Small meshes of curves and their role in the analysis of optimal meshes[1]

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Abstract. An (m,n)–*mesh* is a pair $(\mathcal{B},\mathcal{R})$ of families of closed curves in the plane, of sizes m and n, respectively, such that each curve in $\mathcal{B}$ intersects each curve in $\mathcal{R}$. As Richter and Thomassen observed, the minimum number $i^*(m,n)$ of intersections in an (m,n)–mesh is closely related to the crossing number of the Cartesian product $C_m \times C_n$. In their work on intersections of curve systems, Shahrokhi et al. proved general lower bounds for $i^*(m,n)$, and showed that the exact knowledge of $i^*(k,k)$ yields considerably good bounds for $i^*(m,n)$ if $m,n \geq k$, and m is very close to n. Our aim in this paper is to show that comparable (slightly improved) bounds can be obtained by a careful analysis of the nature of the intersections in certain very small $(3,k)$–meshes. The advantage of this approach is that the analysis of $(3,k)$–meshes seems to be a far easier task than the exact computation of $i^*(k,k)$ for large values of k.

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1 Introduction

In their work on curve systems [6], Richter and Thomassen introduced the concept of an (m,n)–mesh, and showed its relevance in the study of certain crossing number problems.

An (m,n)–*mesh* is a pair $(\mathcal{B},\mathcal{R})$ of families of closed curves in the plane, of sizes m and n, respectively, such that every curve in $\mathcal{B}$ intersects every curve in $\mathcal{R}$. Following Richter and Thomassen, let $i^*(\mathcal{B},\mathcal{R})$ denote the total number of intersections in the mesh $(\mathcal{B},\mathcal{R})$, and let $i^*(m,n)$ denote the minimum number of intersections in an (m,n)–mesh. We remark that it is assumed that no point belongs to more than two curves in the mesh.

Richter and Thomassen proved that the crossing number $cr(C_m \times C_n)$ of the Cartesian product of the cycles of sizes m and n satisfies $cr(C_m \times C_n) \geq i^*(m,n) - mn$. As proved in [6], the exact computation of $i^*(m,n)$ for certain small values of m and n yields the exact crossing number of $C_m \times C_n$ for those values of m and n. However, the exact computation of $i^*(m,n)$ seems to be an increasingly difficult task, even for relatively small values of m and n. To this day, the exact value of $i^*(8,8)$ remains unknown.

Using an ingenious argument in combination with the dual of Dilworth's Theorem, Shahrokhi et al. gave general lower bounds for $i^*(m,n)$ [7]. In the same paper, they show how an easy counting argument yields remarkably better bounds if n is close to m. More precisely, they proved that $i^*(m,n) \geq 5mn/3$, if $3 \leq m \leq n \leq 5(m-1)/4$. This bound is obtained using the equality $i^*(6,6) = 60$ [1]. The better bound $i^*(m,n) \geq 12mn/7$ (also for n sufficiently close to m) can be obtained by using $i^*(7,7) = 84$ [2].

Our aim in this paper is to focus on lower bounds for $i^*(m,m)$. The bound $i^*(m,m) \geq 12m^2/7$ (the best possible bound obtainable using the techniques in [7]) is derived by counting the number of $(7,7)$–meshes contained in an (m,m)–mesh, using the equality $i^*(7,7) = 84$, and then estimating how many times each intersection is (over–)counted. Although the calculation is simple and quick enough, the amount of work behind it is magnified by the great effort involved in the calculation of $i^*(7,7)$ [1, 2].

Moreover, the natural way to improve the bound $i^*(m,m) \geq 12m^2/7$ is to calculate $i^*(k,k)$ for larger values of k, and then to apply a counting argument as the one described above. Based on the experience so far, calculating $i^*(k,k)$ for $k \geq 8$ is a daunting task, and the associated improvement is not very significant (assuming that $i^*(k,k)$ is $k^2 + (k-2)k$, as conjectured, this yields the bound $i^*(m,m) \geq ((2k-2)/k)m^2$ for $m \geq k$).

Our goal is to present an alternative way to obtain lower bounds for $i^*(m,m)$. Our approach is based on a deeper analysis of the nature of the intersections in certain small $(3,k)$–meshes. Based on this approach, we prove the following result.

