On the crossing number of complete graphs

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In Geometric Graph Theory the edges are straight segments. The techniques of Geometric Graph Theory do not apply (in general) to Topological Graph Theory.
In Geometric Graph Theory the edges are *straight segments.*
In Geometric Graph Theory the edges are **straight segments**.

The techniques of Geometric Graph Theory do **not** apply (in general) to Topological Graph Theory.
What we did

We adapted Geometric Graph Theory techniques to attack an old (open) problem in Topological Graph Theory.
Two different *drawings* of the same graph
Two different *drawings* of the same graph

Drawing with 9 crossings  Drawing with 1 crossing
Crossing numbers of graphs

- $G$ can be drawn with 1 crossing
- $G$ cannot be drawn with 0 crossings

Drawing of graph $G$
Crossing numbers of graphs

- $G$ can be drawn with 1 crossing
- $G$ cannot be drawn with 0 crossings

$\implies$

The crossing number of $G$ is 1
We write $\text{cr}(G) = 1$
Do we know the crossing number $\text{cr}(K_n)$ of $K_n$?

No. We only know it for very small values of $n$ ($n \leq 12$).
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Do we have good estimates (close lower and upper bounds)?

Not really. Well, more or less... if you believe in computer-assisted proofs. We do not have any good “computer-free” estimates.
Do we know the crossing number $\text{cr}(K_n)$ of $K_n$?
No. We only know it for very small values of $n$ ($n \leq 12$).

Do we have good estimates (close lower and upper bounds)?
Not really. Well, more or less... if you believe in computer-assisted proofs. We do not have any good “computer-free” estimates.

So... what’s new? What have we done?
We now know (2013) the exact crossing number of $K_n$, for all $n$, for two very important classes of drawings. We did it by “importing” techniques from discrete geometry.
Outline

1 Crossing number of $K_n$: the conjecture

2 Our results

3 The proof: ideas from discrete geometry

4 Concluding Remarks
On the crossing number of complete graphs
Crossing number of $K_n$: the conjecture
An artist drawing graphs with few crossings

Anthony Hill (1930–)
English artist, painter and relief-maker from the “constructivist tradition”.
On the crossing number of complete graphs
Crossing number of $K_n$: the conjecture
An artist drawing graphs with few crossings
On the crossing number of complete graphs
Crossing number of $K_n$: the conjecture
An artist drawing graphs with few crossings

1977
On the crossing number of complete graphs
Crossing number of $K_n$: the conjecture
An artist drawing graphs with few crossings

1957
On the crossing number of complete graphs

Crossing number of $K_n$: the conjecture

An artist drawing graphs with few crossings
Late 1950’s

These are drawings of $K_3$, $K_4$, $K_5$, $K_6$, $K_7$, $K_8$, and $K_9$. 
Together with American painter John Ernest, Anthony Hill investigated drawings of $K_n$ — aiming to minimize the number of crossings.
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Eventually, Hill approached graph theorist Frank Harary to write a paper with his findings.
On the crossing number of complete graphs
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Hill’s method

Place $n/2$ vertices on top lid, $n/2$ on bottom lid, and join each pair using geodesics:
Hill’s method

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Hill’s method

Counting the number of crossings from Hill’s method:

Hill (1958)

The complete graph $K_n$ can be drawn with exactly

$$Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

crossings.
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crossings.

Therefore,

Hill (1958)

The crossing number $\text{cr}(K_n)$ of $K_n$ satisfies

$$\text{cr}(K_n) \leq Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n - 1}{2} \right\rfloor \left\lfloor \frac{n - 2}{2} \right\rfloor \left\lfloor \frac{n - 3}{2} \right\rfloor.$$
On the crossing number of complete graphs
Crossing number of $K_n$: the conjecture
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A conjecture is born...

Hill's Conjecture

$$\text{cr}(K_n) = Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$
In a *cylindrical* drawing:

- The vertices are placed on the rims of the cylinder; and
- No edge crosses the rims (every edge is contained in the top lid, the bottom lid, or in the side of the cylinder).
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**Cylindrical crossing number** $\text{Cyl-Cr}(G)$ of a graph $G$

Minimum number of crossings in a cylindrical drawing of $G$. 
Cylindrical drawings, cylindrical crossing number

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Minimum number of crossings in a cylindrical drawing of $G$.

