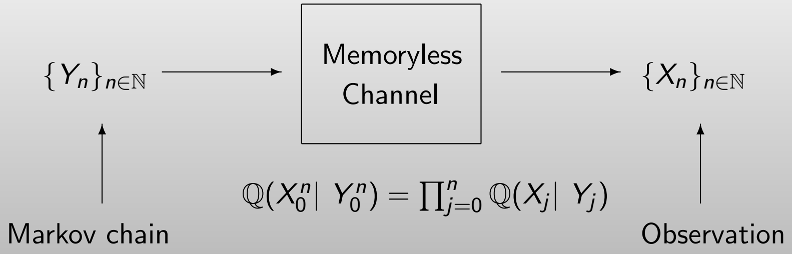

$\{Y_n\}_{n\in\mathbb{N}}$ → Memoryless Channel → $\{X_n\}_{n\in\mathbb{N}}$

$$\mathbb{Q}(X_0^n|\ Y_0^n) = \prod_{j=0}^n \mathbb{Q}(X_j|\ Y_j)$$

Markov chain

Observation

The joint process $\{(Y_n, X_n)\}_{n\in\mathbb{N}}$ is a regular Markov chain,

$$\mathbb{P}((Y_n, X_n)|\ (Y_{n+1}, X_{n+1})) = \mathbb{P}(Y_n|\ Y_{n+1})\mathbb{Q}(X_n|\ Y_n),$$

the HMP is the projection on the second coordinate.

A letter projection $\pi : B \to A$ (many–to–one) and a regular Markov chain $\{Y_n\}_{n\in\mathbb{N}}$ in B define a HMP

$$\begin{tikzcd}[column sep=small]
(R & \subset B^{\mathbb{N}}, & \mu_{\psi}) \arrow[rr, "\pi"] & & (S & \subset A^{\mathbb{N}}, & \nu) \\
\text{TMC} \arrow[u] & & \text{Markovian} \arrow[u] & & \text{Sofic} \arrow[u] & & \text{Gibbsian?} \arrow[u]
\end{tikzcd}$$

Here

$$\begin{aligned}
\mathbb{P}((Y_0^n, X_0^n) = b_0^n) &\quad\rightsquigarrow\quad \mu_{\psi}[b_0^n] \\
\mathbb{P}(X_0^n = a_0^n) &\quad\rightsquigarrow\quad \nu[a_0^n] := \sum_{b_0^n \in \pi^{-1}\{a_0^n\}} \mu_{\psi}[b_0^n]
\end{aligned}$$

with $\psi(b) = \log \mathbb{P}((Y_0, X_0) = b_0 | \, (Y_1, X_1) = b_1)$

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Simplifying framework

$$\begin{tikzcd}
(B_M \arrow[r, phantom, "\subset"] & B^{\mathbb{N}}, \mu_\psi) \arrow[r, "\pi"] & (A_N \arrow[r, phantom, "\subset"] & A^{\mathbb{N}}, \nu) \\
\text{TMC} \arrow[u] & \text{Markovian} \arrow[u] & \text{TMC} \arrow[u] & \text{Gibbsian?} \arrow[u]
\end{tikzcd}$$

For each $aa' \in A \times A$ such that $N(a, a') = 1$ (admissible in A_N) define $\mathcal{M}_{aa'} : \pi^{-1}\{a\} \times \pi^{-1}\{a'\} \to [0, 1]$ such that

$$\mathcal{M}_{aa'}(b, b') = e^{\psi(bb'\cdots)} \equiv \log \mathbb{P}((Y_0, X_0) = b |\ (Y_1, X_1) = b'),$$

and $\mathcal{N}_a : \pi^{-1}\{a\} \to [0, 1]$ such that

$$\mathcal{N}_a(b) = \mu_\psi[b].$$

With this

$$\nu[a_0^n] = \mathbb{1}^\dagger_{a_0} \left(\prod_{j=0}^{n-1} \mathcal{M}_{a_i a_{i+1}} \right) \mathcal{N}_{a_n}$$

