The Crossing Number of P(N,3)

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Abstract. It is proved that the crossing number of the Generalized Petersen Graph $P(3k+h, 3)$ is $k+h$ if $h \in \{0, 2\}$ and $k+3$ if $h = 1$, for each $k \geq 3$, with the single exception of $P(9, 3)$, whose crossing number is 2.

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1. Introduction

A few years ago Fiorini [5] claimed to have determined the crossing numbers of certain families of Generalized Petersen Graphs. Unfortunately, his paper contains one serious mistake that invalidates the principal results. Our aim in this article is to present correct proofs of the main statements in Fiorini's paper, and to extend these results by determining the crossing numbers of a family of graphs for which Fiorini had only claimed upper and lower bounds.

Our main result is the following.

Theorem 1. *The crossing number of the Generalized Petersen Graph $P(3k+h,3)$ is $k+h$ if $h \in \{0,2\}$ and $k+3$ if $h=1$, for each $k \geq 3$, with the single exception of $P(9,3)$, whose crossing number is 2.*

The Generalized Petersen Graph $P(m,\ell)$ is obtained in the following way. Let $C = (v_0,\ldots,v_{m-1})$ be an m–cycle, and let $v'_0,\ldots,v'_{m-1}$ be vertices not in C. Let ℓ be a positive integer. For each i, join v_i to v'_i by an edge e^s_i, and join v'_i to $v'_{i+\ell}$ by an edge (indices are read modulo m). The graph thus obtained is the *Generalized Petersen Graph* $P(m,\ell)$. The cycle C is the *principal* cycle of $P(m,\ell)$, and the edges e^s_i are the *spokes* of the graph.

Generalized Petersen Graphs have been studied in several different contexts. The book [6] contains a section on them. Alspach [1] has determined which are Hamiltonian.

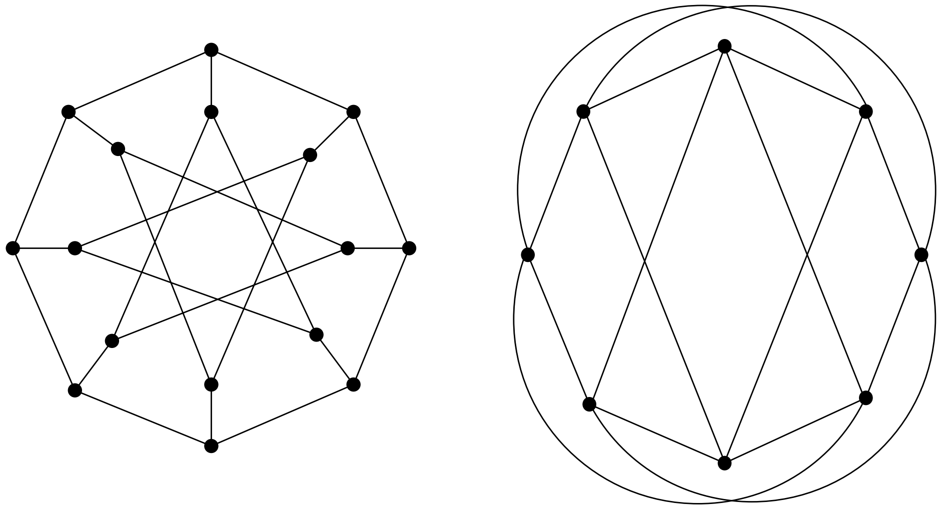
To prove Theorem 1, we need to analyze certain drawings of graphs related to $P(m,\ell)$. The *Derived Generalized Petersen Graph* $P'(m,\ell)$ is the graph obtained by contracting the spokes of $P(m,\ell)$. The m–cycle in $P'(m,\ell)$ whose edges correspond to the edges in the principal cycle of $P(m,\ell)$ will also be called the *principal* cycle of $P'(m,\ell)$. Figure 1 shows drawings of the Generalized Petersen Graph $P(8,3)$ and its corresponding Derived Generalized Petersen Graph $P'(8,3)$.

We remark that the Derived Generalized Petersen Graph $P'(m,\ell)$ is also the circulant $C(m,\{1,\ell\})$, where, in general, the circulant $C(m,S)$ is the graph with vertex set $\{0,1,\ldots,m-1\}$ and vertices i,j are joined by an edge if and only if $|i-j| \in S$. (So we can assume $S \subseteq \{0,1,\ldots,m-1\}$.)

A drawing of $P'(m,3)$ is an *n–drawing* if the edge rotation around each vertex consists of two edges in the principal cycle followed by two edges not in the principal cycle. Thus the drawing of $P'(8,3)$ in Figure 1(b) is not an n–drawing. We denote by $cr_n(P'(m,3))$ the minimum number of crossings in an n–drawing of $P'(m,3)$. We remark that an optimal n–drawing might have adjacent edges crossing.

We follow Fiorini's strategy to prove Theorem 1. Thus, we proceed by induction on k for each h. As shown below, it is straightforward to deal with the inductive step if we consider drawings of $P(3k+h,3)$ in which at least one spoke is crossed. On the other hand, if we take a drawing of $P(3k+h,3)$ in which no spoke is crossed, then there is no simple way to make use of the induction hypothesis. When faced with a drawing of this kind we proceed to contract the spokes, thus obtaining an n–drawing of $P'(3k+h,3)$ with the same number of crossings as the drawing of $P(3k+h,3)$. Finally we show, by a separate inductive process, that $cr_n(P'(3k+h,3))$ is at least $k+h$ if $h \in \{0,2\}$, and at least $k+3$ if $h=1$.

This paper is organized as follows. In Section 2 we reduce the proof of Theorem 1 to the proof of Theorem 2, which gives the mentioned lower bound for $cr_n(P'(3k+h,3))$. Most of the rest of the article is devoted to the proof of Theorem 2. In Section 3 we describe the strategy to prove Theorem 2. Section 4 contains the first part of the proof of Theorem 2, and in Section 5 we carry out a construction that is central for the rest of the proof. Sections 6 and 7 give the final part of the proof. In Section 8 we indicate the error that invalidates the main results in [5]. Section 9 contains some final remarks.



(a) A drawing of $P(8,3)$ (b) A drawing of $P'(8,3)$

Figure 1

This article contains part of the Ph.D. thesis work [10] of the second author, written under the supervision of the first. There are some laborious special cases required that we omit here. We refer the interested reader to [10] for all the details.

