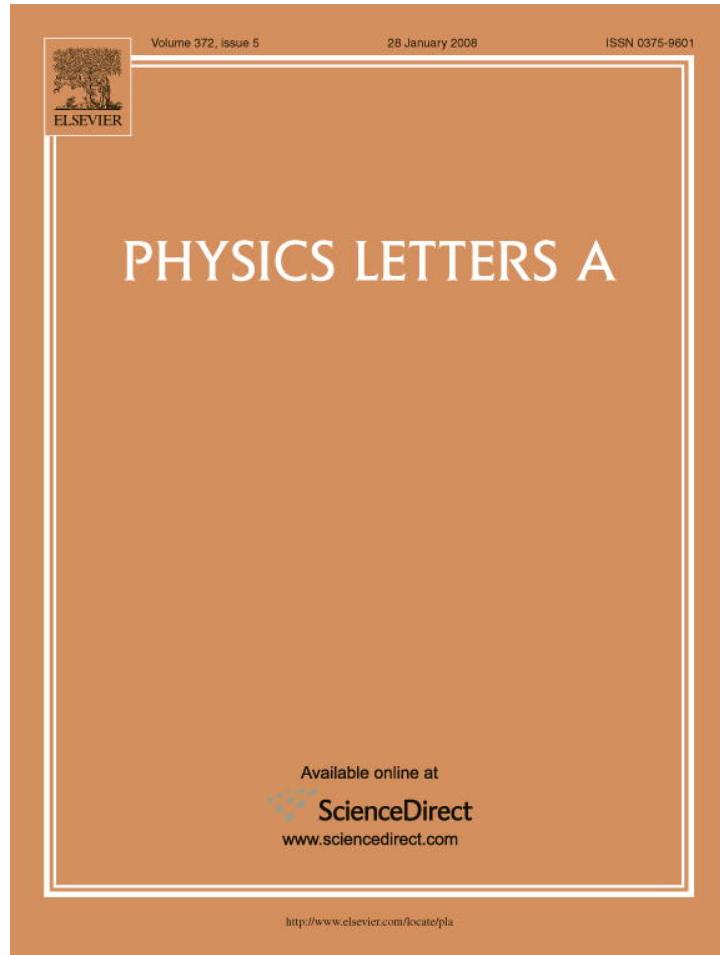


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# Closing the detection loophole in the Bell test for local models based upon three hidden variables

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## Abstract

Locality and fair sampling are proved to be contradictory assumptions in hidden variable models of the Bell test that are based upon a 3-dimensional sample space. This result makes the class of 3-dimensional hidden variable models incompatible with quantum mechanics in the ideal case, independently of detection efficiencies.

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The Bell test for the singlet state [1] entails the measurement of the spin projections (or polarizations) of two spatially separated spin-1/2 particles (or photons) that have been prepared in the singlet state. In local hidden variable models with two outcomes (“spin-up” or “spin-down”) [2] the spin correlations obey an upper bound [3,4] that is exceeded experimentally [5–7]. On the contrary, experimental spin correlations are consistent with quantum theory and with local hidden variable models with three outcomes [8]. The latter are justified by the fact that experiments have very low detection efficiencies [1] and the extra third outcome is interpreted as a sample being missed by the detectors. The situation is known as the detection loophole in the Bell test. The loophole, as sized by the Clauser–Horne inequality [9], is as wide as about 83% of relative joint detection efficiency [8–10]. The “most economical class” of local hidden variable models with three outcomes are based upon a 3-dimensional sample space [8,11]. Here we give a rigorous proof that the “most economical class” is incompati-

ble with quantum mechanics in the ideal case on the basis of an irresolvable antagonism between locality and fair sampling. Our result, in the ideal case, does not extend to models with dimension  $n > 3$  as proved by the 4-dimensional model in Ref. [12]. Neither to the non-ideal case, as proved by the class of models introduced in [13].

The EPR-Bohm setup of the Bell test includes two analyzer/detector (a/d) assemblies, located at distant sites denoted  $A$  and  $B$ . The analyzers have polarizers oriented in directions specified by the unit vectors  $a$  and  $b$ . The outcome at site  $A$  is  $m \in \{-1, 0, 1\}$  (correspondingly  $m'$  at  $B$ ). In a local hidden variable (LHV) model, outcomes depend on hidden variables. The (a fortiori) hidden variables are two angles  $(\phi, \theta)$ , with the “objective physical interpretation” of specifying the direction of “spin” by the unit vector  $s = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ . To exploit the detection loophole in the ideal case [2,8], at least one more hidden variable  $t$  is necessary. It is assumed to take value in the interval  $[0, 1]$ . For our discussion variable  $t$  has no other “objective physical interpretation” than allowing data rejection. The sample space  $\Gamma$ , extended to include  $t$ , is the unit ball. A sample corresponds to a point  $(s, t) \in \Gamma$ . The particle flying towards site  $A$  carries the “physical properties”  $(s, t)$ , and  $(-s, t)$  the one flying towards site  $B$ .