Ansatz. Suppose ν is a BGM with potential $\phi : A_N \to \mathbb{R}$ ($\nu = \nu_\phi$), then the Gibbs inequality

$$\exp(-K_\phi) \leqslant \frac{\nu_\phi[a_o^n]}{\exp\left(\sum_{j=0}^{n} \phi(\sigma^j a) - (n+1)P_\phi\right)} \leqslant \exp(K_\phi)$$

suggests

$$\phi(a) = \lim_{n\to\infty} \log\left(\frac{\nu_\phi[a_0^{n+1}]}{\nu_\phi[a_1^{n+1}]}\right) \rightsquigarrow \lim_{n\to\infty} \log \mathbb{P}(Y_0 = a_0 |\ Y_1^n = a_1^n)$$

$$\phi(a) = \lim_{n\to\infty} \log\left(\frac{\mathbb{1}_{a_0}^{\dagger}\left(\prod_{j=0}^{n-1} \mathcal{M}_{a_j a_{j+1}}\right)\mathcal{N}_{a_n}}{\mathbb{1}_{a_1}^{\dagger}\left(\prod_{j=1}^{n-1} \mathcal{M}_{a_j a_{j+1}}\right)\mathcal{N}_{a_n}}\right).$$

For each $a \in A$ let

$$\Delta_a := \left\{ x \in (0,1)^{\pi^{-1}\{a\}} : |x|_1 := \sum_{b \in \pi^{-1}\{a\}} x_b = 1 \right\}$$

For aa' admissible in A_N, the matrix $\mathcal{M}_{aa'}$ is row allowable if for each $x \in \Delta_{a'}$,

$$F_{aa'}x := \frac{\mathcal{M}_{aa'}x}{|\mathcal{M}_{aa'}x|_1} \in \Delta_a.$$

Assuming $\mathcal{M}_{aa'}$ is row allowable for each aa' admissible in A_N, then our ansatz for the potential $\phi : A_N \to \mathbb{R}$,

$$\phi(a) = \lim_{n\to\infty} \log \left(\frac{\mathbb{1}_{a_0}^{\dagger} \left(\prod_{j=0}^{n-1} \mathcal{M}_{a_i a_{i+1}} \right) \mathcal{N}_{a_n}}{\mathbb{1}_{a_1}^{\dagger} \left(\prod_{j=1}^{n-1} \mathcal{M}_{a_i a_{i+1}} \right) \mathcal{N}_{a_n}} \right),$$

can be rewritten as

$$\phi(a) = \lim_{n\to\infty} \log \left(\mathbb{1}_{a_0}^{\dagger} \mathcal{M}_{a_0 a_1} F_{a_1 a_2} \circ \cdots \circ F_{a_{n-1} a_n}\, x_{a_n} \right),$$

with $x_a := \mathcal{N}_a / |\mathcal{N}_a|_1$ for all $a \in A$

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Let $M : E \times E' \to \mathbb{R}^+$. Define

$$\Delta_E := \left\{ x \in (0,1)^E : \ |x|_1 := \sum_{e \in E} x_e = 1 \right\},$$

and similarly $\Delta_{E'}$. Supply Δ_E with the distance

$$\delta_E(x,y) := \max_{e,f \in E} \log\left(\frac{x_e y_f}{x_f y_e}\right),$$

and similarly for $\Delta_{E'}$.

Suppose $M : E \times E' \to \mathbb{R}^+$ is row allowable, i. e.,

$$Fx := \frac{Mx}{|Mx|_1} \in \Delta_E \quad \text{for each } x \in E'.$$

Then define

$$\tau(M) := \frac{1 - \sqrt{\Phi(M)}}{1 + \sqrt{\Phi(M)}},$$

with

$$\Phi(M) = \begin{cases} \min\limits_{\substack{e,f \in E \\ e',f' \in E'}} \dfrac{M(e,e')M(f,f')}{M(e,f')M(f,e')} & \text{if } T > 0 \\ 0 & \text{if } T \geqslant 0\,. \end{cases}$$

Theorem (Contractiveness[2])

Suppose that $M : E \times E' \to \mathbb{R}^+$ *is row allowable, and let* $F : \Delta_{E'} \to \Delta_E$ *be such that*

$$Fx := \frac{Mx}{|Mx|_1} \quad \text{for each } x \in \Delta_{E'}.$$

Then, $\delta_E(Fx, Fy) \leqslant \tau(M)\delta_{E'}(x, y)$, *for all* $x, y \in \Delta_{E'}$.

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[2] Adapted from Seneta

Theorem (Perron–Frobenius[3])

Suppose that $M : E \times E \to \mathbb{R}^+$ *is primitive i. e., it exists* $\ell \in \mathbb{N}$ *such that* $M^\ell > 0$. *Then its maximal eigenvalue* ρ_M *is simple, and it has unique right and left eigenvectors,* $v_M \in \Delta_M$ *and* w_M *such that* $w_M^\dagger v_M = 1$ *respectively. Furthermore, for each* $x \in \Delta_E$ *there exists a constant* C_x *such that*

$$e^{-C_x \tau^{\lfloor n/\ell \rfloor}} \times (w_M^\dagger x) \rho_M^n v_M \leqslant M^n x \leqslant e^{C_x \tau^{\lfloor n/\ell \rfloor}} \times (w_M^\dagger x) \rho_M^n v_M,$$

for each $n \in \mathbb{N}$.

