# NEARLY–LIGHT CYCLES IN EMBEDDED GRAPHS AND CROSSING–CRITICAL GRAPHS

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ABSTRACT. We find a lower bound for the proportion of face boundaries of an embedded graph that are nearly–light (that is, they have bounded length and at most one vertex of large degree). As an application, we show that every sufficiently large $k$–crossing–critical graph has crossing number at most $2k + 23$.

## 1. Introduction

It is quite natural to inquire about the existence of "light" subgraphs in a given family $\mathcal{G}$ of graphs. Recall that if $H$ is a subgraph of $G$, then the *weight* $w(H)$ *of* $H$ *in* $G$ is the sum of the valences in $G$ of the vertices in $H$. If there is an integer $w$ such that every graph $G$ in $\mathcal{G}$ that contains a subgraph isomorphic to $H$ contains one such subgraph with weight at most $w$ in $G$, then $H$ is *light* in $\mathcal{G}$. Most research on light subgraphs has focused on the case in which $\mathcal{G}$ is a family of graphs embedded in some compact surface (see for instance [1, 2, 6, 7, 8, 9, 12, 13].

Although under certain conditions one can guarantee the existence of light cycles in embedded graphs (see [5]), this is not always the case: every cycle in a wheel either contains a hub vertex (which can have arbitrarily high degree), or is arbitrarily long (as long as the degree of the hub).

In view of this, a natural way to proceed in this context is to inquire about the existence of "nearly–light" cycles. Let $\ell, \Delta$ be positive numbers. A cycle $C$ in a graph $G$ is $(\ell, \Delta)$*–nearly–light* if the length of $C$ is at most $\ell$, and at most one vertex of $C$ has degree greater than $\Delta$. If $G$ is embedded, we define an $(\ell, \Delta)$*–nearly–light face boundary* similarly, with the convention that an edge that is traversed twice in the boundary walk of a face contributes in two to the length of that face boundary.

In [14], Richter and Thomassen investigated the existence of nearly–light cycles, and proved that every planar graph has at least one $(5, 11)$–nearly–light face boundary. One of the aims in this work is to refine this statement, and show that plane (moreover, embedded) graphs have not one but many nearly–light face boundaries.

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**Theorem 1.** *Let* $0 < \varepsilon < 1/6$*, and let* $G$ *be a simple connected graph with minimum degree at least* 3*, embedded in a (orientable or nonorientable) surface of Euler characteristic* $\chi$*. Let* $F(G)$ *denote the set of faces of* $G$*. Then* $G$ *contains at least* $(2\chi - 1) + \left(\frac{1}{4} - \frac{3\varepsilon}{2}\right)|F(G)|$ *face boundaries that are* $(6, 2/\varepsilon)$*–nearly–light.*

The problem of the existence of nearly–light cycles is raised and attacked in [14] in the context of crossing–critical graphs. We recall that the *crossing number* $\mathrm{cr}(G)$ of a graph $G$ is the minimum number of pairwise crossings of edges in a drawing of $G$ in the plane. A graph $G$ is $k$*–crossing–critical* if its crossing number is at least $k$, but $\mathrm{cr}(G - e) < k$ for every edge $e$ of $G$.

In [14], the existence of a nearly–light cycle is used to prove that every $k$–crossing–critical graph has crossing number at most $2.5k + 16$. As we show below, Theorem 1 implies the following statement on the crossing numbers of sufficiently large crossing–critical graphs.

**Theorem 2.** *For each* $k > 0$ *there is an* $n(k)$ *with the following property. If* $G$ *is a* $k$*–crossing–critical graph with at least* $n(k)$ *vertices of degree greater than two, then* $\mathrm{cr}(G) \leq 2k + 23$*.*

We note that the condition in this statement on the degrees of the vertices (greater than two) is unavoidable, since subdivisions of edges change neither the crossing number of a graph nor its criticality.

Crossing–critical graphs are objects of natural interest whose structure has been the object of recent investigations (see for instance [4]). Moreover, upper bounds for the crossing number of crossing–critical graphs also have an important application. Indeed, as Richter and Thomassen observed, their bound $\mathrm{cr}(G) \leq 2.5k + 16$ for $k$–crossing–critical graphs implies that if $H$ is an arbitrary graph with $\mathrm{cr}(H) = k$, then there is an edge $e$ in $H$ such that $\mathrm{cr}(H - e) \geq (2k - 37)/5$. Along the same lines, it is readily checked that our Theorem 2 implies the following.

**Corollary 3.** *For each* $k > 0$ *there is an* $n(k)$ *with the following property. If* $H$ *has at least* $n(k)$ *vertices of degree greater than two, and* $\mathrm{cr}(H) = k$*, then* $H$ *has an edge* $e$ *such that* $\mathrm{cr}(H - e) \geq (k - 26)/2$*.* □

Very recently, Fox and Toth improved this result for sufficiently dense graphs [3].
We prove Theorems 1 and 2 in Sections 2 and 3, respectively.