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For each vector  $a$ , outcome at site  $A$  is given by the random variable  $\xi_a : \Gamma \rightarrow \{-1, 0, 1\}$ . A sample  $(s, t)$  with outcome  $\xi_a(s, t) = 1$  ( $-1$ ) means that the spin-up (spin-down) detector has been triggered. While if  $\xi_a(s, t) = 0$ , means that the sample has been missed by the detector. Outcome  $\xi_a$  defines the following events:  $S_a(\pm 1) := \{\xi_a = \pm 1\}$  and  $S_a(0) := \{\xi_a = 0\}$ , subsets of  $\Gamma$  such that  $S_a(1) \cup S_a(-1) \cup S_a(0) = \Gamma$ . They generate the algebra  $\mathcal{S}_a$  of events. Joint outcomes  $\xi_{a,b}$ , valued in  $\{-1, 0, 1\} \times \{-1, 0, 1\}$ , and the events they define are discussed later on.

A probability over  $\mathcal{S}_a$  is specified by a probability measure  $P$  with a density function  $\lambda(s, t)$  over  $\Gamma$  such that  $P(S) = \int_S \lambda(s, t) dt d\Omega / 4\pi$ , where  $d\Omega$  is the element of solid angle. There is no loss of generality in assuming that  $P$  is independent of vector  $a$ .<sup>1</sup>

Rotational invariance of the singlet state requires, for individual a/d assemblies, that the probabilities  $P(S_a(m))$  should not depend on vector  $a$ . A way to keep it fixed is to assume that  $P$  is spherically symmetric and that the subset  $S_a(m)$  moves rigidly within the ball  $\Gamma$  as vector  $a$  is moved on the unit sphere. I.e., we assume that

$$\begin{cases} (a) \text{ Measure } P \text{ is specified by the density} \\ \quad d\lambda = \mu(t) dt d\Omega / 4\pi. \\ (b) \text{ Index } \mathbb{1}_{S_a(m)}(s, t) \text{ is a function of } (a \cdot s, t). \end{cases} \quad (1)$$

Notice that the relevant quantity is the probability measure of  $S_a(m)$ , thus we are not losing generality in adopting (1) since, otherwise, the (homeomorphic) deformation of  $S_a(m)$ , as a function of vector  $a$ , might be compensated by choosing a probability density  $\lambda(s, t)$  which is not spherically symmetric as to make  $P(S_a(m))$  a constant, independent of  $a$  (the formal argument is similar to the one given in footnote 1). Anyway, condition (1) is one of the assumptions that specify our representative of the class of 3-dimensional LHV models. We have, by condition (1)(b) that  $\xi_a(s, t) = \xi(s \cdot a, t)$  and the corresponding subset  $S_a(m) = \{\xi_a = m\}$  is cylindrically symmetric about  $a$  and that  $p_A := P(S_a(1) \cup S_a(-1))$ , the probability to get a particle at site  $A$ , is a constant independent of vector  $a$ .

Locality in a LHV model requires that subsets  $\{\xi_{a,b} = (m, m')\}$  for joint events to be the intersection of independent events at each one of the distant a/d assemblies. Assuming that sites  $A$  and  $B$  are fully equivalent, their outcomes for the sample  $(s, t)$  are  $\xi(s \cdot a, t)$  and  $\xi(-s \cdot b, t) = \xi_{-b}(s, t)$ , respectively. The corresponding algebras of individual events are  $\mathcal{S}_a$  and  $\mathcal{S}_{-b}$ , with elements generated by  $S_a(m) = \{\xi_a = m\}$  at site  $A$  and by  $S_{-b}(m') = \{\xi_{-b} = m'\}$  at site  $B$ . Thus, the algebra of joint events is  $\mathcal{S}_a \vee \mathcal{S}_{-b}$ , generated by  $\{\xi_{a,b} = (m, m')\} = S_a(m) \cap S_{-b}(m')$ , which are determined by the index function  $\mathbb{1}_{S_a(m)} \mathbb{1}_{S_{-b}(m')}$ , the product of individual index functions.

