Preservation of Gibbsianity under amalgamation of symbols, or Lumped Gibbs measures are Gibbsian [1]

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[1] Joint work with Jean–René Chazottes, École Polytechnique, Palaiseau

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The Plan

Framework

Let A be a finite alphabet. Supply $A^{\mathbb{N}}$ with the distance $d : A^{\mathbb{N}} \times A^{\mathbb{N}} \to \mathbb{R}^+$ such that

$$d(\mathbf{a}, \mathbf{c}) = \exp\left(-\max\left\{n \in \mathbb{N} :\ \mathbf{a}_0^n = \mathbf{c}_0^n\right\}\right),$$

and consider the dynamics $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ such that

$$\sigma \mathbf{a}_n = \mathbf{a}_{n+1}\ \forall\, n \in \mathbb{N}.$$

For each $\mathbf{a} \in A^{\mathbb{N}}$ and $n \in \mathbb{N}$ let $[\mathbf{a}_0^n] := \{\mathbf{c} \in A^{\mathbb{N}} : \mathbf{c}_0^n = \mathbf{a}_0^n\}$

Consider $\psi \to \mathbb{R}$ such that

$$\mathrm{var}_{\mathrm{n}}\psi := \max\{|\psi(\mathbf{a}) - \psi(\mathbf{c})| : \mathbf{a}_0^{\mathrm{n}} = \mathbf{c}_0^{\mathrm{n}}|\} \to 0 \text{ when } \mathrm{n} \to \infty.$$

A σ–invariant Borel measure μ is Gibbsian with potential ψ if there are constants $K \geq 0$ and $P \in \mathbb{R}$ such that

$$\exp(-K) \leq \frac{\mu[\mathbf{a}_0^n]}{\exp\left(\sum_{j=0}^n \psi(\sigma^j \mathbf{a}) - (n+1)P\right)} \leq \exp(K)$$

for each $\mathbf{a} \in A^{\mathbb{N}}$ and $n \in \mathbb{N}$.

A regular potential $\psi : A^{\mathbb{N}} \to \mathbb{R}$ is one such that $\sum_{n\geq 0} \mathrm{var_n}\psi < \infty$.

Theorem (Bowen, Ruelle, and others)

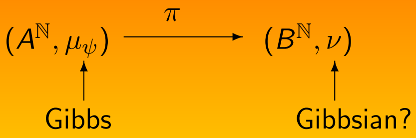
If $\psi : A^{\mathbb{N}} \to \mathbb{R}$ is regular then there exists a unique σ–invariant probability measure μ_ψ for which there are constants $P \in \mathbb{R}$ and $K \geq 0$, such that

$$\exp(-K) \leq \frac{\mu_\psi[a_o^n]}{\exp\left(\sum_{j=0}^{n} \psi(\sigma^j a) - (n+1)P\right)} \leq \exp(K)$$

for each $a \in A^{\mathbb{N}}$ and $n \in \mathbb{N}$.

The Problem

Let A and B be finite alphabets with $\#A > \#B$ and $\pi : A \to B$ a surjective map. Extend coordinatewise this map to $\pi : A^{\mathbb{N}} \to B^{\mathbb{N}}$ such that $(\pi \mathbf{a})_n = \pi(\mathbf{a}_n)$ for all $\mathbf{a} \in A^{\mathbb{N}}$ and $n \in \mathbb{N}$.



If $\mathrm{var_n}\psi \leq \mathrm{C}\theta^{\mathrm{n}}$ for some $C > 0$ and $\theta \in [0,1)$, then ψ is Hölder continuous, and if $\mathrm{var_n}\psi = 0$ for all $n \geq r$, then ψ is of range r.

Theorem (Chazottes & U.)

Let A, B and π be as above. If $\psi : A^{\mathbb{N}} \to \mathbb{R}$ is of range r then $\nu = \mu_\psi \circ \pi^{-1}$ is Gibbsian with potential $\phi : B^{\mathbb{N}} \to \mathbb{R}$ such that

$$\phi(\mathbf{b}) := \lim_{n\to\infty} \log\left(\frac{\nu[\mathbf{b}_0^n]}{\nu[\mathbf{b}_1^n]}\right).$$

Furthermore, there is a constant $C > 0$ such that

$$\mathrm{var_n}\phi \leq \mathrm{C}\left(1 - \exp\left(-\sum_{\mathrm{n=0}}^{\mathrm{r}} \mathrm{var_n}\psi\right)\right)^{\mathrm{n/r}}.$$

Theorem (Lumped Gibbsian is Gibbsian)

Let A, B and π be as above. If $\sum_{r\geq 0} r^{2+\alpha}\mathrm{var}_r\phi < \infty$ for some $\alpha > 0$, then $\nu := \mu_\psi \circ \pi^{-1}$ is Gibbsian, with potential $\phi : B^{\mathbb{N}} \to \mathbb{R}$ such that

$$\phi(\mathbf{b}) := \lim_{n\to\infty} \log\left(\frac{\nu[\mathbf{b}_0^n]}{\nu[\mathbf{b}_1^n]}\right).$$