Hill’s drawings are cylindrical...
Cylindrical crossing number of $K_n$

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Hill’s drawings are cylindrical. . . therefore:

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For crossing number purposes, drawing a graph on the sphere and on the plane makes little difference: a graph can be drawn on the plane with $k$ crossings if and only if it can be drawn on the sphere with $k$ crossings.
Plane, sphere... who cares

For **crossing number** purposes, drawing a graph on the sphere and on the plane makes little difference: a graph can be drawn on the plane with $k$ crossings if and only if it can be drawn on the sphere with $k$ crossings.

**Proof**: From plane to sphere, use one-point compactification; from sphere to plane, punch a little hole in a face of the drawing.
2-disk drawings: Blažek and Koman (1962)

The Blažek-Koman way

Draw a graph on the sphere by putting all vertices on the equator. Then draw the edges so that no edge crosses the equator.
The Blažek-Koman way

Draw a graph on the sphere by putting all vertices on the equator. Then draw the edges so that **no edge crosses the equator**.

Equatorial drawing of $K_4$
The Blažek-Koman way

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Equatorial drawing of $K_4$
2-disk drawings: Blažek and Koman (1962)

Now cut along the equator, and flatten each hemisphere to obtain two disks: one gets the 2-disk representation of the same drawing.
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2-disk drawings of $K_n$

**Drawing $K_n$ the Blažek-Koman way**

Put in one disk the edges with positive slope, and in the other disk the edges with negative slope.
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2-disk drawing of $K_8$
2-disk drawings of $K_n$

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2-disk drawing of $K_8$
On the crossing number of complete graphs
Crossing number of $K_n$: the conjecture
The Blažek-Koman construction

From 2 disks to 2 pages
From 2 disks to 2 pages

2-page drawing of $K_8$
2-page drawings, 2-page crossing number

2-page drawings

The vertices are on a line, and each vertex is completely contained in the upper halfplane or in the lower halfplane (each halfplane is a page).
2-page drawings, 2-page crossing number

2-page drawings
The vertices are on a line, and each vertex is completely contained in the upper halfplane or in the lower halfplane (each halfplane is a page).

2-page crossing number
The 2-page crossing number \(2\text{-PAGE-CR}(G)\) of \(G\) is the minimum number of crossings in a 2-page drawing of \(G\).
Counting the number of crossings in the 2-page Blažek-Koman construction, one gets the same number as in Hill’s construction (!)

\[ Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor. \]
2-page crossing number of $K_n$: upper bound

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Outline

1. Crossing number of $K_n$: the conjecture
2. Our results
3. The proof: ideas from discrete geometry
4. Concluding Remarks
On the crossing number of complete graphs

Our results

\[ Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor \]
On the crossing number of complete graphs

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Hill (1958); Blažek-Koman (1962)

The 2-page crossing number 2-PAGE-CR(\(K_n\)) and the cylindrical crossing number CYL-CR(\(K_n\)) of \(K_n\) satisfy

\[ 2\text{-PAGE-CR}(K_n) \leq Z(n) \quad \text{CYL-CR}(K_n) \leq Z(n) \]
Our results

\[ Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n - 1}{2} \right\rfloor \left\lfloor \frac{n - 2}{2} \right\rfloor \left\lfloor \frac{n - 3}{2} \right\rfloor \]

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The 2-page crossing number \( 2\text{-PAGE-CR}(K_n) \) and the cylindrical crossing number \( C_{\text{YL-CR}}(K_n) \) of \( K_n \) satisfy

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\[ 2\text{-PAGE-CR}(K_n) = Z(n) \quad C_{\text{YL-CR}}(K_n) = Z(n) \]
This gives support to Hill’s Conjecture... perhaps the only “valid” support so far!

Hill’s Conjecture

\[
\text{cr}(K_n) = Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.
\]

Our theorem settles Hill’s Conjecture for 2-page and cylindrical drawings.
Outline

1. Crossing number of $K_n$: the conjecture
2. Our results
3. The proof: ideas from discrete geometry
4. Concluding Remarks
In a *rectilinear drawing*, every edge is a *straight segment*.