<sup>1</sup> We are assuming that events in the finite algebra  $\mathcal{S}_a$  are simple subsets of the unit ball and that the probability measure  $P$  is absolutely continuous respect to Lebesgue's. Thus, for any other absolutely continuous probability measure  $P_a$ , having algebra  $\mathcal{S}'$  of events, we may always find a self-homeomorphism  $T_a : \Gamma \rightarrow \Gamma$  such that  $T_a^{-1} S \in \mathcal{S}'$  and  $P(S) = P_a(T_a^{-1} S) = P_a T_a^{-1}(S)$  for each  $S$ , i.e.,  $P = P_a T_a^{-1}$ .

Rotational invariance and locality imply that the joint probabilities  $P(S_a(m) \cap S_{-b}(m'))$  are not functions of vectors  $a$  and  $b$ , separately. They only depend on the angle  $\alpha$  between  $a$  and  $b$ . Indeed, by assumption (1),  $S_a(m)$  and  $S_b(m')$ , as subsets of the unit ball  $\Gamma$ , are cylindrically symmetric about axes  $a$  and  $b$ , respectively. In this way the “volume”  $P(S_a(m) \cap S_{-b}(m'))$  of the intersection of two individual events is a function of angle  $\alpha$ , only.

One further condition any LHV model must fulfill is that the probability measure  $P(S_a(m) \cap S_{-b}(m')) > 0$  for every  $\alpha \in (0, \pi)$ . We are assuming  $P$  absolutely continuous respect to Lebesgue's. Then we must require the intersection  $S_a(m) \cap S_{-b}(m')$  to have positive Lebesgue measure (its geometric shape is irrelevant) as to be able to give it a probability measure arbitrarily fixed by means of density  $\mu(t)$ . So, there is no loss of generality in adopting Pearle's random variable [8]

$$\xi(a \cdot s, t) = \begin{cases} 1, & a \cdot s > \cos(t\pi/2), \\ -1, & -a \cdot s > \cos(t\pi/2), \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Finally, assumptions (1) and (2) specify our representative of the class of 3-dimensional LHV models. Let us emphasize that (1) and (2) do not represent any loss of generality.

Joint probabilities are given in terms of the density function  $\mu$  by the integral

$$\begin{aligned} p_{m,m'}(\alpha) &:= \int_0^1 dt \mu(t) I_{m,-m'}(\alpha, t) \\ &\equiv P(S_a(m) \cap S_{-b}(m')), \end{aligned} \quad (3)$$

where

$$I_{m,m'}(\alpha, t) := \int_{\Sigma} \frac{d\Omega}{4\pi} \mathbb{1}_{S_a(m)} \mathbb{1}_{S_b(m')} \quad (4)$$

( $\Sigma$  denotes the unit sphere). Using random variable (2) to specify the subsets  $S_a(1)$  and  $S_b(1)$  in the integral (4) with  $m = m' = 1$  and integrating in the angles, (3) yields

$$p_{1,-1}(\alpha) = \frac{\pi}{2} \int_{\alpha/\pi}^1 du \int_0^1 dz \cos(u\pi/2) h(z \cos(u\pi/2)) \sqrt{1-z^2} \quad (5)$$

where

$$h(x) := \frac{2}{\pi} \frac{\mu(u)}{\sin(u\pi/2)} \geq 0, \quad \text{with } x = x(u) = \cos(u\pi/2). \quad (6)$$

Normalization of density  $\mu$  implies that the new density function  $h$  is normalized too,  $\int_0^1 dt h(t) = 1$ . Main steps leading to (5) are followed in Appendix A.

Formulae (5) and (6) constitute the functional relationship between  $p_{1,-1}$  and  $\mu$  that is implied by rotational invariance, locality and the equivalence of sites  $A$  and  $B$ .

Last task to accomplish is to make  $p_{1,-1}(\alpha)$  coincide with quantum theory and the experimental counting rates. And then to invert (5) to get the probability density  $\mu$ , which at the moment is the only “free parameter” left. We are thus compelled

to digress from LHV models and discuss about the assumptions involved in the interpretation of the experimental counting rates.