2. Proof of Theorem 1

Since there is a lack of symmetry in the role of h in Theorem 1, we define a function $f : \{0,1,2\} \to \{0,2,3\}$ by the rules $f(0) = 0$, $f(1) = 3$, and $f(2) = 2$. Using this notation, Theorem 1 can be written in the following way.

Theorem 1. *The crossing number of the Generalized Petersen Graph $P(3k+h,3)$ is $k+f(h)$, for each $k \geq 3$, with the single exception of $P(9,3)$, whose crossing number is 2.*

Proof of Theorem 1. As proved in [5], the crossing number of $P(9,3)$ is 2. The drawings in [5] show that there are drawings of: $P(3k,3)$ with exactly k crossings ($k \geq 4$); $P(3k+1,3)$ with exactly $k+3$ crossings ($k \geq 3$); and $P(3k+2,3)$ with exactly $k+2$ crossings ($k \geq 3$). Therefore, $cr(P(3k+h,3)) \leq k+f(h)$, for each $k \geq 3$. Thus, to prove Theorem 1 it suffices to show that $cr(P(3k+h,3)) \geq k+f(h)$, for each $k \geq 3$, if $h \in \{1,2\}$, and that $cr(P(3k,3)) \geq k$ for each $k \geq 4$.

We prove these inequalities by induction on k. The base cases are $cr(P(10,3)) = 6$ for $h = 1$, $cr(P(11,3)) = 5$ for $h = 2$, and $cr(P(12,3)) = 4$ for $h = 0$; these have been verified by computer by Yuanshen Yan [11].

Let $k \geq 4$ if $h \in \{1,2\}$ and let $k \geq 5$ if $h = 0$, and suppose that the inequality $cr(P(3(k-1)+h,3)) \geq (k-1)+f(h)$ holds for every h. We work on two cases separately.

Case 1. *At least one spoke is crossed in $\mathcal{D}$.* First we note that the graph obtained by deleting three consecutive spokes in $P(3k+h,3)$ is a subdivision of $P(3(k-1)+h,3)$).

Suppose that the spoke e_i^s is crossed in $\mathcal{D}$. By deleting the spokes e_{i-1}^s, e_i^s, and e_{i+1}^s, we get a drawing $\mathcal{D}'$ of a subdivision of $P(3(k-1)+h,3)$ with fewer crossings than $\mathcal{D}$. Since by the induction hypothesis $\mathcal{D}'$ has at least $(k-1)+f(h)$ crossings, $\mathcal{D}$ has at least $(k-1)+f(h)+1 = k+f(h)$ crossings.

Case 2. *No spokes are crossed in $\mathcal{D}$.* If we contract the spokes in $\mathcal{D}$, then we obtain an n–drawing $\mathcal{D}'$ of $P'(3k+h,3)$ with the same number of crossings as $\mathcal{D}$. Hence we are done, since by Theorem 2 below $\mathcal{D}'$ has at least $k+f(h)$ crossings.□

Theorem 2. *For each $k \geq 3$, $cr_n(P'(3k+h,3)) \geq k+f(h)$.*

We remark that in the statement of Theorem 2 we do not require k to be at least 4 if $h = 0$. Thus, even though the crossing number of $P(9,3)$ is 2, the minimum number of crossings in an n–drawing of $P'(9,3)$ is at least 3. (This is proved in [10]; we do not provide the proof here.)

In the next section we outline the proof of Theorem 2.

3. Strategy of the Proof of Theorem 2

The strategy to prove Theorem 2 is as follows. We proceed by induction on k for each h. We assume that the statement in Theorem 2 is true for $k' < k$, and take an n–drawing $\mathcal{D}$ of $P'(3k+h,3)$. We show that if $\mathcal{D}$ does not have at least $k+f(h)$ crossings, then we can construct a family $\mathcal{F}$ of edges not in the principal cycle, with the property that there is a one–to–three map from $\mathcal{F}$ to the collection of edges in the principal cycle, such that no edge is in the image of two different edges in $\mathcal{F}$. This induces a partition of a collection of $3|\mathcal{F}|$ edges in the principal cycle into 3–sets, and since there are exactly $3k+h < 3(k+1)$ edges in the principal cycle, $|\mathcal{F}| < k+1$ (Lemma 8).

On the other hand, a simple counting of crossings produces a lower bound on $|\mathcal{F}|$ that depends on the number of crossings in $\mathcal{D}$ (Lemma 10). The assumption that $\mathcal{D}$ has fewer than $k+f(h)$ crossings is then used to show that $|\mathcal{F}| \geq k+1$, contradicting Lemma 8.

The proof of Theorem 2 is divided in three parts. In Section 4 we present the induction hypothesis, and the objective of Section 5 is to develop the necessary tools to obtain and deal with the collection $\mathcal{F}$ described above. Finally, in Sections 6 and 7, we complete the proof. While we provide complete details here for the cases $h = 0$ and $h = 2$, we give the proof for $h = 1$ only when $k \geq 6$. The cases $k = 4$ and 5 are in [10] and can presumably be verified by a computer.

4. First Part of the Proof of Theorem 2

We prove Theorem 2 in three parts for each h. All the results in this section and in Section 5 are valid for every $h \in \{0,1,2\}$.

The proof of Theorem 2 is by induction on k, for each h. The base cases are obtained in the following way. Since any given n–drawing of $P'(3k+h,3)$ can be extended to a drawing of $P(3k+h,3)$ by splitting the vertices to create spokes, $cr_n(P'(3k+h,3)) \geq cr(P(3k+h,3))$. Therefore the base cases $cr(P(10,3)) = 6$ and $cr(P(11,3)) = 5$ mentioned above show

$cr_n(P'(10,3)) \geq 6$ and $cr_n(P'(11,3)) \geq 4$. The inequality $cr_n(P'(9,3)) \geq 3$ is proved in Appendix A in [10]. These three inequalities constitute the base cases for our present induction, for the values $h = 1, 2$, and 0, respectively.

For the remainder of this paper, we shall assume $k \geq 4$, $h \in \{0,1,2\}$, and $cr_n(P'(3(k-1)+h,3) \geq k-1+f(h)$.

