On the crossing number of complete graphs

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A graph with six *vertices* and nine *edges*
A graph with six vertices and nine edges

- The vertex set is \( V = \{ a_1, a_2, a_3, b_1, b_2, b_3 \} \).
- The edge set is \( E = \{ a_1 b_1, a_1 b_2, a_1 b_3, a_2 b_1, a_2 b_2, a_2 b_3, a_3 b_1, a_3 b_2, a_3 b_3 \} \).
- The graph is \( G = (V, E) \).
Drawings of graphs

Two different drawings of the same graph
Drawings of graphs

Two different **drawings** of the same graph

Drawing with 9 crossings  Drawing with 1 crossing
Crossing numbers of graphs

- $G$ can be drawn with 1 crossing
- $G$ cannot be drawn with 0 crossings

Drawing of graph $G$
Crossing numbers of graphs

- $G$ can be drawn with 1 crossing
- $G$ cannot be drawn with 0 crossings

$\implies$

The crossing number of $G$ is 1
We write $\text{cr}(G) = 1$
Would like, of course, to compute the crossing number of every graph! However, \texttt{CrossingNumber} is hard, even for the simplest graphs one can imagine: near-planar (Cabello and Mohar, 2012).
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Want to understand the structure of graphs with large crossing number: “exactly” when does a graph have large crossing number?
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Want to understand the structure of graphs with large crossing number: “exactly” when does a graph have large crossing number?

Want to calculate the crossing number of interesting families of graphs: complete bipartite graphs $K_{m,n}$ and complete graphs $K_n$.

We’ll talk about the crossing number of the complete graphs $K_n$. 
Do we know the crossing number $\text{cr}(K_n)$ of $K_n$?

No. We only know if for very small values of $n$ ($n \leq 12$).
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Do we have good estimates (close lower and upper bounds)?

Not really. Well, more or less...if you believe in computer-assisted proofs. We do not have any good “computer-free” estimates.
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So... what's new? What have we done?
We now know (2013) the exact crossing number of $K_n$, for all $n$, for two very important classes of drawings. We did it by "importing" techniques from discrete geometry.
Outline

1 Crossing numbers: the origins
2 Crossing number of $K_n$: the conjecture
3 Our results
4 The proof: ideas from discrete geometry
5 Concluding Remarks
1 Crossing numbers: the origins

2 Crossing number of $K_n$: the conjecture

3 Our results

4 The proof: ideas from discrete geometry

5 Concluding Remarks
Paul Turán in a concentration camp near Budapest (1944)
“. . . Our work was to bring out bricks from the oven where they were made and carry them on small vehicles which ran on rails in some of several open stores which happened to be empty. Since one could never be sure which store would be available, each oven was connected to each store by rail. After being loaded in the (rather warm) ovens the vehicles run smoothly with not much effort; the only trouble arose at the crossings of two rails. Here the cars jumped out, the bricks fell down; a lot of extra work. . . it occurred to me why on earth did they build the rail system so uneconomically; minimizing the number of crossings the production could be made much more economical.”

Paul Turán
Problem (Turán, 1944)
Which layout has the minimum number of rail crossings?
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What is the crossing number of this kind of graph?
Complete bipartite graph $K_{p,q}$

One group of $p$ vertices, one group of $q$ vertices, and each vertex of the first group is joined to each vertex of the second group.
Turán’s question using graph theory terminology

What is the crossing number of $K_{p,q}$?
On the crossing number of complete graphs
Crossing numbers: the origins
Crossed lines in the brick factory

Drawing of $K_{5,6}$ with 24 crossings
Easy to generalize: $K_{p,q}$ can be drawn with
\[ \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor \] crossings
On the crossing number of complete graphs
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- Thus $\text{cr}(K_{p,q}) \leq \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor$

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Can draw $K_{p,q}$ with fewer crossings???

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A 60-year-old conjecture

$$\text{cr}(K_{p,q}) = \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor$$

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Drawing of $K_{5,6}$ with 24 crossings

Surely the most famous (and important) open crossing numbers problem...
Zarankiewicz’s Theorem Conjecture

For all $p$ and $q$, $\text{cr}(K_{p,q}) = \left\lfloor\frac{p}{2}\right\rfloor\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{q}{2}\right\rfloor\left\lfloor\frac{q-1}{2}\right\rfloor$

“On a Problem of P. Turán Concerning Graphs.”