Main Theorem. *For each $\epsilon > 0$ there exists an $M(\epsilon)$ such that, for every $m \geq M(\epsilon)$, $i^*(m,m) \geq (31/18 - \epsilon)m^2$.*

The advantage of this approach is that $(3,k)$–meshes are, in many aspects, easier objects to deal with than arbitrary (k,k)–meshes. An example of this is the exact calculation of $i^*(3,k)$ for every k, performed by Richter and Thomassen [6], against the considerable amount of work and ingenuity required to calculate $i^*(6,6)$ and $i^*(7,7)$ [1, 2].

A motivation for estimating $i^*(m,m)$ arises from the connection between $i^*(m,n)$ and $\mathrm{cr}(C_m \times C_n)$. As we observed above, Richter and Thomassen proved the inequality $\mathrm{cr}(C_m \times C_n) \geq i^*(m,n) - mn$. On the other hand, a long–standing conjecture states that $\mathrm{cr}(C_m \times$

$C_n) = (m-2)n$, for all m, n such that $n \geq m \geq 3$ [5]. This conjecture was recently proved true for n sufficiently large compared to m [4], and is still open for values of n close to m. The case $m = n$ is of particular interest, since one would expect that proving the conjecture for this case would result in a proof (by induction) of the whole conjecture. Since each lower bound for $i^*(m, m)$ immediately implies a lower bound for $\mathrm{cr}(C_m \times C_m)$, the calculation of lower bounds for $i^*(m, m)$ is a natural approach to the problem of finding good lower bounds for $\mathrm{cr}(C_m \times C_m)$ (we note that there are drawings of $C_m \times C_n$ with exactly $(m-2)n$ crossings, and so $\mathrm{cr}(C_m \times C_n) \leq (m-2)n$; hence the interest in lower bounds for $\mathrm{cr}(C_m \times C_n)$).

This paper is organized as follows. In Section 2, we explain how a deeper knowledge of the types of intersections in certain small $(3, k)$–meshes can be used to obtain lower bounds for $i^*(m, m)$. The required facts about $(3, k)$–meshes are then worked out in two steps. First we establish, in Section 3, a series of facts on certain specific small meshes. This knowledge is exploited in Section 4, where a balance between certain types of intersections in $(3, m)$–meshes is obtained. The proof of the Main Theorem is given in Section 5. Thus, all the required geometric facts are established in Section 3. The rest of the paper deals with purely combinatorial (mostly basic counting) arguments.

2 The role of the $(3, m)$–submeshes of an (m, m)–mesh

Throughout this paper, the curves in the first and second families of every mesh will be colored blue and red, respectively.

An intersection point p of two curves R, R' is *tangential* if the rotation scheme in a small neighborhood around p consists of two arcs of R followed by two arcs of R'.

An (m, m)–mesh $(\mathcal{B}, \mathcal{R})$ is *optimal* if $i^*(\mathcal{B}, \mathcal{R}) = i^*(m, m)$. In an optimal mesh, all the curves are simple, and if two curves of the same color intersect, then they do so in at least two points [6]. Since we are interested in submeshes of optimal meshes, we assume that every mesh we deal with satisfies these conditions.

Let $(\mathcal{B}, \mathcal{R})$ be an (m, m)–mesh. Each of the $\binom{m}{3}^2$ different $(3, 3)$–submeshes of $(\mathcal{B}, \mathcal{R})$ has at least 12 intersections [6]. Each such intersection appears in either $m\binom{m}{3}$ or $\binom{m-1}{2}^2$ $(3, 3)$–submeshes, depending on whether the intersection is monochromatic or bichromatic, respectively. Thus, $i^*(\mathcal{B}, \mathcal{R}) \geq 12\binom{m}{3}^2 / \max\{m\binom{m}{3}, \binom{m-1}{2}^2\} = 4m^2/3$.

We remark that, in the bound just obtained, our ignorance of the specific numbers of monochromatic and bichromatic intersections forced us to adopt a worst case scenario, namely, that all the intersections are bichromatic, since it is in this case that each intersection is counted $\binom{m-1}{2}^2 = \max\{m\binom{m}{3}, \binom{m-1}{2}^2\}$ times (note that we are assuming $m \geq 6$).