5. The Main Construction

It will be very convenient for our subsequent work to colour the edges in the drawing $\mathcal{D}$ of $P'(3k+h,3)$. This graph has two kinds of edges, namely the edges in the principal cycle, which will be coloured blue, and those which correspond to edges joining vertices v_i' in $P(3k+h,3)$, which will be coloured red.

Each red edge r joins two vertices that are joined by a path $P(r)$ of length 3 consisting of blue edges. We say that the blue edges in $P(r)$ are *dominated* by the red edge r. We denote by $C(r)$ the 4–cycle obtained by adding r to its dominated path $P(r)$.

An edge in $\mathcal{D}$ is *clean* if it is not crossed by any other edge in the drawing.

We denote the vertices in $P'(3k+h,3)$ by v_i, for $0 \leq i \leq 3k+h-1$. The edge (v_i, v_j) joins v_i to v_j, the blue edge b_i is (v_i, v_{i+1}), and the red edge r_i is (v_i, v_{i+3}). Thus, the blue 3–path dominated by r_i is $P(r_i) = (v_i, v_{i+1}, v_{i+2}, v_{i+3})$ and contains the edges b_i, b_{i+1}, b_{i+2}. We remark that indices are always read modulo $3k+h$.

The main objectives of this section are the construction and analysis of the collection $\mathcal{F}$ mentioned in Section 3. We emphasize that this family is constructed under the assumption that $cr(\mathcal{D}) < k + f(h)$, as otherwise Theorem 2 follows. First we establish Propositions 3, 4, 5, 6 and 7 to obtain information about clean red edges and their dominated edges. Once we prove these results, we describe an inductive process from which $\mathcal{F}$, a collection of clean red edges, is obtained. Finally, we give in Lemmas 8 and 10 upper and lower bounds for the size of $\mathcal{F}$.

Proposition 3. *Suppose that $cr(\mathcal{D}) < k + f(h)$. Let r be a clean red edge in $\mathcal{D}$. Then $P(r)$ is crossed at most once, and if it is crossed, then it is crossed by a red edge adjacent to r.*

Proof. Let r_i be a clean red edge in $\mathcal{D}$. It is not difficult to check by deleting the edges in $P(r_i)$ and contracting r_i, we obtain a drawing $\mathcal{D}'$ of $P'(3(k-1)+h,3)$. Depending on the rotations around v_i and v_{i+3}, $\mathcal{D}'$ might or might not be an n–drawing. If $\mathcal{D}'$ is an n–drawing, then $P(r_i)$ can have no crossings, as otherwise $cr(\mathcal{D}) \geq k + f(h)$. Therefore we can assume that $\mathcal{D}'$ is not an n–drawing.

Since $\mathcal{D}'$ is not an n–drawing, the rotation around the vertex of contraction v is red–blue–red–blue. In this case, adding a single crossing in a small neighbourhood of v produces an n–drawing of $P'(3(k-1)+h,3)$. It follows that $P(r_i)$ has at most one crossing, since otherwise $cr(\mathcal{D}) \geq k + f(h)$.

Now suppose $P(r_i)$ has one crossing. Since $P'(3k+h,3)$ is 4–regular, for each j there exists an open disc neighbourhood $\mathcal{N}(v_j)$ of v_j such that the drawing $\mathcal{D}$ restricted to $\mathcal{N}(v_j)$ consists of four rays going out from v_j.

The drawing $\mathcal{D}$ induces a drawing $\mathcal{D}_i$ of $C(r_i)$. Since r_i is clean, it is contained in the boundary of two open faces F_1 and F_2 of $\mathcal{D}_i$. The edge r_{i-3} is incident with v_i, and so it intersects either $\mathcal{N}(v_i) \cap F_1$ or $\mathcal{N}(v_i) \cap F_2$. We can assume without loss of generality that r_{i-3} intersects $\mathcal{N}(v_i) \cap F_1$.

The rotation around the contracted vertex in $\mathcal{D}'$ is red–blue–red–blue. It is not difficult to check that this rotation scheme in $\mathcal{D}'$ implies that r_{i+3} intersects $\mathcal{N}(v_{i+3}) \cap F_2$.

Now we show that the edge that crosses $P(r_i)$ is red. It is easy to see that there is a red path P from v_i to v_{i+3} that does not contain r_i. This path contains both r_{i-3} and r_{i+3}, and since r_{i-3} intersects F_1 and r_{i+3} intersects F_2, P must cross $P(r_i)$. Since all the edges in P are red and at every other vertex we have a red–red–blue–blue rotation, a red edge crosses $P(r_i)$.

Finally, we show that $P(r_i)$ is crossed by one of r_{i-3} and r_{i+3}. Suppose that $P(r_i)$ is crossed by a red edge different from r_{i-3} and r_{i+3}. Since r_{i-3} and r_{i+3} are contained in F_1 and F_2, respectively, each of F_1 and F_2 contains at least one vertex. We claim that this implies that a blue edge crosses $P(r_i)$.

Suppose that there are vertices u_1 and u_2 contained in F_1 and F_2 respectively. Each of u_1 and u_2 is incident with two blue edges, and since the blue edges form a cycle, there are edge–disjoint blue paths P_B and P'_B joining u_1 to u_2. Since only two of the blue edges incident with vertices in $C(r_i)$ are not in $C(r_i)$, one of P_B and P'_B crosses an edge in $P(r_i)$.□

The proof of the following result is contained in the proof of Proposition 3. This statement is a useful characterization of the drawing of $C(r)$ induced by $\mathcal{D}$, for each clean red edge r.

Proposition 4. *Suppose that $cr(\mathcal{D}) < k + f(h)$. Let r be a clean red edge. Then the cycle $C(r)$ has no self–crossings, and all the vertices not in $C(r)$ are contained in the same component of $\mathbb{R}^2 \setminus C(r)$.*□

The edge b_{i+1} is the *central* edge of the path $P(r_i) = (v_i, v_{i+1}, v_{i+2}, v_{i+3})$.

Proposition 5. *Suppose that $cr(\mathcal{D}) < k + f(h)$. Let r_i, r_j be distinct clean red edges in $\mathcal{D}$. Then $P(r_i)$ and $P(r_j)$ have at most one common edge. If $P(r_i)$ and $P(r_j)$ have a common edge b, then b is not the central edge of either $P(r_i)$ nor $P(r_j)$. Thus, j is either $i-2$ or $i+2$.*

Proof. It suffices to show that the central edge of $P(r_i)$ is not contained in $P(r_j)$.