Let  $r_A$  and  $r_B$  be the individual counting rates at sites  $A$  and  $B$ , respectively. Let  $r_{AB}$  be the counting rate of coincident events. We have that  $(r_A + r_B)/2 \geq r_{AB}$ . The relative joint efficiency is the ratio

$$0 < \varepsilon(a, b) = \frac{r_{AB}}{(r_A + r_B)/2} \leq 1.$$

Counting rates measured in experiments are consistent with  $\varepsilon(a, b) = \varepsilon$  being a constant independent of both vectors,  $a$  and  $b$ . We have retrieved by the hand the Innsbruck data plotted in Ref. [14] and found a relative joint efficiency  $\varepsilon \approx 0.13$  with a variance not greater than 3% for the “scanblue” data. The “bluesine” data has  $\varepsilon \approx 0.14$  with a variance smaller than 1%. Furthermore, the experimental counting rates show that the individual rates  $r_A$  and  $r_B$  are constants (consistent with the combined assumption of rotational invariance (for the singlet state) and non-signaling), as well as the rate of coincidences,  $r_{AB}$ .

The foregoing discussion allows us to make the identification

$$\varepsilon = \frac{1}{p_A} \sum_{(m, m') \neq 0} p_{m, m'}(\alpha) = \frac{1}{p_B} \sum_{(m, m') \neq 0} p_{m, m'}(\alpha),$$

where the probabilities  $p_B = p_A := \sum_{m \neq 0, m'} p_{m, m'}(\alpha)$  are independent of the angle  $\alpha$ . To reproduce the experimental observation that the relative joint efficiency  $\varepsilon$  is a constant too, we have to impose the condition

$$\sum_{(m, m') \neq 0} p_{m, m'}(\alpha) =: \eta, \quad \text{a constant}, \quad (7)$$

on the joint detection efficiency  $\eta$ . Notice that (7) is a necessary condition for us to admit the fair sampling assumption. However, (7) actually is imposed by experimental data on us.

Let  $q_{m, m'}(\alpha)$  denote the probabilities given by quantum theory in the ideal case. The specification of the probabilities in the LHV model is fulfilled by requiring further that, when conditioned to the fair ensemble of registered data,  $p_{m, m'}(\alpha)/\eta = q_{m, m'}(\alpha)$ . For  $m = -m' = 1$  the condition is

$$p_{1, -1}(\alpha) = \eta \frac{1}{2} \cos^2(\alpha/2). \quad (8)$$

We have every thing to go for the probability density  $\mu$ . First notice that locality is supporting (5) while (8) is a necessary condition on fair sampling. Equating the derivatives of (5) and (8) results in the equation

$$\int_0^1 dz \sqrt{1-z^2} h(z \cos(\alpha/2)) = \eta \sin(\alpha/2) \quad (9)$$

for the unknown density function  $h$ . The inverse of integral transforms of the type (9) is found in Ref. [8]. Applying it to (9) yields the result

$$\frac{\pi}{2\eta} h(x) = 2 - \frac{x^2}{1-x^2} - \frac{3}{2} x \ln \frac{1+x}{1-x}, \quad x \in [0, 1].$$

This solution for  $h$  is not a density function since it is negative for  $x > 0.62$ , contradicting (6).

The conclusion is immediate: fair sampling and locality are contradictory assumptions in LHV models that intend to reproduce the results of quantum mechanics in the ideal case and that are based upon a 3-dimensional sample space. In such models the functional relationship (5) implied by locality is too restrictive to allocate fair-sampling too. The revealed antagonism between locality and fair sampling explains why LHV models found in the literature [8,11] that are based upon a 3-dimensional sample space have joint detection efficiencies that vary sensibly with  $\alpha$ , the angle between polarizers. On the basis of this antagonism is that the class of 3-dimensional LHV models (which, by definition, hold the locality assumption) is not compatible with quantum mechanics in the ideal case.

The Clauser–Horne inequality [9] on LHV models in its simplest form,  $\varepsilon \leq 2(\sqrt{2} - 1) \approx 83\%$ , assumes fair sampling. It is not valid for 3-dimensional LHV models. Instead, the basic inequality in Ref. [9] produces the following necessary condition,

$$3\varepsilon(\alpha) \cos^2(\alpha/2) - \varepsilon(3\alpha) \cos^2(3\alpha/2) \leq 2, \quad \alpha \in [0, \pi], \quad (10)$$

on LHV models, without assuming fair sampling. As a practical test of locality (10) is very demanding on relative joint efficiencies. We did a numerical exploration of the LHV model in Ref. [11] and determined optimal parameter values (without any physical significance) that put the left-hand side of (10), numerically, very close to the upper bound and put the relative efficiency  $\varepsilon(\alpha)$  itself very close to 83%. This optimal version of model [11] shows us that inequality (10) is a tight one. Whence, we conjecture that (10) is both, necessary and sufficient. The locality condition (10) is deprived of the fair sampling assumption.