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[3] Adapted from Seneta

Corollary (Convergence)

Suppose that $M : E \times E \to \mathbb{R}^+$ *is primitive, let* $v_M \in \Delta_E$ *be the unique right eigenvector associated to the the maximal eigenvalue of* M*, and define* $F : \Delta_E \to \Delta_E$ *as before. Then, for each* $x \in \Delta_M$ *there is a constant* C_x *such that*

$$\delta_E(F^{\circ n}x, v_M) \leqslant C_x \tau^{\lfloor n/\ell \rfloor} \delta_E(x, v_M),$$

for each $n \in \mathbb{N}$*.*

The potential at periodic points.

$$(B_M \subset B^{\mathbb{N}}, \mu_\psi) \xrightarrow{\pi} (A_N \subset A^{\mathbb{N}}, \nu)$$

Suppose that for $a \in \mathrm{Per}_p(A_N)$ the matrix $\mathcal{M}_{\sigma a} := \mathcal{M}_{a_1 a_2}\mathcal{M}_{a_2 a_3} \cdots M_{a_{p-1} a_0}$ is primitive. Then,

$$\phi(a) := \log(|\mathcal{M}_{a_0 a_1} v_{\sigma a}|_1),$$

where $v_{\sigma a} \in \Delta_{a_1}$ is the unique eigenvector associate to the maximal eigenvalue of $\mathcal{M}_{\sigma a}$.

Results

Full support

Topological Markov chain

The entropy

Lost of gibbsianess

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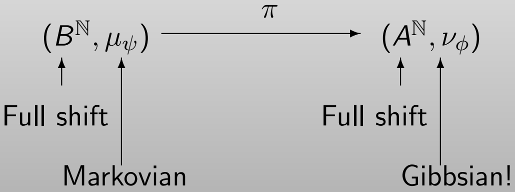

$$\begin{tikzcd}
(B^{\mathbb{N}}, \mu_\psi) \arrow[r, "\pi"] & (A^{\mathbb{N}}, \nu_\phi)
\end{tikzcd}$$

Full shift

Markovian

Full shift

Gibbsian!

⇝Finite range potential

Let $S \subset B^N$ be a subshift. The potential $\psi : S \to \mathbb{R}$ is of finite range if there exists $r \in \mathbb{N}$ such that

$$\psi(b) = \psi(c) \text{ whenever } b_0^r = c_0^r.$$

The minimal such r is the range of the potential.

A Markov measure a BGM for a potential of range $r = 1$.

Theorem (Full shift version)

Let $\psi : B^{\mathbb{N}} \to \mathbb{R}$ be a finite range potential, and $\pi : B^{\mathbb{N}} \to A^{\mathbb{N}}$ a letter–to–letter projection. Let μ_ψ be the unique BGM measure associated to ψ. Then, measure $\nu = \mu_\psi \circ \pi^{-1}$ is a BGM associate to the potential $\phi : A^{\mathbb{N}} \to \mathbb{R}$ such that

$$\phi(a) := \lim_{n\to\infty} \log\left(|\mathcal{M}_{a_0 a_1} F_{a_1 a_2} \circ \cdots \circ F_{a_{n-1} a_n}\, x_{a_n}|_1\right),$$

for each $a \in A^{\mathbb{N}}$.

Sketch of the proof for $r = 1$.

- Each elementary cycle $c \in \cup_{m=2}^{\#A+1} A^m$ defines a positive matrix

$$\mathcal{M}_c := \prod_{i=0}^{|c|-2} \mathcal{M}_{c_i c_{i+1}} : \pi^{-1}\{c_0\} \times \pi^{-1}\{c_0\} \to \mathbb{R}^+.$$

- The transformation $F_c : \Delta_{c_0} \to \Delta_{c_0}$ corresponding to the matrix $\mathcal{M}_c$ is a contraction with (projective) contraction coefficient $\tau_c \in (0,1)$.