Suppose that the central edge of $P(r_i) = (v_i, v_{i+1}, v_{i+2}, v_{i+3})$ is in $P(r_j)$. Since $r_i \neq r_j$, the paths $P(r_i)$ and $P(r_j)$ are not equal, and so j is either $i-1$ or $i+1$. For the sake of definiteness, we assume $j = i+1$. Therefore $P(r_j) = (v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4})$.

By Proposition 4, neither $C(r_i)$ nor $C(r_j)$ has a self–crossing, and, from Proposition 3, they do not cross each other.

By Proposition 4, all the vertices of $P'(3k+h, 3)$ not in $C(r_i) \cup C(r_j)$ lie in the same face of $\mathcal{D}(C(r_i) \cup C(r_j))$. By Proposition 3, the edge r_{i+2} does not cross either $C(r_i)$ or $C(r_j)$, so all the other vertices are either in the face bounded by $C(r_i)$ or in the face bounded by $C(r_j)$.

In the first case, r_{i+4} must cross $C(r_i)$, while in the second case r_{i-1} must cross $C(r_j)$. Both of these possibilities contradict Proposition 3.□

For any 2–path (v_i, v_{i+1}, v_{i+2}) the vertex v_{i+1} is the *interior* vertex of the path.

Proposition 6. *Suppose that $cr(\mathcal{D}) < k + f(h)$. Let r_i, r_j be distinct clean red edges in $\mathcal{D}$. Suppose further that $P(r_i)$ and $P(r_j)$ have exactly one common edge b. Then the red edge going from the interior vertex of $P(r_i) - b$ to the interior vertex of $P(r_j) - b$ crosses a blue edge.*

Proof. Let r_i, r_j be clean red edges, and suppose that $cr(\mathcal{D}) < k + f(h)$. By Proposition 5, j is either $i-2$ or $i+2$. For the sake of definiteness we assume that $j = i+2$.

By Proposition 3, $C(r_i)$ and $C(r_{i+2})$ do not cross each other, and by Proposition 4 neither of these cycles has a self–crossing.

The edges r_{i+1} and r_{i+4} do not cross either of $C(r_i)$ or $C(r_j)$ and, therefore, all the vertices of $P'(3k+h,3)$ not in $C(r_i) \cup C(r_j)$ must be in the face F of $\mathcal{D}(C(r_i) \cup C(r_j))$ with both r_i and $r_j = r_{i+2}$ in its boundary. Now, as we traverse the path $Q = (v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5})$, F appears on opposite sides of Q at v_{i+1} and v_{i+4}.

Since $\mathcal{D}$ is an n-drawing, r_{i+1} is on the same side of Q as r_{i+4}. As the blue edges form a cycle, r_{i+1} must cross a blue edge.□

As we explained in Section 3, our goal is, based on the assumption that $cr(\mathcal{D}) < k + f(h)$, to find a family $\mathcal{F}$ of clean red edges such that there is a one–to–three map from $\mathcal{F}$ to the edges in the blue cycle, with the property that no blue edge is in the image set of two different edges in $\mathcal{F}$. The family $\mathcal{F}$ is obtained at the end of an inductive process that we now describe.

Let us suppose that $cr(\mathcal{D}) < k + f(h)$, and let $\mathcal{R}_c$ denote the set of all clean red edges. We set $\mathcal{F}_1^0 = \mathcal{F}_2^0 = \emptyset$ and $\mathcal{F}_3^0 = \mathcal{R}_c$, and we assume that after m steps we have partitioned the set of clean red edges into collections $\mathcal{F}_1^m, \mathcal{F}_2^m$, and $\mathcal{F}_3^m$, with $\mathcal{F}_3^m$ not empty. The $(m+1)$–st step consists of taking one edge r in $\mathcal{F}_3^m$ and letting $\mathcal{F}_1^{m+1} = \mathcal{F}_1^m \cup \{r\}$. Then we define $\mathcal{F}_2^{m+1}$ as the union of $\mathcal{F}_2^m$ with the set of edges in $\mathcal{F}_3^m \setminus \{r\}$ which have a dominated edge that is also dominated by r, and we let $\mathcal{F}_3^{m+1} = \mathcal{R}_c \setminus (\mathcal{F}_1^{m+1} \cup \mathcal{F}_2^{m+1})$. Since at every step we take at least one element from the third collection, there exists a smallest integer $m_{\mathcal{D}}$ such that $\mathcal{F}_3^{m_{\mathcal{D}}} = \emptyset$, and so we stop the process after $m_{\mathcal{D}}$ steps, and set $\mathcal{F} = \mathcal{F}_1^{m_{\mathcal{D}}}$.

Remark. Whenever we make reference to $\mathcal{F}$, we suppose that $cr(\mathcal{D}) < k + f(h)$, since $\mathcal{F}$ is constructed under this assumption. Further, $\mathcal{F}$ has at least one element, since there is at least one clean red edge. (To see this, we note that if every red edge is crossed, then there are at least $\lceil (3k+h)/2 \rceil$ crossings, and this number is at least $k + f(h)$ for every $k \geq 4$.)

The main property of $\mathcal{F}$ is obvious and is recorded in our next result.

Proposition 7. *There exists a one–to–three map from $\mathcal{F}$ to the set of blue edges, with the property that no blue edge belongs to the image sets of different elements in $\mathcal{F}$.*

Since there are $3k+h$ blue edges and $3k+h < 3(k+1)$, Proposition 7 implies the following.

Lemma 8. *There are at most k elements in $\mathcal{F}$.*

The last two results in this section involve inequalities which relate the size of $\mathcal{F}$ to the sizes of the collections in a partition $\{\mathcal{R}_b, \mathcal{R}_r, \mathcal{R}_c\}$ of the set $\mathcal{R}$ of red edges. We let $\mathcal{R}_b$ be the subset of $\mathcal{R}$ consisting of all red edges which cross a blue edge, we define $\mathcal{R}_r$ as the set of all red edges which do not cross any blue edge but that cross at least one red edge, and we let $\mathcal{R}_c$ be the set of all clean red edges. We use the following inequality in the proof of Lemma 10.

Lemma 9. $|\mathcal{F}| \geq |\mathcal{R}_c| - |\mathcal{R}_b|$.