## 2. Nearly–light face boundaries in embedded graphs

In this section we show that the technique used in the proof of Theorem 1 in [14] can be refined to give a proof of Theorem 1. For an embedded graph $G$, we let $V(G), E(G)$, and $F(G)$ denote the sets of vertices, edges, and faces of $G$, respectively.

*Proof of Theorem 1.* As in [14], for each face $f$ of $G$ let the *weight* $w(f)$ be the sum $\sum_{v \sim f}(1/d(v))$, where $d(v)$ denotes the degree of vertex $v$ and $v \sim f$ means that $v$ is incident with $f$. Thus, for each face $f$, $w(f) \le l(f)/3$, where $l(f)$ denotes the length of the boundary of $f$.

As in the proof of Theorem 1 in [14], we note that $\sum_{f\in F(G)} w(f) = |V(G)|$, and $\sum_{f\in F(G)} l(f) = 2|E(G)|$. Thus, Euler's formula implies that $\sum_f \{w(f) - l(f)/2+1\} \ge \chi$ (note that strict inequality may occur, as there may be a noncontractible curve in the surface that does not intersect the embedded graph $G$).

Let us say that a face $f$ is *good* if $w(f) - l(f)/2 + 1 > -1/6 + \varepsilon$.

We complete the proof by showing that the following statements hold.

(A) For each good face $f$, the face boundary of $f$ is $(6, 2/\varepsilon)$–nearly–light.

(B) There are at least $(2\chi - 1) + \big(1/4 - 3\varepsilon/2\big)|F(G)|$ good faces.

Let $f$ be a good face, and suppose that $l(f) > 6$ . Since $-1/6+\varepsilon < w(f)-l(f)/2+1$, and $w(f) \le l(f)/3$, then $-1/6 + \varepsilon < -l(f)/6 + 1 \le -7/6 + 1 = -1/6$, contradicting the assumption $\varepsilon > 0$. Thus $l(f) \le 6$. Now suppose that at least two vertices $v$ incident with $f$ have $d(v) > 2/\varepsilon$. Therefore $w(f) < (l(f) - 2)/3 + 2(\varepsilon/2) = (l(f) - 2)/3 + \varepsilon$. Since $-1/6 + \varepsilon < w(f) - l(f)/2 + 1$, it follows that $-1/6 + \varepsilon < l(f)/3 - 2/3 + \varepsilon - l(f)/2 + 1 = -l(f)/6 + 1/3 + \varepsilon$. Hence $l(f) < 3$, contradicting the assumption that $G$ is simple. Hence at most one vertex incident with $f$ has degree greater than $2/\varepsilon$. This proves (A).

Let $D(G)$ denote the set of good faces. Now $\sum_{f\in D(G)}\{w(f) - l(f)/2 + 1\} + \sum_{f\in (F(G)\setminus D(G))}\{w(f) - l(f)/2 + 1\} \ge \chi$. By definition, each $f \in (F(G) \setminus D(G))$ satisfies $w(f) - l(f)/2 + 1 \le -1/6 + \varepsilon$. On the other hand, every face $f$ has $w(f) - l(f)/2 + 1 \le 1/2$. Thus $|D(G)|/2 + (|F(G)| - |D(G)|)(-1/6 + \varepsilon) \ge \chi$. An easy manipulation then yields that $|D(G)| > \big(\frac{(1/6)-\varepsilon}{(2/3)-\varepsilon}\big)|F(G)| + \chi/(2/3 - \varepsilon)$. Hence $|D(G)| > \big(1/4 - 3\varepsilon/2\big)|F(G)| + \chi/(2/3 - \varepsilon)$.

We finally note that $0 < \varepsilon < 1/6$ implies that, if $\chi \le 0$, then $\chi/(2/3 - \varepsilon) \ge 2\chi > 2\chi - 1$. On the other hand, if $\chi > 0$ then $\chi = 1$ or $2$, and so $\chi > 0$ implies $\chi/(2/3-\varepsilon) > 2\chi-1$. It follows that regardless of the sign of $\chi$, $\chi/(2/3-\varepsilon) > 2\chi-1$. Therefore $|D(G)| > \big(1/4 - 3\varepsilon/2\big)|F(G)| + (2\chi - 1)$. This proves (B). □

## 3. Crossing–critical graphs

In this section we prove Theorem 2. The proof has two main ingredients. First we show (Lemma 4) that large crossing–critical graphs have $(6, 12)$–nearly–light cycles. Then we invoke a result (Lemma 5) whose proof is implicit in the proof of Theorem 3 in [14], namely that the existence of a nearly–light cycle in a crossing–critical graph yields an upper bound for the crossing number of the graph.