Our result, in the ideal case of condition (8), does not extend to LHV models with more than three hidden variables. The example in Ref. [12], by using four hidden variables reproduces the quantum spin correlation functions and supports the fair sampling assumption with a constant relative joint efficiency  $\varepsilon = 2/3$ .

Our result neither extends to the non-ideal situation [9] where condition (8) is relaxed with a further parameter  $V \in (0, 1)$  by replacing  $\cos^2(\alpha/2)$  with  $(1 + V \cos \alpha)/2$ . In Ref. [13] Santos provides a class of LHV models based upon two hidden variables that are compatible with quantum mechanics in the non-ideal case. For  $V \approx 1$ , the joint efficiency  $\eta$  for the models in the class by Santos is rather small. It is bounded from above as  $\eta < 2(1 - V)/\pi^2$ .

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## Appendix A. Working out integral (5)

For the random variables (2), subsets  $\{\xi_a = 1\}$  and  $\{\xi_b = 1\}$ , in the integral (4) with  $m = m' = 1$ , intersect the  $t$ -sphere in caps that have a half-opening angle  $t\pi/2$  and are centered at points  $at$  and  $bt$  each. For  $\alpha \geq t\pi$  the caps do not overlap so that  $I_{1,1}(\alpha, t) = 0$ .

When  $\alpha < \pi t$ , subset  $\{\xi_a = 1\} \cap \{\xi_b = 1\}$  has a non-empty intersection with the  $t$ -sphere. To calculate  $I_{1,1}(\alpha, t)$  the coordinate system is chosen with the  $x$ - $y$  plane the same as the plane defined by vectors  $a$  and  $b$ . The  $x$  axis bisects the angle between  $a$  and  $b$ , such that  $a = (a_x, a_y, 0)$  and  $b = (a_x, -a_y, 0)$ . The border line  $\phi_0(\theta)$  of the  $a$ -cap is defined implicitly by the condition

$$\begin{aligned}\cos(t\pi/2) &= a \cdot s = a_x \sin \theta \cos \phi_0 + a_y \sin \theta \sin \phi_0 \\ &= \sin \theta \cos(\alpha/2 - \phi_0).\end{aligned}$$

Just by symmetry, the border lines of the  $a$ - and  $b$ -caps cross at  $\phi_0 = 0$  and  $\theta_0$  given by  $\phi_0(\theta_0) = 0$ . I.e., given by  $\cos(t\pi/2) = \sin \theta_0 \cos(\alpha/2)$ .

The intersection with the  $t$ -sphere of the subset  $\{\xi_a = 1\} \cap \{\xi_b = 1\}$  is described as follows. The  $x$ - $y$  plane splits it into two specular halves. The half above the  $x$ - $y$  plane is divided into specular halves by the  $\phi = 0$  plane. Thus, the integral  $I_{1,1}(\alpha)$  is 4 times the integral over the fourth piece that has points with angle  $\phi \in [\phi_0(\theta), 0]$  for each  $\theta \in [\theta_0, \pi/2]$ . We have that

$$I_{1,1}(\alpha, t) = 4 \frac{1}{4\pi} \int_{\theta_0}^{\pi/2} d\theta \sin \theta \int_{\phi_0(\theta)}^0 d\phi = -\frac{1}{\pi} \int_{\theta_0}^{\pi/2} d\theta \phi_0(\theta) \sin \theta.$$

After an integration by parts we have that

$$\begin{aligned}I_{1,1}(\alpha, t) &= \frac{1}{\pi} \int_{\theta_0}^{\pi/2} d\theta \cot^2 \theta \cos(t\pi/2) \\ &\times \left( 1 - \left( \frac{\cos(t\pi/2)}{\sin \theta} \right)^2 \right)^{-1/2}.\end{aligned}$$

Changing variable of integration  $\theta \mapsto u$ :  $\cos(u\pi/2) = \cos(t\pi/2)/\sin \theta$  we have that

$$I_{1,1}(\alpha, t) = \frac{1}{2} \int_{\alpha/\pi}^t du \sqrt{1 - \left( \frac{\cos(t\pi/2)}{\cos(u\pi/2)} \right)^2}, \quad (\text{A.1})$$

whenever  $\alpha < \pi t$ . Then integral (A.1) is substituted into (3) and the order of integration is exchanged,  $\int_{\alpha/\pi}^1 dt \int_{\alpha/\pi}^t du = \int_{\alpha/\pi}^1 du \int_u^1 dt$ . Finally, we get the double integral (5) once the change of variable of integration  $t \mapsto z$ :  $z = \cos(t\pi/2)/\cos(u\pi/2)$  is done.

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