Proof. First we note that $|\mathcal{R}_c| = |\mathcal{F}_1^{m_{\mathcal{D}}}| + |\mathcal{F}_2^{m_{\mathcal{D}}}|$, and since $\mathcal{F}_1^{m_{\mathcal{D}}} = \mathcal{F}$, it is enough to prove that $|\mathcal{F}_2^{m_{\mathcal{D}}}| \leq |\mathcal{R}_b|$. We prove this last inequality by exhibiting an injection $\mathcal{I}$ from $\mathcal{F}_2^{m_{\mathcal{D}}}$ to $\mathcal{R}_b$.

Let r_i be an edge in $\mathcal{F}_2^{m_\mathcal{D}}$, and let m be the smallest integer such that $r_i \in \mathcal{F}_2^m$. By construction, exactly one of r_{i-2} and r_{i+2} is in $\mathcal{F}_1^m$. If r_{i-2} is in $\mathcal{F}_1^m$, then we let $\mathcal{I}(r_i) = r_{i-1}$; if r_{i+2} is in $\mathcal{F}_1^m$, then we let $\mathcal{I}(r_i) = r_{i+1}$. We claim that if $\mathcal{I}(r_i) = r_{i-1}$, then (a) r_{i-1} is in $\mathcal{R}_b$, and (b) r_{i-1} cannot be $\mathcal{I}(r_j)$ for an edge $r_j \neq r_i$. Similarly, we claim that if $\mathcal{I}(r_i) = r_{i+1}$, then (a) r_{i+1} is in $\mathcal{R}_b$, and (b) r_{i+1} cannot be $\mathcal{I}(r_j)$ for an edge $r_j \neq r_i$.

Suppose that r_{i-2} is in $\mathcal{F}_1^m$, so that $\mathcal{I}(r_i) = r_{i-1}$. Since both r_i and r_{i-2} are clean, Proposition 6 implies r_{i-1} is in $\mathcal{R}_b$. Now if r_{i-1} is $\mathcal{I}(r_j)$ for some r_j, then by definition r_j is in $\mathcal{F}_2^{m_\mathcal{D}}$ and j is either i or $i-2$. It is clear that j cannot be $i-2$, since r_{i-2} is in $\mathcal{F}_1^m$. Therefore r_i is the only edge r_j such that $\mathcal{I}(r_j) = r_{i-1}$.

A similar argument shows that if r_{i+2} is in $\mathcal{F}_1^m$, then r_{i+1} is in $\mathcal{R}_b$ and r_i is the only edge r_j such that that $\mathcal{I}(r_j) = r_{i+1}$.□

Our last result in this section gives a lower bound for $\mathcal{F}$ in terms of $k, cr(\mathcal{D})$, and certain quantities that depend on the crossings between the edges in $\mathcal{R}_b, \mathcal{R}_r$, and $\mathcal{R}_c$. We define ρ as the number of intersections between elements in $\mathcal{R}_b$ and elements in $\mathcal{R}_r$, we let β be the number of blue–blue crossings, and we denote by τ the number of intersections between elements in $\mathcal{R}_b$. We define the *excess* of an element in $\mathcal{R}_b$ as the total number of blue crossings it has minus one, and we denote by ε the sum of the excesses of all the elements in $\mathcal{R}_b$.

Let $G_{\mathcal{R}_r}$ be the graph whose vertex set $V(G_{\mathcal{R}_r})$ is the set of edges in $\mathcal{R}_r$, and in which two vertices are joined by an edge if and only if they cross. Finally, we let $H_{\mathcal{R}_r}$ be the subgraph of $G_{\mathcal{R}_r}$ induced by the set of non–isolated vertices in $G_{\mathcal{R}_r}$.

Lemma 10. $|\mathcal{F}| \geq 3k + h - 2cr(\mathcal{D}) + \rho + 2\beta + 2\tau + 2\varepsilon + (2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|)$.

Proof. First we show that $|\mathcal{R}_b| = cr(\mathcal{D}) - |E(H_{\mathcal{R}_r})| - \rho - \beta - \tau - \varepsilon$. The number of red–red crossings is equal to $|E(H_{\mathcal{R}_r})| + \rho + \tau$, and there are exactly β blue–blue crossings. Therefore the number of blue–red crossings is equal to $cr(\mathcal{D}) - |E(H_{\mathcal{R}_r})| - \rho - \beta - \tau$. Since ε of these crossings occur on red edges which cross more than one blue edge, $|\mathcal{R}_b| = cr(\mathcal{D}) - E(H_{\mathcal{R}_r}) - \rho - \beta - \tau - \varepsilon$.

Since $\{\mathcal{R}_b, \mathcal{R}_r, \mathcal{R}_c\}$ is a partition of the set of red edges, $|\mathcal{R}_c| = 3k + h - (|\mathcal{R}_b| + |\mathcal{R}_r|)$, and so, by Lemma 9, $|\mathcal{F}| \geq |\mathcal{R}_c| - |\mathcal{R}_b| = 3k + h - 2|\mathcal{R}_b| - |\mathcal{R}_r|$. By the definition of $H_{\mathcal{R}_r}$, there are at most $|V(H_{\mathcal{R}_r})| + \rho$ elements in $|\mathcal{R}_r|$, and so $|\mathcal{F}| \geq 3k + h - 2(cr(\mathcal{D}) - |E(H_{\mathcal{R}_b})| - \rho - \beta - \tau - \varepsilon) - (|V(H_{\mathcal{R}_b})| + \rho) = 3k + h - 2cr(\mathcal{D}) + \rho + 2\beta + 2\tau + 2\varepsilon + (2|E(H_{\mathcal{R}_b})| - |V(H_{\mathcal{R}_b})|)$.□

Remark. Each of $(2|E(H_{\mathcal{R}_b})| - |V(H_{\mathcal{R}_b})|)$, ρ, β, τ, and ε is nonnegative. In particular, $|\mathcal{F}| \geq 3k + h - 2cr(\mathcal{D})$.

Lemma 10 is almost all we need in order to prove Theorem 2. In fact, for $h = 0$ it is all we need. For in this case $f(h) = 0$ and Lemma 10 implies $|\mathcal{F}| \geq 3k - 2cr(\mathcal{D})$. Since $|\mathcal{F}| \leq k$ by Lemma 8, we see that $2cr(\mathcal{D}) \geq 2k$, or $cr(\mathcal{D}) \geq k$, which contradicts the assumption that $cr(\mathcal{D}) < k + f(h) = k$. Therefore, $cr_n(P'(3k, 3)) \geq k$.

