Approximation of equilibrium states

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Information and Randomness 2010

Outline

What is an equilibrium state?

The physical system is represented by a metric space X.

The energy landscape by a continuous function $\psi : X \to \mathbb{R}$.

The states are Borel probability measures on X.

An equilibrium state is a Borel probability measure μ satisfying

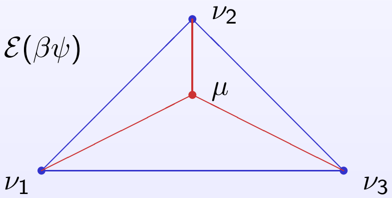
$$\int_X \psi(x)\, d\mu(x) - \frac{1}{\beta}\, h(\mu) = \inf_{\nu \text{ is a state}} \left\{ \int_X \psi(x)\ d\mu(x) - \frac{1}{\beta}\, h(\nu) \right\}$$

where $h(\nu)$ denotes the entropy of the state ν and $\beta = 1/T$ denotes the inverse temperature.

What is an equilibrium state?

$\mathcal{E}(\beta\psi) = \{$equilibrium states at inverse temperature $\beta\}$.

$\mathcal{E}(\beta\psi)$ is a Choquet Simplex.



$$\mu = \int_{\mathcal{I}} \nu \; d\eta(\nu)$$

$$\mathcal{I} := \{\text{extrema of } \mathcal{E}(\beta\psi)\}$$

What is a lattice system?

Given a finite set of possible spin values A, the configuration space is the product $X = A^{\mathbb{Z}}$.
It is a compact metric space with the distance

$$d(x, y) = e^{-\min\{n:\, x_{-n}^{n} \neq y_{-n}^{n}\}}.$$

The spacial translation is the map $\sigma : X \to X$ such that

$$(\sigma x)_n = x_{n+1}.$$

The energy landscape is then defined by a continuous function

$$\psi : X \to \mathbb{R}.$$

What is a lattice system?

We will consider translation–invariant states

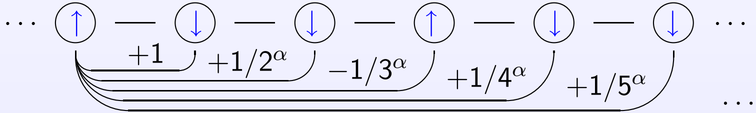
$$\mathcal{E}_\sigma(\beta\psi) = \{\text{translation invariant states in } \mathcal{E}(\beta\psi)\}.$$

In this case

$$\begin{aligned} \mathcal{I}_\sigma(\beta\psi) &= \{\text{the extrema of } \mathcal{E}_\sigma(\beta\psi)\} \\ &= \{\text{all the ergodic measures in } \mathcal{E}_\sigma(\beta\psi)\}. \end{aligned}$$

Example

$$X = \{-1,1\}^{\mathbb{Z}}, \quad \psi(x) = -\frac{1}{2}\sum_{r=1}^{\infty} \frac{x_0\,(x_r + x_{-r})}{r^{\alpha}}$$



Summability & Gibbs measures

What is known for lattice systems?

- (Existence) If ψ is continuous, then $\mathcal{E}_\sigma(\beta\psi) \neq \emptyset$.
- (Uniqueness) For

$$\psi(x) = \frac{1}{2}\sum_{r=0}^{\infty}\left(\psi_r(x_0, x_r) + \psi_r(x_0, x_{-r})\right),$$

  if $2\,\beta\,\sum_{r=1} ||\psi_r|| < 1$, then $\#\mathcal{E}_\sigma(\beta\psi) = 1$.
- (Uniqueness) Let

$$\mathrm{var}_n\psi := \sup\{|\psi(x) - \psi(y)| : \ d(x,y) \leqslant \exp(-n)\}.$$

  If $\sum_{n=0}^{\infty} \mathrm{var}_n\psi < \infty$ then $\#\mathcal{E}_\sigma(\beta\psi) = 1$.

Summability & Gibbs measures

Under the condition $\sum_{n=0}^{\infty} \mathrm{var}_n \psi < \infty$ we have that

$$C^{-1} \leqslant \frac{\mu_{\beta\psi}([x_m^n])}{e^{-\beta\,(U_{m-n}(\sigma^m x)-(n-m)f(\beta))}} \leqslant C$$

Where

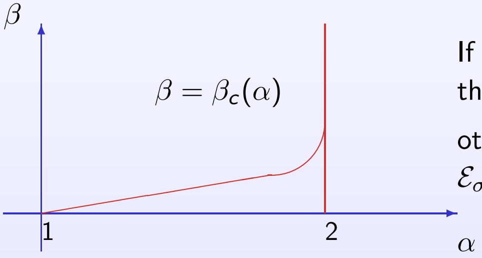
$$f(\beta) := \inf_{\nu \in \mathcal{E}_\sigma(\beta\psi)} \left\{ \mathbb{E}_\nu(\psi) - \frac{1}{\beta}\, h(\nu) \right\}$$

and $U_n(x) = \sum_{k=0}^{n-1} \psi(\sigma^k(x))$.