Zarankiewicz’s Theorem Conjecture

Zarankiewicz’s Statement

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“On a Problem of P. Turán Concerning Graphs.”


Kainen (1965) and Ringel (1966)

Zarankiewicz’s “proof” is wrong.
Zarankiewicz’s Theorem Conjecture

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For all $p$ and $q$, $\text{cr}(K_{p,q}) = \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor$

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Kainen (1965) and Ringel (1966)
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Zarankiewicz’s CONJECTURE (open)
For all $p$ and $q$, $\text{cr}(K_{p,q}) = \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor$
Status on Zarankiewicz’s Conjecture

\[ Z(p, q) := \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p - 1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q - 1}{2} \right\rfloor. \]
Status on Zarankiewicz’s Conjecture

\[ Z(p, q) := \left\lfloor \frac{p^2}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q^2}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor. \]

What is known

- \( \text{cr}(K_{p,q}) = Z(p, q) \) when \( \min\{p, q\} \leq 6. \)
Status on Zarankiewicz’s Conjecture

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What is known

- \( \text{cr}(K_p, q) = Z(p, q) \) when \( \min\{p, q\} \leq 6 \).
- Computer-assisted results (lower bounds) for \( K_{7,n}, K_{8,n}, K_{9,n} \).
Status on Zarankiewicz’s Conjecture

\[ Z(p, q) := \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p - 1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q - 1}{2} \right\rfloor. \]

What is known

- \( \text{cr}(K_p,q) = Z(p, q) \) when \( \min\{p, q\} \leq 6 \).
- Computer-assisted results (lower bounds) for \( K_7, n, K_8, n, K_9, n \).

Yet another reason to work on Zarankiewicz’s Conjecture

Best bounds known for \( \text{cr}(K_n) \) come from the best bounds on \( \text{cr}(K_{m,n}) \).
Crossing numbers: the origins

Crossing number of $K_n$: the conjecture

Our results

The proof: ideas from discrete geometry

Concluding Remarks
On the crossing number of complete graphs
Crossing number of $K_n$: the conjecture
An artist drawing graphs with few crossings

Anthony Hill (1930–)

English artist, painter and relief-maker from the “constructivist tradition”. 
On the crossing number of complete graphs
Crossing number of $K_n$: the conjecture
An artist drawing graphs with few crossings
On the crossing number of complete graphs

Crossing number of $K_n$: the conjecture

An artist drawing graphs with few crossings

1977
On the crossing number of complete graphs
Crossing number of $K_n$: the conjecture
An artist drawing graphs with few crossings

1957
On the crossing number of complete graphs

Crossing number of $K_n$: the conjecture

An artist drawing graphs with few crossings
Late 1950’s

These are drawings of $K_3, K_4, K_5, K_6, K_7, K_8$, and $K_9$. 
Together with American painter John Ernest, Anthony Hill investigated drawings of $K_n$ — aiming to minimize the number of crossings.

Eventually, Hill approached graph theorist Frank Harary to write a paper with his findings.
Hill’s method

Place $n/2$ vertices on top lid, $n/2$ on bottom lid, and join each pair using geodesics:
Hill’s method

Place $n/2$ vertices on top lid, $n/2$ on bottom lid, and join each pair using geodesics:
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Counting the number of crossings from Hill’s method:

Hill (1958)

The complete graph $K_n$ can be drawn with exactly

$$Z(n) := \frac{1}{4} \left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-2}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor$$
crossings.
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crossings.

Therefore,

**Hill (1958)**

The crossing number $cr(K_n)$ of $K_n$ satisfies

$$cr(K_n) \leq Z(n) := \frac{1}{4} \left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-2}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor.$$
A conjecture is born... 

Hill’s Conjecture

\[
\text{cr}(K_n) = Z(n) := \frac{1}{4} \left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-2}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor.
\]
In a *cylindrical* drawing:

- The vertices are placed on the rims of the cylinder; and
- No edge crosses the rims (every edge is contained in the top lid, the bottom lid, or in the side of the cylinder).

**Cylindrical crossing number of a graph $G$**

Minimum number of crossings in a cylindrical drawing of $G$.