Now let us consider the individual contribution of each $(3, m)$–submesh separately. First, let us introduce some notation. Let $(\mathcal{P}, \mathcal{R})$ be a $(3, m)$–mesh. For each 3–set $\mathcal{R}_i$ of $\mathcal{R}$, denote by $\beta_0(\mathcal{P}, \mathcal{R}_i)$ (respectively $\mu_0(\mathcal{P}, \mathcal{R}_i)$) the number of bichromatic (respectively monochromatic) intersections in the $(3, 3)$–mesh $(\mathcal{P}, \mathcal{R}_i)$. Finally, define $\beta(\mathcal{P}, \mathcal{R}) = \sum_{i=1}^{\binom{m}{3}} \beta_0(\mathcal{P}, \mathcal{R}_i)$ and $\mu(\mathcal{P}, \mathcal{R}) = \sum_{i=1}^{\binom{m}{3}} \mu_0(\mathcal{P}, \mathcal{R}_i)$.

Let $(\mathcal{B}, \mathcal{R})$ be an (m, m)–mesh. Let $\{\mathcal{B}_i\}$ be the set of all the different 3–sets of $\mathcal{B}$. By an analogous argument to the one used to derive the bound $i^*(\mathcal{B},\mathcal{R}) \geq 4m^2/3$, we get

$$i^*(\mathcal{B},\mathcal{R}) \geq \sum_{i=1}^{\binom{m}{3}} \left\{ \mu(\mathcal{B}_i,\mathcal{R})/(m\binom{m}{3}) + \beta(\mathcal{B}_i,\mathcal{R})/\binom{m-1}{2}^2 \right\}. \qquad (1)$$

This bound equals $4m^2/3$ if (and only if) $\mu(\mathcal{B}_i,\mathcal{R}) = 0$ and $\beta(\mathcal{B}_i,\mathcal{R}) = 12\binom{m}{3}$ for every i. Our claim is that a little more knowledge about the nature of the intersections (monochromatic and bichromatic) in $(3,m)$–meshes will reveal a balance between $\beta(\mathcal{B}_i,\mathcal{R})$ and $\mu(\mathcal{B}_i,\mathcal{R})$ that will result in the much improved bound in the Main Theorem. The required facts about $(3,m)$–meshes are established in the next two sections.

3 Analysis of types of intersections in certain $(3,3)$– and $(3,5)$–meshes

Our task in this section is to prove a few simple facts on the interplay between bichromatic and monochromatic intersections in some specific $(3,3)$– and $(3,5)$–meshes.

Since we are interested in optimal meshes, we can assume that each pair of simple curves under consideration intersect each other finitely often.

A curve C *separates* the curves C' and C'' if $C' \setminus C$ and $C'' \setminus C$ are both connected and nonempty, and are contained in different components of $\mathbb{R}^2 \setminus C$.

Suppose that the curves C_1, C_2 intersect tangentially at a point p. Let U_p be a closed disc with center p such that the restriction of $C_1 \cup C_2$ to U_p is homeomorphic to $\{(x,y) \in \mathbb{R}^2 | (x^2+y^2 \leq 1)$ and $(x=0$ or $y=0)\}$. Moreover, U_p can be chosen so that its boundary intersects each C_i in exactly two points, say s_i and t_i. For each $i \in \{1,2\}$, let C_i' denote the curve obtained from C_i by removing the part of C_i in the interior of U_p and joining s_i and t_i by a straight segment. Clearly, C_1' and C_2' have one fewer intersection point than C_1 and C_2. We say that C_1' and C_2' are obtained by *smoothly removing* the (tangential) intersection point p.

3.1 Meshes in which the blue curves are pairwise disjoint

Let $(\mathcal{B},\mathcal{R})$ be a $(3,3)$–mesh with no blue–blue intersections. A curve in $\mathcal{R}$ is *of Type 1* if it has at least six bichromatic intersections; *of Type 2* if it has four or five bichromatic intersections; and *of Type 3* if it has exactly three bichromatic intersections.