Rectilinear drawing of $K_5$
Rectilinear crossing number

Rectilinear crossing number $\overline{cr}(G)$ of a graph $G$

Minimum number of crossings in a rectilinear drawing of $G$. 
On the crossing number of complete graphs
The proof: ideas from discrete geometry
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- For $K_n$, it is known that $\overline{cr}(K_n) > cr(K_n)$ for all $n > 10$. 
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- Finding $\overline{cr}(K_n)$ seems harder than $cr(K_n)$! There is no “Hill’s Conjecture” for $\overline{cr}(K_n)$!
The proof: ideas from discrete geometry

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- Finding $\text{cr}(K_n)$ seems harder than $\text{cr}(K_n)$! There is no “Hill’s Conjecture” for $\text{cr}(K_n)$!
- If you want to become famous, find $\text{cr}(K_n)$ — this also answers a question of Sylvester from 1863 (150 years ago): “Sylvester’s Four Point”, from geometric probability.
On the rectilinear crossing number

Even if $\overline{cr}(K_n)$ seems so hard that it does not even have a “Hill’s Conjecture”, we are (in principle!) very close to knowing $\overline{cr}(K_n)$ asymptotically:

**Theorem (Ábrego, Aichholzer, Fernández, Ramos, and S., 2010)**

$$0.37992 < \lim_{n \to \infty} \frac{\overline{cr}(K_n)}{n^4} < 0.38048.$$
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Why are we so close?
k-edges

Take a point set $P$ with $n$ points, and let $D$ be the drawing of $K_n$ it induces. A $k$-edge of $P$ (or of $D$) is an edge whose supporting line leaves exactly $k$ points on one of its sides.
In discrete geometry we have nice tools

$k$-edges

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2-edge (also 1-edge)
On the crossing number of complete graphs
The proof: ideas from discrete geometry

$k$-edges, $E_k$, and $E_{<k}$

$k$-edges and $\overline{\text{cr}}(K_n)$

For a rectilinear drawing $D$:

$$E_k(D) := \text{number of } k\text{-edges of } D.$$
On the crossing number of complete graphs
The proof: ideas from discrete geometry
k-edges, $E_k$, and $E_{<k}$

$k$-edges and $\overline{cr}(K_n)$

For a rectilinear drawing $D$:

$E_k(D) :=$ number of $k$-edges of $D$.

For the rectilinear crossing number $\overline{cr}(K_n)$, the location of the points “is everything”: once you fix the points, the edges are determined (straight segments joining points), and so the number of crossings is determined.
On the crossing number of complete graphs
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$k$-edges and $\text{cr}(K_n)$

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Number of $k$-edges of a drawing determines its number of crossings

For any rectilinear drawing $D$ of $K_n$,

$$\text{cr}(D) = 3 \binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k \cdot (n - 2 - k) E_k(D).$$
Bounding $E_k$ does not help

Since

$$\overline{cr}(D) = 3 \binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k \cdot (n - 2 - k)E_k(D),$$

to bound $\overline{cr}(D)$ it seems a good idea to bound $E_k(D)$. 
Bounding $E_k$ does not help

Since

$$\overline{\text{cr}}(D) = 3 \binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k \cdot (n - 2 - k)E_k(D),$$

to bound $\overline{\text{cr}}(D)$ it seems a good idea to bound $E_k(D)$. But it doesn’t work well... so we introduce:

$$E_{\leq k}(D) := \sum_{j=0}^{k} E_k(D),$$
Bounding $E_k$ does not help

Since

$$\overline{cr}(D) = 3 \binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k \cdot (n - 2 - k)E_k(D),$$

to bound $\overline{cr}(D)$ it seems a good idea to bound $E_k(D)$. But it doesn’t work well... so we introduce:

$$E_{\leq k}(D) := \sum_{j=0}^{k} E_k(D),$$

A simple calculation yields

$$\overline{cr}(D) = \sum_{k<\frac{n-2}{2}} (n - 2k - 1)E_{\leq k}(D) - \frac{3}{4} \binom{n}{3}$$
On the crossing number of complete graphs
The proof: ideas from discrete geometry
k-edges, $E_k$, and $E_{\leq k}$

$E_{\leq k}(D)$ turns out to be the right thing

\[
\overline{cr}(D) = \sum_{k < \frac{n-2}{2}} (n - 2k - 1)E_{\leq k}(D) - \frac{3}{4} \binom{n}{3}
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On the crossing number of complete graphs
The proof: ideas from discrete geometry

$k$-edges, $E_k$, and $E_{<k}$

$E_{\leq k}(D)$ turns out to be the right thing

$$\overline{cr}(D) = \sum_{k < \frac{n-2}{2}} (n - 2k - 1)E_{\leq k}(D) - \frac{3}{4} \binom{n}{3}$$