The cylindrical crossing number of $G$ is

$$\text{Cyl-Cr}(G).$$

Hill’s drawings are cylindrical...
Cylindrical crossing number of $K_n$

Hill (1958)

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Hill’s drawings are cylindrical... therefore:

Hill (1958)

The cylindrical crossing number of $G$ satisfies

$$\text{CYL-CR}(K_n) \leq Z(n) := \frac{1}{4} \left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-2}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor.$$
For crossing number purposes, drawing a graph on the sphere and on the plane makes is no different: a graph can be drawn on the plane with $k$ crossings if and only if it can be drawn on the sphere with $k$ crossings. (*Proof:* From plane to sphere, use one-point compactification; from sphere to plane, punch a little hole in a face of the drawing).
The Blažek-Koman way

Draw a graph (any graph) on the sphere by putting all vertices on the equator. Then draw the edges so that no edge crosses the equator.
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Equatorial drawing of $K_4$
The Blažek-Koman way

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Equatorial drawing of $K_4$
2-disks drawings: Blažek and Koman (1962)

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Equatorial drawing of $K_4$
2-disk drawings: Blažek and Koman (1962)

Now cut along the equator, and flatten each hemisphere to obtain two disks: one gets the 2-disk representation of the same drawing.
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2-disks drawings of $K_n$

**Drawing $K_n$ the Blažek-Koman way**

Put in one disk the edges with negative slope, and in the other disk the edges with positive slope.
2-disks drawings of $K_n$

Drawing $K_n$ the Blažek-Koman way

Put in one disk the edges with negative slope, and in the other disk the edges with positive slope.

2-disk drawing of $K_8$
4-disks drawings of $K_n$

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2-disk drawing of $K_8$
2-disks drawings of $K_n$

Drawing $K_n$ the Blažek-Koman way

Put in one disk the edges with negative slope, and in the other disk the edges with positive slope.
On the crossing number of complete graphs
Crossing number of $K_n$: the conjecture
The Blažek-Koman construction

From 2 disks to 2 pages
From 2 disks to 2 pages

2-page drawing of $K_8$
2-page drawings, 2-page crossing number

2-page drawings

The vertices are on a line, and each vertex is completely contained in the upper halfplane or in the lower halfplane (each halfplane is a page).

Counting the number of crossings in the 2-page Blažek-Koman construction, one gets (!)

\[ Z(n) := \frac{1}{4} \left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-2}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor. \]

Blažek-Koman (1962)

The 2-page crossing number \( 2\text{-PAGE-CR}(K_n) \) of \( K_n \) satisfies

\[ 2\text{-PAGE-CR}(K_n) \leq Z(n) := \frac{1}{4} \left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-2}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor. \]
On the crossing number of complete graphs

Our results

\[ Z(n) := \frac{1}{4} \left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-2}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor. \]

Hill (1958); Blažek-Koman (1962)

The 2-page crossing number \( 2\text{-PAGE-CR}(K_n) \) and the cylindrical crossing number \( \text{Cyl-CR}(K_n) \) of \( K_n \) satisfy

\[ 2\text{-PAGE-CR}(K_n) \leq Z(n) \quad \text{Cyl-CR}(K_n) \leq Z(n). \]


The 2-page crossing number \( 2\text{-PAGE-CR}(K_n) \) and the cylindrical crossing number \( \text{Cyl-CR}(K_n) \) of \( K_n \) satisfy

\[ 2\text{-PAGE-CR}(K_n) = Z(n) \quad \text{Cyl-CR}(K_n) = Z(n). \]
Hill’s Conjecture

\[ \text{cr}(K_n) = Z(n) := \frac{1}{4} \left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-2}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor. \]

So our theorem settles Hill’s Conjecture for 2-page and cylindrical drawings.
Rectilinear drawings

In a *rectilinear drawing* of a graph, every edge is a *straight segment*.
Rectilinear crossing number

Rectilinear crossing number $\overline{cr}(G)$ of a graph $G$

Minimum number of crossings in a rectilinear drawing of $G$. 
Rectilinear crossing number

The usual and the rectilinear crossing number can be arbitrarily different!
Rectilinear crossing number

Rectilinear crossing number $\overline{cr}(G)$ of a graph $G$

Minimum number of crossings in a rectilinear drawing of $G$.