Coexistence

In the example

$$X = \{-1,1\}^{\mathbb{Z}}, \quad \psi(x) = -\frac{1}{2}\sum_{r=1}^{\infty} \frac{x_0\,(x_r + x_{-r})}{r^{\alpha}}$$



If $\beta \leqslant \beta_c(\alpha)$ or $\alpha > 2$
then $|\mathcal{E}_\sigma(\beta\psi)| = 1$

otherwise
$\mathcal{E}_\sigma(\beta\psi) \simeq [0,1]$

Markovian approximations

For each r let $\bar{\psi}_r : A^{2r+1} \to \mathbb{R}$ be such that

$$\bar{\psi}_r(u) := \max\{\psi(x) : \; x_{-r}^{r} = u\}.$$

$\bar{\psi}_r(u)$ admits a unique equilibrium state $\mathcal{M}_{r,\beta}$ for each $\beta \in \mathbb{R}$.

$\mathcal{M}_{r,\beta}$ is a $2r$–step Markov chain with transition probabilities

$$\mathcal{M}_{r,\beta}(x_r | x_{-r}^{r-1}) = \frac{e^{-\beta\bar{\psi}_r(x_{-r}^{r})} \, R_r(x_{-r+1}^{r})}{\rho_r \, R_r(x_{-r}^{r-1})},$$

where ρ_r is the maximal eigenvalue and R_r the associated eigenvector of the transition matrix

$$M_r(x_{-r}^{r-1}, x_{-r+1}^{r}) = e^{-\beta\bar{\psi}_r(x_{-r}^{r})}$$

Atomic approximations

For each $p \in \mathbb{N}$ we consider the atomic measure $\mathcal{A}_{p,\beta}$ such that

$$\mathcal{A}_{p,\beta}(x) = \frac{e^{-\beta U_{p-1}(x)}}{\sum_{y \text{ is } p\text{–periodic}} e^{-\beta U_{p-1}(x)}}$$

for each p–periodic sequence $x \in X$.

Here $U_{p-1}(x) = \sum_{k=0}^{p-1} \psi(\sigma^k(x))$.

Note that $\operatorname{supp}(\mathcal{A}_{p,\beta}) = \{p\text{–periodic sequences in } X\}$.

Convergence

Theorem (with Moreno)

Suppose $\sum_r r^{1+q}\mathrm{var}_r\psi < \infty$, then there is a constant $C > 0$ such that for each β

$$\max\left(D\left(\mathcal{M}_{r,\beta}, \mu_{\beta\psi}\right), D\left(\mathcal{A}_{r^{1+q}}, \mu_{\beta\psi}\right)\right) \leqslant C\,\beta, \sum_{s\geqslant r} s^{1+q}\mathrm{var}_s\psi.$$

Here D is a particular distance compatible with the *weak topology.

Let us remind that

$$\mathrm{var}_n\psi := \sup\{|\psi(x) - \psi(y)| : \ d(x,y) \leqslant \exp(-n)\}.$$

Decay of correlations

Theorem (with Moreno)

Suppose $\sum_r r^{1+q}\mathrm{var}_r\psi < \infty$, then there is a constant $C > 0$ and for each $\ell \in \mathbb{N}$ there exists s_0 such that for each β

$$\left|\frac{\mu_{\beta\psi}([u] \cap \sigma^{-s}[w])}{\mu_{\beta\psi}([u])\,\mu_{\beta\psi}([w])} - 1\right| \leqslant C\,\beta \sum_{r \geqslant s^{1/(1+q)}} r^{1+q}\mathrm{var}_r\psi,$$

for each $u, w \in A^\ell$ and all $s \geqslant s_0$.

In particular, if $f, g : X \to \mathbb{R}$ are locally constant, then

$$|\mathbb{E}_{\beta\psi}(f \circ g^s) - \mathbb{E}_{\beta\psi}(f)\mathbb{E}_{\beta\psi}(g)| \leqslant ||f||\,||g||\,C\,\beta \sum_{r \geqslant s^{1/(1+q)}} r^{1+q}\mathrm{var}_r\psi$$

for eventually all s.

The limit ground state

If ψ is locally constant, then $\mu^*_\psi := \lim_{\beta\to\infty} \mu_{\beta\psi}$ exists (Brémont 2003, Laipledeur 2006).

We have that $\mathrm{supp}(\mu^*_\psi) = \sqcup_{k=1}^{N} \bar{X}_k$, where each $\bar{X}_k$ is a transitive subshift of finite type and $\mu^*_\psi = \sum_{k=1}^{N} c_k\ \mu_k$, where μ_k is the measure of maximal entropy on $\bar{X}_k$.