We have good techniques and tricks to bound $E_{\leq k}(D)$ for any drawing $D$ — that’s why we have such good estimates for $\overline{cr}(K_n)$. 
An intriguing calculation

\[ \overline{cr}(D) = \sum_{k < \frac{n-2}{2}} (n - 2k - 1)E_{\leq k}(D) - \frac{3}{4} \binom{n}{3} \]
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Remember Hill’s Conjecture is that \(cr(K_n)\) equals

\[
Z(n) := \frac{1}{4} \left[ \frac{n}{2} \right] \left[ \frac{n-1}{2} \right] \left[ \frac{n-2}{2} \right] \left[ \frac{n-3}{2} \right].
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$$Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

It is not too hard to show that $E_{\leq k}(D) \geq 3\binom{k+2}{2}$. 
An intriguing calculation

\[ \overline{\text{cr}}(D) = \sum_{k < \frac{n-2}{2}} (n - 2k - 1)E_{\leq k}(D) - \frac{3}{4} \binom{n}{3} \]

Remember Hill’s Conjecture is that \( \text{cr}(K_n) \) equals

\[ Z(n) := \frac{1}{4} \left[ \frac{n}{2} \right] \left[ \frac{n-1}{2} \right] \left[ \frac{n-2}{2} \right] \left[ \frac{n-3}{2} \right] \]

It is not too hard to show that \( E_{\leq k}(D) \geq 3(k+2) \). Using this in the above expression, one gets **EXACTLY**

\[ \overline{\text{cr}}(K_n) \geq Z(n). \]
Curious...

Discrete geometry techniques give **EXACTLY**

\[ \overline{cr}(K_n) \geq Z(n). \]
Curious. . .

Discrete geometry techniques give exactly

\[ \overline{cr}(K_n) \geq Z(n). \]

This “proves Hill’s Conjecture for rectilinear drawings”, but it is not such a big deal — it is known that the rectilinear crossing number \( \overline{cr}(K_n) \) is strictly bigger than \( Z(n) \).
On the crossing number of complete graphs
The proof: ideas from discrete geometry
k-edges, $E_k$, and $E_{<k}$

Curious...

Discrete geometry techniques give EXACTLY

$$\text{cr}(K_n) \geq Z(n).$$

This “proves Hill’s Conjecture for rectilinear drawings”, but it is not such a big deal — it is known that the rectilinear crossing number $\text{cr}(K_n)$ is strictly bigger than $Z(n)$.

What is intriguing is that, working in a totally different context, one gets EXACTLY this ugly number $Z(n)$.

The obvious question

Can we adapt geometric techniques to topological drawings? Is there an equivalent of $k$-edges for topological drawings?
On the crossing number of complete graphs
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It only took us 10 years

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The obvious question

Can we adapt geometric techniques to topological drawings? Is there an equivalent of $k$-edges for topological drawings?

**YES.** And it is very natural . . .
$k$ edges: from geometrical to topological

**Key**: being able to say when a point is “to the left” or “to the right”. How to achieve this in topological drawings?
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$w$ is to the **left** of edge $uv$
Key: being able to say when a point is “to the left” or “to the right”. How to achieve this in topological drawings?

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$w$ is to the left of edge $uv$
Key: being able to say when a point is “to the left” or “to the right”. How to achieve this in topological drawings?

We can say when a point is to the left/right $\Rightarrow$ we can say when an edge is a $k$-edge !!!
On the crossing number of complete graphs
The proof: ideas from discrete geometry
k-edges, $E_k$, and $E_{\leq k}$

Moving on to 2-page drawings

In a rectilinear drawing of $K_n$ we have:

$$\overline{cr}(D) = \sum_{k < \frac{n-2}{2}} (n - 2k - 1)E_{\leq k}(D) - \frac{3}{4} \binom{n}{3}$$
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We now have $k$-edges for topological drawings (and in particular for 2-page drawings). We can define $E_k$, and $E_{\leq k}$, and therefore, for any (topological) drawing of $K_n$:

$$cr(D) = \sum_{k<\frac{n-2}{2}} (n - 2k - 1)E_{\leq k}(D) - \frac{3}{4} \binom{n}{3}$$
A taste of reality

If we can prove $E_{\leq k}(D) \geq 3\binom{k+2}{2}$ for every 2-page drawing $D$

$\implies$

Hill’s Conjecture for 2-page drawings.
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A desperate attempt

Investigate

$$E_{\leq \leq k}(D) := \sum_{j=0}^{k} \sum_{i=0}^{j} E_i(D)$$

instead of $E_{\leq k}(D)$.
In terms of $E_k$ (the number of $k$-edges):

$$cr(D) = 3 \binom{n}{4} - \sum_{k=0}^{[n/2]-1} k \cdot (n - 2 - k)E_k(D)$$

In terms of $E_{\leq k}$:

$$cr(D) = \sum_{k < \frac{n-2}{2}} (n - 2k - 1)E_{\leq k}(D) - \frac{3}{4} \binom{n}{3}$$