- The usual and the rectilinear crossing number can be arbitrarily different!
- For $K_n$, it is known that $\overline{cr}(K_n) > cr(K_n)$ for all $n > 10$. 
Rectilinear crossing number

Rectilinear crossing number $\overline{cr}(G)$ of a graph $G$
Minimum number of crossings in a rectilinear drawing of $G$.

- The usual and the rectilinear crossing number can be arbitrarily different!
- For $K_n$, it is known that $\overline{cr}(K_n) > cr(K_n)$ for all $n > 10$.
- Finding $\overline{cr}(K_n)$ seems harder than $cr(K_n)$! There is no “Hill’s Conjecture” for $\overline{cr}(K_n)$!
Rectilinear crossing number

The usual and the rectilinear crossing number can be arbitrarily different!

For $K_n$, it is known that $\overline{cr}(K_n) > cr(K_n)$ for all $n > 10$.

Finding $\overline{cr}(K_n)$ seems harder than $cr(K_n)$! There is no “Hill’s Conjecture” for $\overline{cr}(K_n)$!

If you want to become famous, find $\overline{cr}(K_n)$ — this also answers a question of Sylvester from 1863 (150 years ago): “Sylvester’s Four Point”, from geometric probability.
On the rectilinear crossing number

Even if $\overline{cr}(K_n)$ seems so hard that it does not even have a “Hill’s Conjecture”, we are (in principle!) very close to knowing $\overline{cr}(K_n)$ asymptotically:

$$0.37992 < \lim_{n \to \infty} \frac{\overline{cr}(K_n)}{n^4} < 0.38048.$$
On the rectilinear crossing number

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$$\frac{0.37992}{0.38048} \approx 0.998.$$
On the rectilinear crossing number

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$$\frac{0.37992}{0.38048} \approx 0.998.$$ 

Why are we so close?
In discrete geometry we have nice tools

$k$-edges

Take a point set $P$ with $n$ points, and let $D$ be the drawing of $K_n$ it induces. A $k$-edge of $P$ (or of $D$) is an edge whose supporting line leaves exactly $k$ points on one of its sides.
In discrete geometry we have nice tools

\(k\)-edges

Take a point set \(P\) with \(n\) points, and let \(D\) be the drawing of \(K_n\) it induces. A \(k\)-edge of \(P\) (or of \(D\)) is an edge whose supporting line leaves exactly \(k\) points on one of its sides.

2-edge (also 1-edge)
**k-edges and $\overline{cr}(K_n)$**

For a rectilinear drawing $D$:

$E_k(D) := \text{number of } k\text{-edges of } D$.

For the rectilinear crossing number $\overline{cr}(K_n)$, the location of the point “is everything”: once you fix the points, the edges are determined (straight segments joining points), and so the number of crossings is determined.

**Number of k-edges of a drawing determines its number of crossings**

For any rectilinear drawing $D$ of $K_n$,

$$\overline{cr}(D) = 3 \binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k \cdot (n - 2 - k)E_k(D).$$
Bounding $E_k$ does not help

Since

$$\overline{cr}(D) = 3 \binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k \cdot (n - 2 - k)E_k(D),$$

to bound $\overline{cr}(D)$ it seems a good idea to bound $E_k(D)$. But it doesn’t work well...

$$E_{\leq k}(D) := \sum_{j=0}^{k} E_k(D),$$

A simple calculation yields

$$\overline{cr}(D) = 3 \binom{n}{4} + 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq k}(D).$$
$E_{\leq k}(D)$ turns out to be the right thing

$$
\text{cr}(D) = 3 \binom{n}{4} + 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq k}(D).
$$
$E_{\leq k}(D)$ turns out to be the right thing

$$
\overline{cr}(D) = 3 \binom{n}{4} + 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq k}(D).
$$

We have good techniques and tricks to bound $E_{\leq k}(D)$ for any drawing $D$ — that’s why we have such good estimates for $\overline{cr}(K_n)$. 
An intriguing calculation

\[ \overline{cr}(D) = 3 \binom{n}{4} + 2 \sum_{k=0}^{\left\lfloor n/2 \right\rfloor - 3} E_{\leq k}(D). \]

Remember Hill's Conjecture is that \( cr(K_n) \) equals

\[ Z(n) := \frac{1}{4} \left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-2}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor. \]

It is not too hard to show that \( E_{\leq k}(D) \geq 3 \binom{k+2}{2} \). Using this in the above expression, one gets EXACTLY

\[ \overline{cr}(K_n) \geq Z(n). \]
Curious. . .