**Lemma 4.** *For each integer $k > 0$, there is an $n(k)$ with the following property. Let $G$ be a simple $k$–crossing–critical graph with minimum degree at least* 3*. Suppose that $|V(G)| \ge n(k)$. Then $G$ contains a $(6, 12)$–nearly–light cycle.*

*Proof.* First we observe that if $G$ is $k$–crossing–critical, then $G$ can be embedded in the orientable surface $\Sigma_k$ of genus $k$ (that is, Euler characteristic $\chi = 2 - 2k$) . This follows since $G$ contains a set of at most $k$ edges whose deletion leaves $G$ planar.

We show that this embedding has a $(6, 12)$–nearly–light face boundary. This completes the proof, as this face boundary contains the required $(6, 12)$–nearly–light cycle.

Apply Theorem 1 to $G$ embedded in $\Sigma_k$, with $\varepsilon = 4/25$. This yields the existence of at least $(2\chi - 1) + (1/4 - 6/25)|F(G)| = (3 - 4k) + (1/4 - 6/25)|F(G)|$ face boundaries that are $(6, 12)$–nearly–light (note that a $(6, 12.5)$–nearly–light face boundary is $(6, 12)$–nearly–light).

We finally note that if $|V(G)|$ is sufficiently large (compared to $k$), then (by Euler's formula) so is $|F(G)|$, and this in turn guarantees that $(3-4k)+(1/4-6/25)|F(G)| \geq 1$. Therefore, if $|V(G)|$ is sufficiently large, then there is a $(6, 12)$–nearly–light face boundary. □

The proof of the first inequality in the following lemma is implicit in the proof of Theorem 3 in [14]. The second inequality follows from the first inequality and the definition of an $(\ell, \Delta)$–nearly–light cycle.

**Lemma 5.** *Let $G$ be a $k$–crossing–critical graph, and let $s > 0$. Suppose that $G$ has a cycle $C$ with a vertex $v$ such that $\sum_{u \in C \setminus \{v\}} (d(u) - 2) \leq s$. Then*

$$\mathrm{cr}(G) \leq 2(k - 1) + s/2.$$

*Thus, if $G$ has an $(\ell, \Delta)$–nearly–light cycle, then*

$$\mathrm{cr}(G) \leq 2(k - 1) + \frac{(\Delta - 2)(\ell - 1)}{2}.$$

□

*Proof of Theorem 2.* Let $G$ be a $k$–crossing–critical graph. By supressing vertices of degree two if necessary (this affects neither the crossing number nor the criticality) we may assume that $G$ has no vertices of degree two or less. Now suppose that $|V(G)| \geq n(k)$, where $n(k)$ is as in Lemma 4. As in the proof of Theorem 3 in [14], we can assume that $G$ is simple, as otherwise $\mathrm{cr}(G) \leq 2(k - 1)$, in which case we are done. Lemma 4 then applies, and yields the existence of a $(6, 12)$–nearly–light cycle in $G$. By applying Lemma 5 we obtain $\mathrm{cr}(G) \leq 2(k - 1) + (10)(5)/2 = 2k + 23$. □

## 4. Concluding Remarks

It is natural to inquire about the tightness of the bound in Theorem 1. How much can the coefficient of $|F(G)|$ be improved by allowing larger values of $\ell$ and $\Delta$? Consider the following construction. Let $H_0$ be a graph isomorphic to $K_{4,4} - e$, and let $u, v$ denote the degree 2 vertices of $H_0$. Now let $G_n$ be obtained by taking $n$ copies of $H_0$, and identifying them along $u$ and $v$. Thus $G_n$ has two vertices of degree $2n$, and $2n$ vertices of degree 3. Moreover, every planar embedding of $G_n$ has

$n$ faces (of size four) incident with both $u$ and $v$, and $2n$ faces (of size three) incident with two degree 3 vertices and exactly one copy of $H_0$. Thus, for every fixed $\Delta$, if $n$ is sufficiently large then exactly two thirds of the faces of any embedding of $G_n$ are $(\ell, \Delta)$–nearly–light. This shows that the coefficient of $|F(G)|$ in Theorem 1 cannot be improved to a value greater than 2/3, regardless of the size of $\Delta$.

On the other hand, the upper bound 2/3 on the coefficient of $|F(G)|$ can be almost attained as a lower bound, as the following statement claims.

**Theorem 6.** *For each $\varepsilon > 0$ and integer $\chi \leq 2$ there exist $\ell_0 := \ell_0(\varepsilon, \chi), \Delta_0 := \Delta_0(\varepsilon, \chi)$, $c := c(\varepsilon, \chi)$ with the following property. Let $G = (V, E)$ be a simple connected graph with minimum degree at least 3, embedded in a surface with Euler characteristic $\chi$. Let $F$ denote the set of faces of $G$. Then $G$ contains at least $\left(\frac{2}{3} - \varepsilon\right)|F| + c$ face boundaries that are $(\ell_0, \Delta_0)$–nearly–light.*

This result can be proved by direct geometrical methods (see [11]). Unfortunately, these arguments are not nearly as neat and elegant as the powerful technique, introduced by Lebesgue in [10], that we used in the proof of Theorem 1.

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