In terms of $E_{\leq\leq k}$, an elementary calculation gives

$$cr(D) = 2 \sum_{k=0}^{[n/2]-3} E_{\leq\leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n - 2}{2} \right\rfloor$$
On the crossing number of complete graphs

The proof: ideas from discrete geometry

Beyond $E_k$ and $E_{<k}$: introducing $E_{<=k}$

Moving up to $E_{<=k}$... works!

\[
\text{cr}(D) = 2^{\left\lfloor \frac{n}{2} \right\rfloor - 3} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 3} E_{\leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n - 2}{2} \right\rfloor
\]

Theorem (´Abrego, Aichholzer, Fernández, Ramos, and S., 2013)

In any 2-page drawing $D$, $E_{\leq k}(D) \geq 3 \left( k + \frac{3}{3} \right)$.

Corollary (Hill’s Conjecture for 2-page drawings)

Every 2-page drawing $D$ satisfies $\text{cr}(D) \geq \mathbb{Z}(n)$. 

Moving up to $E_{\leq \leq k} \ldots$ works!

$$
cr(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq \leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n-2}{2} \right\rfloor
$$

**Theorem (Ábrego, Aichholzer, Fernández, Ramos, and S., 2013)**

In any 2-page drawing $D$,

$$
E_{\leq \leq k}(D) \geq 3 \binom{k+3}{3}.
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On the crossing number of complete graphs

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Moving up to $E_{\leq k}$ . . . works!

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**Theorem (Ábrego, Aichholzer, Fernández, Ramos, and S., 2013)**

In any 2-page drawing $D$,

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**Corollary (Hill’s Conjecture for 2-page drawings)**

Every 2-page drawing $D$ satisfies $\text{cr}(D) \geq Z(n)$. 
On the crossing number of complete graphs

The proof: ideas from discrete geometry

A few words about the proof

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Goal

Show \( \forall \) 2-page drawing \( D \), \( E_{\leq k}(D) \geq 3\binom{k+3}{3} \).
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Proof is by induction on $n$. 
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Show \( \forall \) 2-page drawing \( D \), \[ E_{\le k}(D) \ge 3^{(k+3)/3}. \]

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Goal

Show \( \forall \) 2-page drawing \( D \), \( E_{\leq k}(D) \geq 3^{\binom{k+3}{3}} \).

Proof is by induction on \( n \).

Remove the \( n \)-th vertex, get drawing \( D' \) of \( K_{n-1} \). By induction, we know \( E_{\leq k-1}(D') \geq 3^{\binom{k-1+3}{3}} \).
Remember \( E_{\leq k}(D) = \sum_{j=0}^{k} \sum_{i=0}^{j} E_i(D) \)

**Goal:** Prove \( E_{\leq k}(D) \geq 3^{\binom{k+3}{3}} \) \((*)\)

**Know:** \( E_{\leq k-1}(D') \geq 3^{\binom{k+2}{3}} \)
Remember \( E_{\leq k}(D) = \sum_{j=0}^{k} \sum_{i=0}^{j} E_i(D) \)

**Goal:** Prove \( E_{\leq k}(D) \geq 3\binom{k+3}{3} \) \((*)\)

**Know:** \( E_{\leq k-1}(D') \geq 3\binom{k+2}{3} \)

Elementary counting arguments show that it suffices to prove:

\[
\sum_{j=0}^{k} \#(j\text{-edges in } D' \text{ that are } j\text{-edges in } D) \geq \binom{k+2}{2}
\]
On the crossing number of complete graphs
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2-page drawing of \( K_8 \)

Matrix representation
Prove: \[ \sum_{j=0}^{k} \#(j\text{-edges in } D' \text{ that are } j\text{-edges in } D) \geq \binom{k+2}{2} \]

Key observation: for each blue point \( ij \) the following holds
\[
(\# \text{ blue points to its right } + \# \text{ blue points above it }) = k
\]
\[ \implies ij \text{ is a } k\text{-edge.} \]
$D$ a 2-page drawing of $K_n$

\[ E_{\leq k}(D) \geq 3\binom{k+3}{3} \]
On the crossing number of complete graphs
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\[ D \text{ a 2-page drawing of } K_n \]

\[ \implies \]

\[ E_{\leq k}(D) \geq 3 \binom{k+3}{3} \]

Using

\[ \text{cr}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n-2}{2} \right\rfloor : \]

Theorem (Hill's Conjecture for 2-page drawings)

\[ 2\text{-PAGE-CR}(K_n) = Z(n). \]
Key property of 2-page drawings for the proof

There is a natural way to construct 2-page drawings by “adding points to the infinite face”
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There is a natural way to construct 2-page drawings by “adding points to the infinite face”
This idea of getting a drawing by “adding points to the infinite face” is captured by the concept of *shellability*.