Discrete geometry techniques give $\text{EXACTLY}$

$$
\overline{cr}(K_n) \geq Z(n).
$$
On the crossing number of complete graphs
The proof: ideas from discrete geometry

Curious. . .

Discrete geometry techniques give EXACTLY

$$\overline{cr}(K_n) \geq Z(n).$$

This “proves Hill’s Conjecture for rectilinear drawings”, but it is not such a big deal — it is known that the rectilinear crossing number $\overline{cr}(K_n)$ is strictly bigger than $Z(n)$. 
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Discrete geometry techniques give **EXACTLY**

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\overline{cr}(K_n) \geq Z(n).
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This “proves Hill’s Conjecture for rectilinear drawings”, but it is not such a big deal — it is known that the rectilinear crossing number \(\overline{cr}(K_n)\) is strictly bigger than \(Z(n)\).

What is intriguing is that, working in a totally different context, one gets **EXACTLY** this ugly number \(Z(n)\).

**The obvious question**

Can we adapt geometric techniques to topological drawings? Is there an equivalent of \(k\)-edges for topological drawings?
It only took us 10 years

The obvious question

Can we adapt geometric techniques to topological drawings? Is there an equivalent of $k$-edges for topological drawings?
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Can we adapt geometric techniques to topological drawings? Is there an equivalent of $k$-edges for topological drawings?

**YES.** And it is very natural . . .
Key: being able to say when a point is “to the left” or “to the right”. How to achieve this in topological drawings?
Key: being able to say when a point is “to the left” or “to the right”. How to achieve this in topological drawings?

- $w$ is to the left of edge $uv$
- $x$ is to the right of edge $uv$
Key: being able to say when a point is “to the left” or “to the right”. How to achieve this in topological drawings?

We can say when a point is to the left/right $\Rightarrow$ we can say when an edge is a $k$-edge!!!
Moving on to 2-page drawings

\[ \overline{cr}(D) = 3 \binom{n}{4} + 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq k}(D). \]

We now have \( k \)-edges for topological drawings (and in particular for 2-page drawings). We can define \( E_k \), and \( E_{\leq k} \), and therefore:

\[ 2\text{-PAGE-CR}(D) = 3 \binom{n}{4} + 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq k}(D). \]
On the crossing number of complete graphs
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Moving on to 2-page drawings

$$\overline{cr}(D) = 3 \binom{n}{4} + 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq k}(D).$$

We now have $k$-edges for topological drawings (and in particular for 2-page drawings). We can define $E_k$, and $E_{\leq k}$, and therefore:

$$2\text{-PAGE-CR}(D) = 3 \binom{n}{4} + 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq k}(D).$$

Following the steps of the $\overline{cr}(K_n)$ calculation, if we can prove $E_{\leq k}(D) \geq 3 \binom{k+1}{2}$ for every 2-page drawing we are done! We would get $2\text{-PAGE-CR}(K_n) \geq Z(n)$ — and so $2\text{-PAGE-CR}(K_n) = Z(n)$, proving Hill’s Conjecture for 2-page drawings!
A taste of reality

If we can prove $E_{\leq k}(D) \geq 3\binom{k+1}{2}$ for every 2-page drawing $D$

$\implies$

Hill’s Conjecture for 2-page drawings.
A taste of reality

If we can prove $E_{\leq k}(D) \geq 3\binom{k+1}{2}$ for every 2-page drawing $D$

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Hill’s Conjecture for 2-page drawings.

However... easy examples show $E_{\leq k}(D) \geq 3\binom{k+1}{2}$ does not hold in general...
If we can prove $E_{\leq k}(D) \geq 3\left(\frac{k+1}{2}\right)$ for every 2-page drawing $D$

$\implies$

Hill’s Conjecture for 2-page drawings.

However... easy examples show $E_{\leq k}(D) \geq 3\left(\frac{k+1}{2}\right)$ does not hold in general...