Proposition 1 *Let $(\mathcal{B},\mathcal{R})$ be a $(3,3)$–mesh such that $\mathcal{B}$ is a disjoint family, and such that no curve in $\mathcal{B} \cup \mathcal{R}$ separates two blue curves. Suppose that all the curves in $\mathcal{R}$ are of Types 2 or 3. Then either two red curves of Type 3 intersect each other in at least four points, or there are two red curves, at least one of which is of Type 2, that intersect each other in at least two points.*

Proof. Four cases need to be analyzed separately, depending on exactly how many curves are of Type 2. Their proofs are quite similar to each other. For brevity, we analyze in detail the case where all curves are of Type 2, and omit the proofs of the other three cases.

We can assume without any loss of generality that for each $B \in \mathcal{B}$, the other blue curves in $\mathcal{B}$ are contained in the unbounded region of $\mathbb{R}^2 \setminus B$.

Seeking a contradiction, suppose that the curves in $\mathcal{R} = \{R, R', R''\}$ are pairwise disjoint. Since R'' has fewer than six bichromatic intersections, only one component C'' of $\mathbb{R}^2 \setminus R''$ has non–empty intersection with every curve in $\mathcal{B}$. Thus, both R and R' are contained in C''. Similarly, both R' and R'' are contained in the same component C of $\mathbb{R}^2 \setminus R$, and R and R'' are contained in the same component C' of $\mathbb{R}^2 \setminus R'$.

Let D'' be the component of $\mathbb{R}^2 \setminus R''$ different from C''. Since each curve in $\mathcal{R}$ has either four or five bichromatic intersections, it follows that there are either one or two pieces (simple arcs) of blue curves contained in D''. Each of these arcs forms, together with an arc in R'', a digon that does not intersect other curves in $(\mathcal{B}, \mathcal{R})$. Thus, the result of contracting each such digon to a single point is a new $(3,3)$–mesh in which the red curve derived from R'' has exactly three bichromatic intersections.

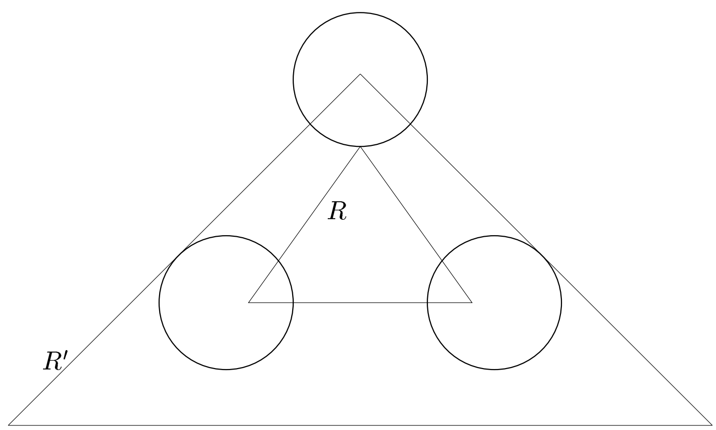

FIGURE 1. Illustration of the proof of Proposition 1. The blue curves are circular–shaped, and the red curves R, R' are triangular–shaped. Since R and R' have 5 and 4 bichromatic intersections, respectively, it follows that they are both indeed of Type 2. It is impossible to add a red curve R'', also of Type 2, without intersecting either R or R'. Indeed, a red curve disjoint from both R and R' would necessarily be of Type 1 (that is, it would have at least 6 bichromatic intersections).

By performing similar operations on the digons that involve R and R', we obtain a $(3,3)$–mesh in which each red curve has exactly three bichromatic intersections, and in which there are no monochromatic intersections. This contradicts the fact that every $(3,3)$–mesh has at least 12 intersections [6]. ∎

3.2 Meshes with exactly two blue–blue intersections

Proposition 2 *Let $(\mathcal{B}, \mathcal{R})$ be a $(3,3)$–mesh with exactly two blue–blue intersections, and let B_0 be a (blue) curve, disjoint from the curves in $\mathcal{B}$, such that $(\mathcal{B} \cup \{B_0\}, \mathcal{R})$ is a $(4,3)$–mesh. Suppose that no curve in $\mathcal{B} \cup \{B_0\} \cup \mathcal{R}$ separates two blue curves. Then the following statements hold.*

(i) Suppose that each red curve has exactly three bichromatic intersections with the curves in $\mathcal{B}$. Then there are two red curves that intersect each other at least 4 times.

(ii) Suppose that no red curve has 6 or more intersections with the curves in $\mathcal{B}$, and suppose that no red curve intersects B_0 four or more times. Then the red curves are not pairwise disjoint.