**Shellable drawings**

Drawing $D$ of $K_n$ is *$s$-shellable* if there exist a subset $S = \{v_1, v_2, \ldots, v_s\}$ of the vertices and a region $R$ of $D$ with the following property: for all $1 \leq i \leq j \leq s$, if $D_{ij}$ is the drawing obtained from $D$ by removing $v_1, v_2, \ldots, v_{i-1}, v_{j+1}, \ldots, v_s$, then $v_i$ and $v_j$ are on the boundary of the region of $D_{ij}$ that contains $R$. 
Theorem (Ábrego, Aichholzer, Fernández, Ramos, and S., 2013)

If $D$ is an $s$-shellable drawing of $K_n$ with $s \geq n/2$, then

$$E_{\leq \leq k}(D) \geq 3 \binom{k + 3}{3}.$$
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Corollary (Hill’s Conjecture for shellable drawings)

If $D$ is an $s$-shellable drawing of $K_n$ with $s \geq n/2$, then

$$cr(D) \geq Z(n).$$
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If $D$ is an $s$-shellable drawing of $K_n$ with $s \geq n/2$, then

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Observation (easy)

- 2-page drawings of $K_n$ are $n$-shellable.
- Cylindrical drawings of $K_n$ are $n/2$-shellable.
- Monotone drawings of $K_n$ are $n$-shellable.
Corollary (Hill's Conjecture for shellable drawings)

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Corollary

Every cylindrical or 2-page drawing of $K_n$ has at least $Z(n)$ crossings.
Corollary (Hill’s Conjecture for shellable drawings)

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- 2-page drawings of $K_n$ are $n$-shellable.
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Corollary

Every cylindrical or 2-page drawing of $K_n$ has at least $Z(n)$ crossings.

Corollary (Hill’s Conjecture for cylindrical and 2-page drawings)

$$\text{2-PAGE-CR}(K_n) = \text{CYL-CR}(K_n) = Z(n).$$
Outline

1. Crossing number of $K_n$: the conjecture
2. Our results
3. The proof: ideas from discrete geometry
4. Concluding Remarks
In terms of $E_k$ (the number of $k$-edges):

$$
cr(D) = 3 \binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k \cdot (n - 2 - k)E_k(D)
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In terms of $E_{\leq k}$:

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cr(D) = \sum_{k < \frac{n-2}{2}} (n - 2k - 1)E_{\leq k}(D) - \frac{3}{4} \binom{n}{3}
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In terms of $E_{\leq\leq k}$:

$$
cr(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq\leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n - 2}{2} \right\rfloor
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In terms of $E_k$ (the number of $k$-edges):

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Can prove this for any drawing, to settle Hill’s Conjecture???
Can prove this for **any** drawing, to settle Hill’s Conjecture???

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\text{cr}(D) = 2 \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 3} E_{\leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n - 2}{2} \right\rfloor.
\]
Can prove this for any drawing, to settle Hill’s Conjecture???

$$\text{cr}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n-2}{2} \right\rfloor.$$ 

No! We would need to prove that for any drawing $D$ of $K_n$,

$$E_{\leq k}(D) \geq 3 \binom{k+3}{3},$$

but we know it isn’t true!
However... 

\[ \text{cr}(D) = 2E_{\leq \leq \lfloor n/2 \rfloor} - 2(D) - \frac{1}{8}n(n-1)(n-3). \]

All drawings we know satisfy

\[ E_{\leq \leq k}(D) \geq 3\binom{k+4}{4}. \]
However... 

\[ \text{cr}(D) = 2E_{\leq \leq \lfloor n/2 \rfloor} - 2(D) - \frac{1}{8}n(n - 1)(n - 3). \]

All drawings we know satisfy 

\[ E_{\leq \leq k}(D) \geq 3 \binom{k + 4}{4}. \]

If we can prove that all drawings of \( K_n \) satisfy this, the full Hill’s Conjecture follows!
Thank you for your attention!