A desperate attempt

What if instead of $E_{\leq k}(D)$ we investigate $E_{\leq \leq k}(D)$?
In terms of $E_k$ (the number of $k$-edges):

$$2\text{-PAGE-CR}(D) = 3 \binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k \cdot (n - 2 - k)E_k(D)$$

In terms of $E_{\leq k}$:

$$2\text{-PAGE-CR}(D) = 3 \binom{n}{4} + 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq k}(D)$$

In terms of $E_{\leq \leq k}$, an elementary calculation gives

$$2\text{-PAGE-CR}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq \leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n - 2}{2} \right\rfloor$$
On the crossing number of complete graphs
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Moving up to \( E_{\leq \leq k} \ldots \) works!

\[
2\text{-PAGE-CR}(D) = 2 \left\lfloor \frac{n}{2} \right\rfloor - 3 \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} E_{\leq \leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n - 2}{2} \right\rfloor
\]

Maybe don’t use \( 2\text{-PAGE-CR} \) but simply \( \text{cr} \)
Moving up to $E_{\leq \leq k} \ldots$ works!

$$2\text{-PAGE-CR}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq \leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n-2}{2} \right\rfloor$$

Maybe don’t use $2\text{-PAGE-CR}$ but simply cr

**Theorem (Ábrego, Aichholzer, Fernández, Ramos, and S. (2013))**

In any 2-page drawing $D$,

$$E_{\leq \leq k}(D) \geq 3 \binom{k+3}{3}.$$
Moving up to $E_{\leq \leq k}$... works!

$$\text{2-PAGE-CR}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq \leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n-2}{2} \right\rfloor$$

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**Theorem (Ábrego, Aichholzer, Fernández, Ramos, and S. (2013))**

In any 2-page drawing $D$,

$$E_{\leq \leq k}(D) \geq 3 \binom{k+3}{3}.$$

**Corollary (Hill’s Conjecture for 2-page drawings)**

Every 2-page drawing $D$ satisfies

$$\text{2-PAGE-CR}(D) \geq Z(n).$$
A few words about the proof

**Goal:** Show $\forall$ 2-page drawing $D$, $E_{\leq k}(D) \geq 3\left(\frac{k + 3}{3}\right)$.

Proof is by induction on $n$.
HERE goes a drawing of $K_8$, and the 8-th point should be removable.
Remove the $n$-th vertex, get drawing $D'$ of $K_{n-1}$. By induction, we know $E_{\leq k-1}(D') \geq 3\left(\frac{(k-1) + 3}{3}\right)$.
Here recycle the picture from $K_8$ to $K_7$

**Goal:** $E_{\leq k}(D) \geq 3\binom{k+3}{3}$

**Know:** $E_{\leq k-1}(D') \geq 3\binom{k+2}{3}$

Remember $E_{\leq k}(D) = \sum_{j=0}^{k} \sum_{i=0}^{j} E_i(D)$ \hspace{1cm} (*)

In $D$, two types of edges:

(I) incident with vertex $n$;

(II) non-incident with vertex $n$.

Type I contribute to $E_{\leq k}(D)$ in $2\binom{k+2}{2}$. 
On the crossing number of complete graphs
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\[ E_{\leq k}(D) \geq E_{\leq k-1}(D') + 2^{k+2} + \sum_{j=0}^{k} \#(j - \text{edges in } D' \text{ that are } j\text{-edges in } D) \]
The proof: ideas from discrete geometry

\[ E_{\leq k}(D) \geq E_{\leq k-1}(D') + 2\binom{k+2}{2} + \sum_{j=0}^{k} \#(j - \text{edges in } D' \text{ that are } j\text{-edges in } D) \]

Suppose
\[ \sum_{j=0}^{k} \#(j - \text{edges in } D' \text{ that are } j\text{-edges in } D) \geq \binom{k+2}{2} \] (*)

Now \( E_{\leq k}(D') \geq 3\binom{k+2}{2} \) (induction hypothesis), so
\( E_{\leq k}(D) \geq 3\binom{k+3}{3} \) (our final goal).
So it all reduces to proving (*).
Goal: prove the following

\[ \sum_{j=0}^{k} \#(j \text{- edges in } D' \text{ that are } j\text{-edges in } D) \geq \binom{k+2}{2} \quad (*) \]