Furthermore,

$$\mathrm{var_n}\phi \leq \mathrm{C}\sum_{\mathrm{r}\geq\sqrt{\mathrm{n}}} \mathrm{r}^{1+\alpha}\mathrm{var_r}\psi.$$

Refined Perron–Frobenius

Let $M : E \times E' \to (0, \infty)$. Define

$$\Delta_E := \left\{ x \in (0,1)^E : \ |x|_1 := \sum_{e \in E} x_e = 1 \right\},$$

and similarly for $\Delta_{E'}$. Supply Δ_E with the distance

$$\delta_E(x, y) := \max_{e,f \in E} \log\left(\frac{x_e y_f}{x_f y_e}\right),$$

and similarly for $\Delta_{E'}$.

Theorem (Contractiveness[2])

Suppose that $M : E \times E' \to (0,\infty)$ *is and let* $F : \Delta_{E'} \to \Delta_E$ *be such that*

$$Fx := \frac{Mx}{|Mx|_1} \quad \text{for each } x \in \Delta_{E'}.$$

Then, $\delta_E(Fx, Fy) \le \tau(M)\delta_{E'}(x,y)$, *for all* $x, y \in \Delta_{E'}$, *where*

$$\tau(M) := \frac{1-\sqrt{\Phi(M)}}{1+\sqrt{\Phi(M)}},$$

with

$$\Phi(M) = \min_{\substack{e,f\in E\\ e',f'\in E'}} \frac{M(e,e')M(f,f')}{M(e,f')M(f,e')}.$$

[2] Adapted from Seneta

Sketch of the Proof

It is enough to proof that

$$\mathbf{b} \mapsto \phi(\mathbf{b}) := \lim_{n\to\infty} \log\left(\frac{\nu[\mathbf{b}_0^n]}{\nu[\mathbf{b}_1^n]}\right)$$

exists for each $\mathbf{b} \in B^{\mathbb{N}}$, and defines a continuous function such that $\sum_{n=0}^{\infty} \mathrm{var_n}\phi < \infty$.

Now, for each $r \in \mathbb{R}$ let $\psi_r : A^{\mathbb{N}} \to \mathbb{R}$ be such that

$$\psi_r(\mathbf{a}) := \max\{\psi(\mathbf{c}) : \mathbf{c}_0^r = \mathbf{a}_0^r\}$$

Denote by μ_{ψ_r} the Gibbs measure associated to ψ_r, and let $\nu_r := \mu_{\psi_r} \circ \pi^{-1}$. It is well known that $\lim_{r\to\infty} \nu_r[\mathbf{b}_0^n] = \nu[\mathbf{a}_0^n]$ for each $\mathbf{a} \in A^{\mathbb{N}}$ and $n \in \mathbb{N}$.

We have proved that (Chazottes, Ramirez & U.),

$$\left|\log\left(\frac{\nu_r[\mathbf{b}_0^n]}{\nu_r[\mathbf{b}_1^n]}\right) - \log\left(\frac{\nu[\mathbf{b}_0^n]}{\nu[\mathbf{b}_1^n]}\right)\right| \leq D\sum_{s\geq r} s^{1+\alpha}\mathrm{var}_\mathrm{s}\psi,$$

for some $D > 0$ and $s_\psi := \sum_{n\geq 0}\mathrm{var}_\mathrm{n}\psi$, as long as $n < r^{1+\alpha}$.

We also known (Chazottes & U.) that it exists a Hölder continuous function $\phi_r : B^{\mathbb{N}} \to \mathbb{R}$ such that

$$\left|\log\left(\frac{\nu_r[\mathbf{b}_0^n]}{\nu_r[\mathbf{b}_1^n]}\right) - \phi_r(\mathbf{b})\right| \leq D' r\left(1 - e^{-s_\psi}\right)^{n/r}$$

for some $D' > 0$.

It follows that $\phi_r \to \phi$ uniformly, and

$$|\phi(\mathbf{b}) - \phi_r(\mathbf{b})| \leq C' \sum_{s \geq r} s^{1+\alpha} \mathrm{var_s} \psi$$

with $C' = 4 \max(D, D')$.

It also follows that

$$|\phi(\mathbf{b}) - \phi(\mathbf{d})| \leq C \sum_{r \geq \sqrt{n}} s^{1+\alpha} \mathrm{var_r} \psi$$

for $\mathbf{b}, \mathbf{d} \in B^{\mathbb{N}}$ such that $\mathbf{b}_0^n = \mathbf{d}_0^n$.

Final Comments

- Hölder to Hölder?
- Loss of Gibbsianity?

¡Muchas Gracias!

Presentation on line at
`http://www.ifisica.uaslp.mx/~ugalde`