6. Refining the Bound

The proofs for $h = 2$ and $h = 1$ require two further observations, which we give in this section. We will use them in the next section to show that there is a j such that both r_j and r_{j+3} are clean, i.e. there is a pair of adjacent clean red edges. We comment here that the choice of $\mathcal{F}$ is fairly arbitrary. If there is an adjacent pair of clean red edges r_j and r_{j+3}, we can require these to be among those chosen to be in $\mathcal{F}$, simply by, at the first two steps in obtaining $\mathcal{F}$, picking first r_j and then picking r_{j+3}. Thus, if we prove that there is a pair of adjacent clean red edges, then we can assume such a pair occurs in $\mathcal{F}$.

Let $I = \{i \mid \{r_i, r_{i+3}\} \subseteq \mathcal{F}\}$. Recall β counts the number of self- crossings of the blue cycle. Let $\gamma = \rho + 2\beta + 2\tau + 2\varepsilon + (2|E(H_{\mathcal{R}_r})| - |v(H_{\mathcal{R}_r})|$. We have the following important fact.

Lemma 11. $\gamma \geq 2\beta + |I|$.

Proof. It clearly suffices to prove that $\rho + 2\tau + 2\varepsilon + (2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|) \geq |I|$. Let $\gamma' = \rho + 2\tau + 2\varepsilon + (2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|)$. The main point is the following.

Claim 12. *Suppose $cr(\mathcal{D}) < k + f(h)$ for the drawing $\mathcal{D}$ of $P'(3k+h, 3)$ and suppose further that r_i and r_{i+3} are both clean red edges. Then: (a) r_{i+1} crosses r_{i+2}; (b) either r_{i+1} crosses two blue edges or r_{i+1} crosses r_{i-1}; and (c) either r_{i+2} crosses two blue edges or r_{i+2} crosses r_{i+4}.*

We shall show that each of the crossings involving r_{i+1} described in Claim 12 necessarily contribute at least 1 to γ'. Further, it is clear that if $i, j \in I$ and $i \neq j$, then the crossings involving r_{i+1} and r_{j+1} described in Claim 12 are all different. The immediate conclusion from these two sentences is that $\gamma' \geq |I|$.

So to prove Lemma 11, it suffices to prove Claim 12 and to show that each of the crossings described in Claim 12 necessarily contribute at least 1 to γ'. We deal with the second part here; the proof of Claim 12 is given below.

If r_{i+1} crosses two blue edges, then it has positive excess and, therefore, contributes at least one to ε, and so at least two to γ'. Thus, we need to show that if we have the crossings r_{i+1} with r_{i-1}, r_{i+1} with r_{i+2} and r_{i+2} with r_{i+4}, that one of these contributes to γ'.

More generally suppose r, r', r'' are red edges such that r crosses both r' and r''. We see from the exhaustive list below that these two crossings contribute at least 1 to γ'.

(1) $(r, r', r'') \in (\mathcal{R}_r, \mathcal{R}_r, \mathcal{R}_r)$: These are all vertices of $H_{\mathcal{R}_r}$, with r having degree at least 2. Ignoring the two crossings r with r' and r with r'' will reduce γ by at least 1. More precisely, if we delete these two edges from $H_{\mathcal{R}_r}$, $|E(H_{\mathcal{R}_r})|$ decreases by 2, and $|V(H_{\mathcal{R}_r})|$ decreases by at most 3, so that $2|(E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|$ decreases by at least 1.
(2) $(r, r', r'') \in (\mathcal{R}_r, \mathcal{R}_r, \mathcal{R}_b)$: In this case, we get a contribution of 1 to ρ.
(3) $(r, r', r'') \in (\mathcal{R}_b, \mathcal{R}_r, \mathcal{R}_r)$: In this case, we get a contribution of 2 to ρ.
(4) $(r, r', r'') \in (\mathcal{R}_r, \mathcal{R}_b, \mathcal{R}_b)$: In this case, we get a contribution of 2 to ρ.
(5) $(r, r', r'') \in (\mathcal{R}_b, \mathcal{R}_r, \mathcal{R}_b)$: In this case, we get a contribution of 1 to ρ and 1 to τ.
(6) $(r, r', r'') \in (\mathcal{R}_b, \mathcal{R}_b, \mathcal{R}_b)$: In this case, we get a contribution of 2 to τ.

Therefore, the proof of Lemma 11 will be completed by proving Claim 12, which is done next.□

Proof of Claim 12. (a) By Proposition 3, the edges r_{i+1} and r_{i+2} do not cross either the 4-cycle $C(r_i)$ consisting of r_i and three blue edges or the analogous 4-cycle $C(r_{i+3})$. It is now clear that they must cross each other.

(b) We show that if r_{i+1} does not cross the blue cycle twice, then it crosses r_{i-1}.

Since $\mathcal{D}$ is an n-drawing, the ends of the edge r_{i+1} are in different regions determined by the blue cycle than the ends of r_i and r_{i+3}. Since the two ends of r_{i+1} are in the same region determined by the blue cycle, it follows that if r_{i+1} crosses the blue cycle, it must do so at least twice. Therefore, we can assume r_{i+1} does not cross the blue cycle at all.

The cycle $C = (v_i, b_i, v_{i+1}, r_{i+1}, v_{i+4}, b_{i+3}, v_{i+3}, r_{i+3}, v_i)$ is drawn by $\mathcal{D}$ with no self-crossings. Further, the vertices v_{i-1} and v_{i+2} are on different sides of C. Therefore, r_{i-1} crosses C. We know that r_{i-1} does not cross r_i (it is clean), b_i or b_{i+3} (by Proposition 3). Therefore, r_{i-1} must cross r_{i+1}.

(c) Similarly, r_{i+2} must also cross r_{i+4}.□

7. Finishing the cases $h = 2$ and $h = 1$

We are assuming that we have a drawing $\mathcal{D}$ of $P'(3k+h, 3))$ having at most $k-1+f(h)$ crossings. By Lemmas 10 and 11, $|\mathcal{F}| \geq 3k + h - 2cr(\mathcal{D}) + |I| + 2\beta$. Assuming $cr(\mathcal{D}) \leq k - 1 + f(h)$, we see that $|\mathcal{F}| \geq k + 2 + h - 2f(h) + |I| + 2\beta$.