Tool: Matrix representation of 2-page drawings

Key observation

Take blue point \( ij \) in matrix. \# blue points to its right + \# blue points above it = \( k \) \( \implies \) \( ij \) is a \( k \)-edge.
$D$ a 2-page drawing of $K_n$

$E_{\leq k}(D) \geq 3(k+3)$. Using

$$cr(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n-2}{2} \right\rfloor :$$

Theorem (Hill’s Conjecture for 2-page drawings)

$2\text{-PAGE-CR}(K_n) = Z(n)$. 
Key property of 2-page drawings for the proof

There is a natural way to construct 2-page drawings by “adding points to the infinite face”
On the crossing number of complete graphs
The proof: ideas from discrete geometry

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There is a natural way to construct 2-page drawings by “adding points to the infinite face”
This idea of getting a drawing by “adding points to the infinite face” is captured by the concept of \textit{shellability}.

\textbf{Shellable drawings}

Drawing $D$ of $K_n$ is \textit{s-shellable} if there exist a subset $S = \{v_1, v_2, \ldots, v_s\}$ of the vertices and a region $R$ of $D$ with the following property: for all $1 \leq i \leq j \leq s$, if $D_{ij}$ is the drawing obtained from $D$ by removing $v_1, v_2, \ldots, v_{i-1}, v_{j+1}, \ldots, v_s$, then $v_i$ and $v_j$ are on the boundary of the region of $D_{ij}$ that contains $R$. 
On the crossing number of complete graphs
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\[
\text{cr}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq k}(D) - \frac{1}{2} \binom{n}{2} \lfloor \frac{n-2}{2} \rfloor :
\]

**Theorem Ábrego, Aichholzer, Fernández, Ramos, and S. (2013)**

If \( D \) is an \( s \)-shellable drawing of \( K_n \) with \( s \geq n/2 \), then

\[
E_{\leq k}(D) \geq 3 \binom{k+3}{3}.
\]

**Corollary (Hill’s Conjecture for shellable drawings)**

If \( D \) is an \( s \)-shellable drawing of \( K_n \) with \( s \geq n/2 \), then

\[
\text{cr}(D) \geq Z(n).
\]
Corollary (Hill's Conjecture for shellable drawings)

If $D$ is an $s$-shellable drawing of $K_n$ with $s \geq n/2$, then

$$cr(D) \geq Z(n).$$

Observation (easy)

- 2-page drawings of $K_n$ are $n$-shellable.
- Cylindrical drawings of $K_n$ are $n/2$-shellable.
- Monotone drawings of $K_n$ are $n$-shellable.

Corollary

Every cylindrical or 2-page drawing of $K_n$ has at least $Z(n)$ crossings.

Corollary (Hill’s Conjecture for cylindrical and 2-page drawings)

$$2\text{-PAGE-CR}(K_n) = \text{CYL-CR}(K_n) = Z(n).$$
On the crossing number of complete graphs

1. Crossing numbers: the origins
2. Crossing number of $K_n$: the conjecture
3. Our results
4. The proof: ideas from discrete geometry
5. Concluding Remarks
$D$ a drawing of $K_n$:

\[ \text{cr}(D) = 3 \binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k \cdot (n - 2 - k) E_k(D). \]

\[ \text{cr}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n-2}{2} \right\rfloor. \]

The second one is used to prove Hill’s bound for $\overline{\text{cr}}(K_n)$.

The third one is used to prove Hill’s conjecture for shellable drawings (2-page, cylindrical).

Can (3) be used for any drawing, to settle Hill’s Conjecture???
Can the following be used for any drawing, to settle Hill’s Conjecture???

$$\text{cr}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n-2}{2} \right\rfloor.$$ 

No! We would need to prove that for any drawing $D$ of $K_n$,

$$E_{\leq k}(D) \geq 3 \binom{k + 3}{3},$$

but we know it isn’t true!

However... 

$$\text{cr}(D) = 2E_{\leq \lfloor n/2 \rfloor - 2}(D) - \frac{1}{8} n(n-1)(n-3).$$

All drawings we know satisfy

$$E_{\leq \leq \leq k}(D) \geq 3 \binom{k + 4}{4}.$$ 

If we can prove that all drawings of $K_n$ satisfy this, the full Hill’s Conjecture follows!