On a crossing number result of Richter and Thomassen*,†

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15 September 1999

Abstract. We show that if G is a graph minimal with respect to having crossing number at least k, and G has no vertices of degree 3, then G has crossing number at most $2k + 35$.

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* Revised version for the "Notes" Section of JCTB

† Originally submitted under the title "Remarks Upon Minimal Graphs with Crossing Number at Least k"

Richter and Thomassen proved that if G is minimal with respect to having crossing number at least k, then the crossing number $cr(G)$ of G is at most $2.5k + 16$ [1]. Our aim in this note is to observe that if G has no vertices of degree 3, then the proportionality constant in this bound can be improved to 2.

THEOREM 1. *Let G be a graph minimal with respect to having crossing number at least k. Suppose that G has no vertices of degree 3. Then $cr(G) \leq 2k + 35$.*

Proof. Our proof is largely based on the proof of Theorem 3 in [1].

As proved in Theorem 3 in [1], we can assume G is simple and has no vertices of degree 2. Since G clearly cannot have vertices of degree 1, it follows that we can assume G has minimum degree at least 4.

Let t_0 be least possible such that there is a set E of t_0 edges such that $G - E$ is planar. Let C be a cycle and v a vertex as in Theorem 2 in [1]. Thus $\sum_{u \in V(C) \setminus \{v\}} (d(u) - 2) \leq t_0 + 36$. Let $e = vw$ be an edge of C incident with v, and let P be the path $C - e$. Since the minimum degree of G is at least 4, it follows that there are at most $(t_0 + 36)/2$ edges in P.

By minimality of G, there is a drawing of $G - e$ with at most $k - 1$ crossings. In this drawing some edges in P may cross other edges in P. We regard the drawing of P as a planar graph H with vertices of degrees 2 and 4. Let P' be a path in H joining v and w.

There are two ways of drawing e close to P', one for each side of P'. It is readily checked that the total number of crossings in these two drawings of e is at most $(t_0 + 36) + 2cr(P)$, where $cr(P)$ is the number of crossings of edges of P in the drawing of $G - e$.

Removing from $G - e$ the (at most $(t_0 + 36)/2$) edges of P leaves a drawing with at most $k - 1 - cr(P)$ crossings. Therefore, there is a set of at most $1 + (t_0 + 36)/2 + (k - 1 - cr(P))$ edges whose removal from G leaves a planar graph. By the definition of t_0, this implies that $cr(P) \leq k - t_0/2 + 18$.

Thus, the number of crossings in the two drawings of e is at most $(t_0 + 36) + 2cr(P) \leq 2k + 72$, and so e can be drawn with at most $k + 36$ crossings. Since $G - e$ is drawn with at most $k - 1$ crossings, it follows that G can be drawn with at most $2k + 35$ crossings. ∎

Theorem 1 implies the following versions of Corollaries 1 and 2 in [1].

COROLLARY 1. *Let G be a graph with crossing number k. Suppose that G has no vertices of degree 3. Then there is an edge e of G such that $cr(G - e) \geq (k - 37)/2$.*

COROLLARY 2. *Let G be a simple graph minimal with respect to having crossing number at least k. If all vertices of G have degree at least 7, then G has at most $4k + 58$ vertices.*

REFERENCES

[1] R.B. Richter and C. Thomassen, Minimal Graphs with Crossing Number at Least k, *J. Combin. Theory Ser. B* **58** (1993), 217–224.