Sketch of the proof

- To each $a \in A^{\mathbb{N}}$ it corresponds a sequence of elementary cycles $c^{(0)} c^{(1)} \cdots$ so that

  $$a = a_0 \left[l^{(0)} c^{(0)} r^{(0)} \right] \left[l^{(1)} c^{(1)} r^{(1)} \right] \cdots \left[l^{(k)} c^{(k)} r^{(k)} \right] \cdots ,$$

  with $l^{(n)} c^{(n)} r^{(n)} \in A^{\# A + 1}$.
- For each $k \geqslant 0$, the linking map $F_{r l^{(k)}} : \Delta_{c_0^{(k+1)}} \to \Delta_{c_0^{(k)}}$ corresponding to the product matrix

  $$\mathcal{M}_{c_0^{(k)} r^{(k)} l^{(k+1)} c_0^{(k+1)}} : \pi^{-1} \left\{ c_0^{(k)} \right\} \times \pi^{-1} \left\{ c_0^{(k+1)} \right\} \to \mathbb{R}_0^+ ,$$

  is non–expansive. The same holds for the ending map $F_{l^{(0)}} : \Delta_{c_0^{(0)}} \to \Delta_{a_1}$.

Sketch of the proof

- Let

$$x_n := F_{l(0)} \circ F_{c(0)} \circ F_{rl(0)} \circ \cdots \circ F_{rl(k-1)} \circ F_{c(k)} \circ F_{r(k)} \circ R_n \; x_{a_n},$$

where $k = \lfloor n/(\#A+1) \rfloor$ and

$$R_n := F_{a_{k(\#A+1)}a_{k(\#A+1)+1}} \circ F_{a_{n-1}a_n} : \Delta_{a_n} \to \Delta_{a_{k(\#A+1)}}$$

is the n–starting map, which is also non–expansive.

- The sequence $\{x_n\}_{n \geqslant \#A+1}$ in Δ_{a_1} satisfies

$$\delta_{a_1}(x_n, x_m) \leqslant C \left(\max_{c \text{ elementary cycle}} \tau_c \right)^k,$$

for some constant C.

Sketch of the proof

- With this we have

  $$\phi(a) := \lim_{n\to\infty} \log(|\mathcal{M}_{a_0 a_1} x_n|_1) \text{ exits.}$$

  Furthermore, there exists $C' > 0$ and $\theta \in [0,1)$ such that

  $$|\phi(a) - \phi(\bar{a})| \leqslant C'\theta^n,$$

  for each $\bar{a} \in A^{\mathbb{N}}$ such that $a_0^n = \bar{a}_0^n$.
- Finally, the Ruelle–Bowen Theorem implies that if

  $$\phi(a) := \lim_{n\to\infty} \log\left(\frac{\nu[a_0^n]}{\nu[a_1^n]}\right)$$

  exists and is Hölder, then $\nu = \nu_\phi$.

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Theorem (TMC Version)

Let B_M be a specified TMC and $\pi : B \to A$ a letter projection.

Suppose that for each aa' admissible in $\pi(B_N)$, the matrix $M_{aa'} : \pi^{-1}\{a\} \times \pi^{-1}\{a'\} \to \{0,1\}$ such that $M_{aa'}(bb') = M(b,b')$ is row allowable.

Let $\mu : \mathcal{B}(B_M) \to [0,1]$ be a Markov measure, and suppose that for each $a \in \mathrm{Per}_p(\pi(B_M))$ with $1 \leqslant p \leqslant \#B$, the product matrix $\mathcal{M}_a := \prod_{i=0}^{p-1} \mathcal{M}_{a_i a_{i+1}} : \pi^{-1}\{a\} \times \pi^{-1}\{a'\} \to [0,1]$ is positive.

Then there exists a primitive matrix $N : A \times A \to \{0,1\}$ such that $\pi(B_M) = A_N$, and a Hölder continuous potential $\phi : A_N \to \mathbb{R}$ such that $\nu = \mu \circ \pi^{-1}$ is the BGM associate to ϕ.

The proof follows the same idea, and the induced potential is given by

$$\begin{aligned}\phi(a) &:= \lim_{n\to\infty}\log\left(\frac{\nu[a_0^n]}{\nu[a_1^n]}\right)\\ &\equiv \lim_{n\to\infty}\log\left(\frac{\mathbb{1}_{a_0}^{\dagger}\left(\prod_{j=0}^{n-1}\mathcal{M}_{a_ia_{i+1}}\right)\mathcal{N}_{a_n}}{\mathbb{1}_{a_1}^{\dagger}\left(\prod_{j=1}^{n-1}\mathcal{M}_{a_ia_{i+1}}\right)\mathcal{N}_{a_n}}\right).\end{aligned}$$

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$$\begin{tikzcd}
(B_M, \mu_\psi) \arrow[rr, "\pi"] & & (A_N, \nu_\phi) \\
\text{TMC} \arrow[u] & & \text{TMC} \arrow[u] \\
\text{Markovian} + \text{hypotheses} \arrow[uu] & & \text{Gibbsian!} \arrow[uu]
\end{tikzcd}$$