Theorem (with Chazottes and Gambaudo)

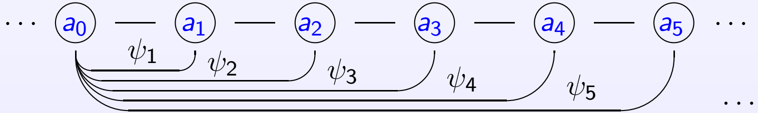
There exists to compute the coefficients c_k. Furthermore, there exists β_0 and $C > 0$ such that

$$|\mu_{\beta\psi}[x_0^n] - c_k \mu_k [x_0^n]| \leqslant e^{-\beta C}$$

for each $\beta \geqslant \beta_0$ all $x \in X$ such that $[x_0^n] \cap \bar{X}_k \neq \emptyset$.

Refinements for pair interactions

$$X = A^{\mathbb{Z}}, \quad \psi(x) = \frac{1}{2}\sum_{r=0}^{\infty}(\psi_r(x_0, x_r) + \psi_r(x_0, x_{-r}))$$



Convergence for pair interactions

Proposition (with Moreno and Salgado)

Let us suppose that $\psi(x) = \frac{1}{2}\sum_{r=0}^{\infty}\left(\psi_r(x_0, x_r) + \psi_r(x_0, x_{-r})\right)$ *and that* $\sum_{s=1}^{r} s\,\mathrm{var}_1\psi_s \leqslant c\,\log(r)$ *for some* $c > 0$*.*
If $\beta < 1/c$ *then* $\lim_{r\to\infty}\mu_r$ *exists and*

$$D(\mu_r, \mu_{\beta\psi}) \leqslant C\,\beta\,r^{-(1-c\beta)/3}$$

for some constant $C > 0$*.*

In the example

$$X = \{-1, 1\}^{\mathbb{Z}}, \quad \psi(x) = -\frac{1}{2}\sum_{r=1}^{\infty}\frac{x_0\,(x_r + x_{-r})}{r^{\alpha}}$$

If $\alpha > 2$ then μ_r converge to $\mu_{\beta\psi}$ for all β.
If $\alpha = 2$ then μ_r converge to $\mu_{\beta\psi}$ for all $\beta < 1/2$.
The classical criterium ensures uniqueness for $\beta < 3/\pi^2$

The transition regime

Theorem (with Moreno and Salgado)

For $\psi(x) = -\frac{1}{2}\sum_{r=1}^{\infty}(x_0\,(x_r + x_{-r}))r^{-\alpha}$ *with* $\alpha > 4/3$, *if* $\limsup_{r\to\infty} \lambda_r/\rho_r < 1$, *then* $\lim_{r\to\infty} \mu_r$ *exists and*

$$D(\mu_r, \mu_{\beta\psi}) \leqslant Cr^{1-3\alpha/4}$$

for some constant $C > 0$.

Here $\lambda_r = \max\{|\mu| : \ \mu \in \mathrm{spec}(M_r) \setminus \{\rho_r\}\}$.

From a theorem by Weil we have

$$\frac{\lambda_r}{\rho_r} \leqslant \frac{\sigma_r^{(1)}\sigma_r^{(2)}}{\rho_r^2},$$

where $\sigma_r^{(1)}$ y $\sigma_r^{(2)}$ are the two largest singular values of M_r.

Singular values can be easily computed, and are such that

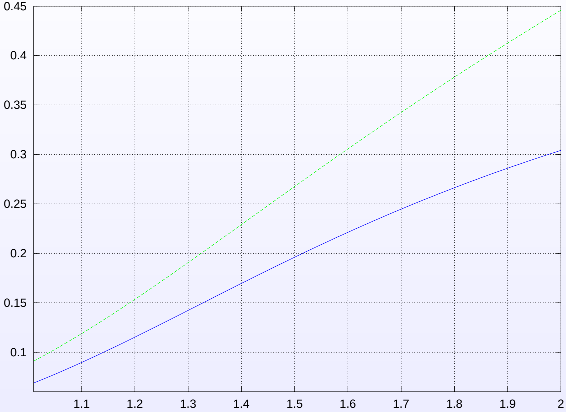
$$\lim_{r\to\infty} \sigma_r^{(i)} = 2\cosh(2\beta\zeta(a)),$$

for $i = 1, 2$.

As a byproduct of the Perron–Frobenius theorem we have the lower bound

$$\rho_r \geqslant 2\cosh\left(\beta\sum_{k=1}^{r}(-1)^{k+1}\ k^{-\alpha}\right).$$

The critical line



We have the following lower bound for the critical line

$$\beta^* = \beta^*(\alpha): \ \text{satisfying} \cosh(2\beta^*\zeta(\alpha)) = 2\cosh(\beta^*\mathcal{Z}(\alpha)),$$

where $\mathcal{Z}(\alpha) := \sum_{k=1}^{\infty}(-1)^{k+1}k^{-\alpha}$.

Final remarks

- All estimates are based on a refinement of the classical Perron–Frobenius Theorem.
- In all cases we can estimate the decay of correlations for the limit measure from the decay of correlations of the Markovian approximations.
- The Markovian and atomic approximations "sense" the phase transition.

¡Muchisimas Gracias!