The edges in $\mathcal{F}$ dominate $3|\mathcal{F}|$ blue edges. Therefore, $3(k+2+h-2f(h)+|I|+2\beta)$ blue edges are dominated. Since there are $3k+h$ blue edges, this means there are $6f(h) - 6 - 2h - 3|I| - 6\beta$ undominated blue edges.

For $h = 2$, $f(h) = 2$ and there are at most $2-3|I|$ blue edges not dominated by members of $\mathcal{F}$. Thus, there are at most 2 undominated blue edges. Since $k \geq 4$, there are at least 12 blue edges. At least one of the two segments of the blue cycle between the undominated edges has at least 6 edges, and, therefore, $|I| > 0$. By Lemmas 10 and 11, $|\mathcal{F}| \geq k+h+2-2f(h)+|I| \geq k+2+2-4+1 = k+1$, contradicting Lemma 8. We conclude that $cr_n(P'(3k+2, 3)) \geq k+2$.

For $h = 1$, as mentioned earlier, we shall only prove $cr(P'(3k+1, 3)) \geq k+3$ under the additional assumption that $k \geq 6$.

Let $\mathcal{D}$ be a drawing of $P'(3k+1, 3)$ and suppose $cr(\mathcal{D}) \leq k+2$. We have that $|\mathcal{F}| \geq k-3+\gamma$ and that $\gamma \geq 2\beta$. Thus, if $\beta \geq 2$, then $|\mathcal{F}| \geq k+1$, contradicting Lemma 8. Therefore, we can assume $\beta \leq 1$. We begin by proving that $I \neq \emptyset$.

Proposition 13. *If $k \geq 6$, then $I \neq \emptyset$.*

Proof. Suppose first that $|\mathcal{F}| \geq k-1$. (This happens, for instance, when $\beta = 1$.) There are at least $3k-3$ dominated edges and, therefore, at most 4 undominated edges. Since $k \geq 6$, there are at least 19 edges in the blue cycle, so at least 15 of them are dominated. Since $15/4 > 3$, at least one segment between consecutive undominated blue edges has more than three (and so at least 6) blue edges, implying the existence of adjacent clean red edges in $\mathcal{F}$.

Now suppose $|\mathcal{F}| < k-1$. Recall from Lemma 10 that $|\mathcal{F}| \geq 3k+1-cr(\mathcal{D})+\rho+2\beta+2\tau+2\varepsilon+(2|E(H_{\mathcal{R}_r})|-|V(H_{\mathcal{R}_r})|)$. Since $cr(\mathcal{D}) \leq k+2$ and each of ρ, β, τ, ε and $2|E(H_{\mathcal{R}_r})|-|V(H_{\mathcal{R}_r})|$ is non-negative, it follows that $\beta = \tau = \varepsilon = 0$ and $\rho + (2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|) \leq 1$.

Let X denote the set of all edges $r \in \mathcal{R}_r$ such that r crosses more than one red edge. In general, $|X| \leq (2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|)$, so the preceding paragraph implies $|X| \leq 1$.

We consider the nature of each of the $3k+1$ adjacencies r_j and r_{j+3} in terms of which sets $\mathcal{R}_r$, $\mathcal{R}_b$ and $\mathcal{R}_c$ contain r_j and r_{j+3}. Each edge r_j is involved in two such adjacencies, namely with r_{j-3} and r_{j+3}.

Let $r \in \mathcal{R}_r \setminus X$. Then there is a unique red edge r' that crosses r. Since $\beta = 0$, there are no blue-blue crossings and since $r \in \mathcal{R}_r$, r does not cross any blue edge. Therefore, the 4-cycle $C(r)$ consisting of r and three blue edges is drawn by $\mathcal{D}$ in the plane without self-intersections. Since the blue cycle is drawn without self- crossings, there is a component (the "outside") of $\mathbb{R}^2 \setminus C(r)$ that is incident with all the vertices of $P'(3k+1, 3)$ and the other component (the "inside") is incident only with the vertices of $C(r)$. The edge r' has one end in each of these two components.

Because $\mathcal{D}$ is an n-drawing, the other red edge r'' incident with vertex at the end of r' inside of $C(r)$ must also start inside $C(r)$. The other end of r'' is outside $C(r)$ and so r'' must cross $C(r)$. Since r'' cannot be a second red edge crossing r (remember $r \notin X$), r'' must cross a blue edge, so $r'' \in \mathcal{R}_b$.

The conclusion is that there are at least $|\mathcal{R}_r| - |X|$ adjacencies that are $\mathcal{R}_r - \mathcal{R}_b$ adjacencies. Since $|\mathcal{R}_r| \geq |V(H_{\mathcal{R}_r})|$, we have at least $|V(H_{\mathcal{R}_r})| - |X|$ adjacencies that are $\mathcal{R}_r - \mathcal{R}_b$. This leaves at most $3k + 1 - |V(H_{\mathcal{R}_r})| + |X|$ adjacencies.

Evidently, $|\mathcal{R}_r| + |\mathcal{R}_b| + |\mathcal{R}_c| = 3k + 1$. From the proof of Lemma 10, we see that $|\mathcal{R}_b| = cr(\mathcal{D}) - |E(H_{\mathcal{R}_r})| - \rho$ (since $\beta = \tau = \varepsilon = 0$). It is also plain that $|\mathcal{R}_r| \leq |V(H_{\mathcal{R}_r})| + \rho$. Therefore, $|\mathcal{R}_c| \geq 3k + 1 - |\mathcal{R}_r| - |\mathcal{R}_b| \geq 3k + 1 - (cr(\mathcal{D}) - |E(H_{\mathcal{R}_r})| - \rho) - (|V(H_{\mathcal{R}_r})| + \rho)$. Using the fact that $cr(\mathcal{D}) \leq k+2$, we conclude that $|\mathcal{R}_c| \geq (2k-1) + (|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|)$.