Under the hypotheses of one of the previous theorem, the entropy of the induced measure $\nu_\phi := \mu \circ \pi^{-1}$ satisfies

$$h_{\nu_\phi}(\sigma) = -\lim_{p\to\infty} \frac{\sum_{a\in\mathrm{Per}_p(A_N)} \phi(a) e^{\sum_{j=0}^{p-1} \phi(\sigma^j a)}}{\sum_{a\in\mathrm{Per}_p(A_N)} e^{\sum_{j=0}^{p-1} \phi(\sigma^j a)}}$$

Notice that

$$\phi(a) := \log(|\mathcal{M}_{a_0 a_1} v_{\sigma a}|_1),$$

where $v_{\sigma a} \in \Delta_{a_1}$ is the unique eigenvector associate to the maximal eigenvalue of $\mathcal{M}_{\sigma a} := \mathcal{M}_{a_1 a_2}\mathcal{M}_{a_2 a_3} \cdots M_{a_{p-1} a_0}$.

Furthermore, we can find constants[4] $C > 0$ and $\theta \in [0,1)$ such that

$$\left| h_{\nu_\phi}(\sigma) + \frac{\sum_{a \in \mathrm{Per}_p(A_N)} \phi(a) e^{\sum_{j=0}^{p-1} \phi(\sigma^j a)}}{\sum_{a \in \mathrm{Per}_p(A_N)} e^{\sum_{j=0}^{p-1} \phi(\sigma^j a)}} \right| \leqslant C\theta^{\sqrt{p}}.$$

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[4] Adapted from Ramirez–Chazottes–Ugalde 2005

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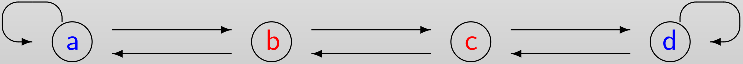
Example



$$\begin{tikzcd}(B_M, \mu) \arrow[d, "\pi"] \\ (A_N, \nu)\end{tikzcd}$$

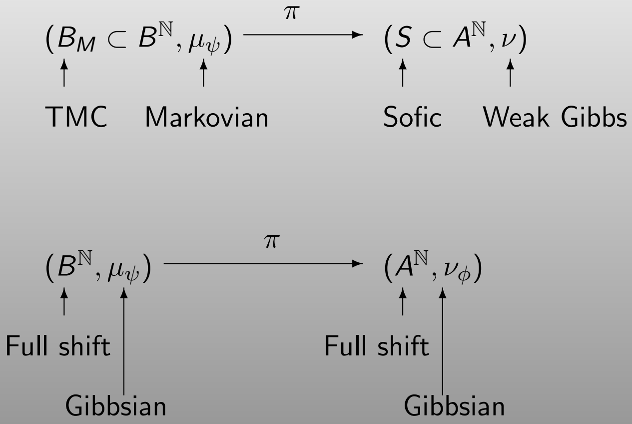
$$\mathcal{M} := \left(\begin{array}{cc|cc} 0 & \gamma & 1-\gamma-\beta & \beta \\ 1/2 & 0 & \beta & 1/2-\beta \\ \hline 1/2-\beta & 1/2+\beta-\gamma & \gamma & 0 \\ \beta & 1/2-\beta & 0 & 1/2 \end{array}\right)$$

The matrix $\mathcal{M}$ is double stochastic, it defines a Markov measure μ in A_M. The induced measure is not gibbsian since

$$|\mathcal{M}_{00} F_{00}^{\circ n} x_0|_1 = \begin{cases} (2\gamma + 1)/4 & \text{if } n \text{ is even,} \\ \gamma/(2\gamma + 1) & \text{if } n \text{ is odd.} \end{cases}$$

Hence, for $\gamma \neq 1/2$, the value $\phi(0000\cdots)$ cannot be defined.

To be done



$$\begin{tikzcd}(B_M \subset B^{\mathbb{N}}, \mu_\psi) \arrow[r, "\pi"] & (S \subset A^{\mathbb{N}}, \nu)\end{tikzcd}$$

TMC
Markovian
Sofic
Weak Gibbs

$$\begin{tikzcd}(B^{\mathbb{N}}, \mu_\psi) \arrow[r, "\pi"] & (A^{\mathbb{N}}, \nu_\phi)\end{tikzcd}$$

Full shift
Gibbsian
Full shift
Gibbsian

¡Muchas Gracias!

Presentation on line at
`http://www.ifisica.uaslp.mx/~ugalde`