If there are no adjacent clean red edges, then there are at least $2|\mathcal{R}_c|$ adjacencies other than the ones of type $\mathcal{R}_r - \mathcal{R}_b$ estimated above. Therefore, either there is a pair of adjacent clean red edges or $2|\mathcal{R}_c| \leq 3k+1 - |V(H_{\mathcal{R}_r})| + |X|$. The following chain of inequalities shows there must be a pair of clean red edges:

$$\begin{aligned}
2|\mathcal{R}_c| &\geq 2\left((2k-1) + (|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|)\right) \\
&= 4k - 2 + 2(|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|) \\
&= 4k - 2 + (2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|) - |V(H_{\mathcal{R}_r})| \\
&\geq (3k+1) + (k-3) - |V(H_{\mathcal{R}_r})| \\
&> (3k+1) - |V(H_{\mathcal{R}_r})| + |X|. \square
\end{aligned}$$

Now we move on to the other half of the argument.

Proposition 14. *If $I \neq \emptyset$, then $cr(\mathcal{D})$ has at least $k+3$ crossings.*

Proof. We begin by noting that if some red edge is crossed at least three times, then we are done. For if r is a red edge with at least three crossings, then we can choose the indexing so that $r = r_{3k+1}$. Then, for each $j = 0, 1, \ldots, k-1$, the three red edges r_{3j+1}, r_{3j+2} and r_{3j+3}, together with the blue cycle, are a subdivision of $K_{3,3}$. Thus, there is a crossing among these edges.

Suppose, first, that $\beta \geq 2$. Then $|\mathcal{F}| \geq k - 3 + 2\beta \geq k + 1$, contradicting Lemma 8.

Next suppose $\beta = 1$. Then $|\mathcal{F}| \geq k - 1 + |I|$ and so there are at least $3k - 3 + 3|I|$ dominated blue edges. Since $I \neq \emptyset$, this implies there is at most one undominated blue edge. But this in turn implies $|I| \geq k$, which implies $|\mathcal{F}| > k$, contradicting Lemma 8.

Finally, we assume $\beta = 0$. We begin by noting that if some red edge is crossed at least three times, then we are done. For if r is a red edge with at least three crossings, then we can choose the indexing so that $r = r_{3k+1}$. Then, for each $j = 0, 1, \ldots, k-1$, the three red edges r_{3j+1}, r_{3j+2} and r_{3j+3}, together with the blue cycle, are a subdivision of $K_{3,3}$. Thus, there is a crossing among these edges. Since $\beta = 0$, this crossing must involve a red edge and so be different for each j. Thus, there are at least k crossings from these $K_{3,3}$'s plus the three crossings of r, for a total of $k+3$ crossings.

Let r_j and r_{j+3} be adjacent clean red edges. By Claim 12, either one of r_{j+1} or r_{j+2} has at least three crossings or they cross each other, r_{j+1} crosses r_{j-1} and r_{j+2} crosses r_{j+4}. If the former, then we are done by the preceding paragraph. So we assume the latter.

Now delete r_{j+1} (with its two crossings) and consider the k $K_{3,3}$'s using three consecutive red edges and the blue cycle. Each has at least one crossing, yielding at least k further crossings, for a total of at least $k+2$ crossings. Thus, there can be no crossings not counted, or else we are done.

If r_{j+6} has a crossing, then delete r_{j+6} to get the $K_{3,3}$'s using $r_{j+7}, r_{j+8}, r_{j+9}, \ldots, r_j, r_{j+1}, r_{j+2}$, and $r_{j+3}, r_{j+4}, r_{j+5}$, respectively. There are at least k crossings counted among the $K_{3,3}$'s, plus the crossing of r_{j+6}, plus the two crossings r_{j-1} with r_{j+1} and r_{j+2} with r_{j+4}, for a total of at least $k+3$. So we may assume r_{j+6} is clean.

In this case, r_{j+3} and r_{j+6} make a pair of adjacent clean red edges, so, as for r_j and r_{j+3}, the edges r_{j+4} and r_{j+5} cross, which gives the $k+3$rd crossing (they are in different $K_{3,3}$'s relative to the deletion of r_{j+1}).□

8. The Error in Fiorini's Argument

Fiorini [5] attempts an induction as follows. If some edge e of the blue cycle B is in a blue–blue crossing, then delete e and two successive edges from B. Now contract the red edge joining the resulting degree–3 vertices to obtain a drawing of $P'(3(k-1)+h,3)$. This is correct; what is not is the assumption that this drawing has fewer crossings than the drawing of $P'(3k+h,3)$. What is missing is that the contracted edge should have no crossings.

9. Comments

An important motivation for investigating the crossing numbers of the Generalized Petersen Graphs is the following. It follows from the work in [2] and [7] that for the random cubic graph G, $cr(G) \geq k|V|^2$. At the time of writing, the "mesh of trees" M_n has cn^2 vertices and its crossing number is $c'n^2 \log(n)$ [8,3]. It would be interesting to find a family of cubic graphs whose crossing numbers are proportional to $|V|^{1+\varepsilon}$, for some $\varepsilon > 0$.

Fiorini also claimed [5] that the crossing number of the Derived Generalized Petersen Graph $P'(3k+h,3)$ is $k+h$ for every $h \in \{0,1,2\}$. Unfortunately, the error pointed out in Section 8 also invalidates the proof of this statement. However, we expect that the techniques developed above can be used to show that the crossing number of $P'(3k+h,3)$ is indeed $k+h$ for every h.

As noted in [5], the graphs $P(3k+1,k), P(3k+1,2k+1)$, and $P(3k+1,3k-2)$ have the same crossing number as $P(3k+1,3)$, and each of $P(3k+2,k+3), P(3k+2,2k+1)$, and $P(3k+2,3k-1)$ has the same crossing number as $P(3k+2,3)$. Therefore it follows from Theorem 1 that $cr(P(3k+1,k)) = cr(P(3k+1,2k+1)) = cr(P(3k+1,3k-2)) = k+3$, and that $cr(P(3k+2,k+3)) = cr(P(3k+2,2k+1)) = cr(P(3k+2,3k-1)) = k+2$.

Exoo et al. [4] proved that $cr(P(m,2))$ is 0 for each even $m \geq 4$, and $cr(P(m,2))$ is 3 for each odd $m \geq 7$. They also proved that the crossing numbers of $P(3,2)$ and $P(5,2)$ are 0 and 2, respectively. Fiorini [5] proved that $cr(P(8,3) = 4$; a simple proof can be found in [9]. Except for $P(m,2), P(m,3)$, the graphs in the above paragraph, and for a few isolated cases [F,Y], the crossing number of the Generalized Petersen Graph $P(m,\ell)$ remains unknown.

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