Proof. We assume without any loss of generality that the curves B, B' in $\mathcal{B}$ intersect each other exactly twice, and that the other blue curve, say B'', is disjoint from both B and B'.

Since there are exactly two blue–blue intersections, there is exactly one component C of $\mathbb{R}^2 \setminus (B \cup B' \cup B'')$ whose boundary intersects all three blue curves.

Proof of (i). Since each red curve has exactly three bichromatic intersections, all red curves are contained in $C \cup (B \cup B' \cup B'')$.

We can assume without any loss of generality (throughout the Proof of (i)) that B'' (and therefore also C) is contained in a bounded region of $\mathbb{R}^2 \setminus (B \cup B')$. Seeking a contradiction, suppose that every two red curves have either 0 or 2 points in common.

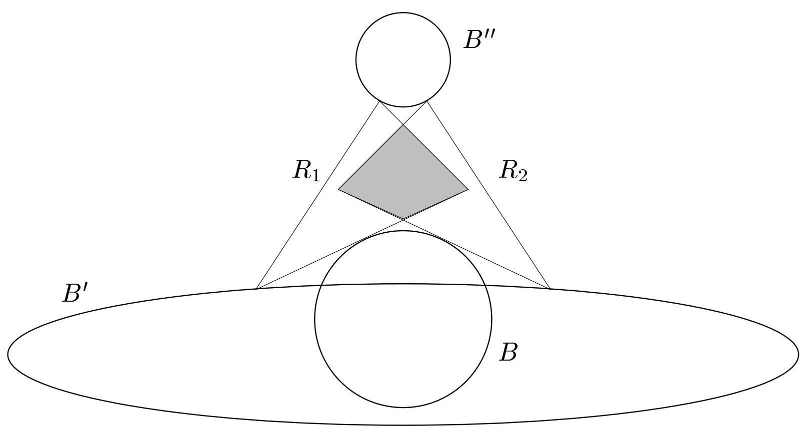

FIGURE 2. Illustration of the proof of (i) in Proposition 2. The red curves R_1, R_2 (triangles) intersect each blue curve exactly once. If R_1 and R_2 intersect in exactly two points, and (by assumption) neither R_1 nor R_2 separates two blue curves, then R_1 and R_2 bound a digon D (shaded region) that intersects no blue curve. The intersections between R_1 and R_2 then can be removed (first transform them into tangential intersections, and then smoothly remove them) without losing the mesh property of $(\mathcal{B}, \mathcal{R})$.

Suppose that the red curves R_1, R_2 intersect each other exactly twice. We now describe how to obtain from $(\mathcal{B}, \mathcal{R})$ a $(3,3)$–mesh where all red–red intersections are tangential, and with the same number of blue–blue and bichromatic intersections as $(\mathcal{B}, \mathcal{R})$.

Since no red curve separates B'' from B or B', it follows that R_1 and R_2 bound a red–red digon D whose interior contains no blue points. Let D_1 and D_2 denote the subarcs of D that belong to R_1 and R_2, respectively, and define $R_1' = (R_1 \setminus D_1) \cup D_2$, and $R_2' = (R_2 \setminus D_2) \cup D_1$. Clearly, R_1' and R_2' have exactly two tangential intersections.

By performing this operation as many times as necessary (at most 3), we obtain a $(3,3)$–mesh with the required properties. If each tangential intersection p is smoothly removed (recall the definition at the beginning of this section), then the result is a $(3,3)$–mesh with two blue–blue intersections and exactly nine bichromatic intersections. This contradicts the fact that no $(3,3)$–mesh has fewer than 12 intersections [6].

Proof of (ii). We can assume without any loss of generality (throughout the Proof of (ii)) that B'' (and therefore also C) is contained in the unbounded region of $\mathbb{R}^2 \setminus (B \cup B')$.

Let R_1, R_2, R_3 denote the red curves in $\mathcal{R}$. By assumption, no R_i separates two blue curves. To complete the Proof of (ii), it suffices to derive a contradiction from the assumption that the following hold: (a) no R_i has 6 or more intersections with the curves in $\mathcal{B}$; (b) the red curves are pairwise disjoint; and (c) B_0 intersects no red curve more than twice (since we are only interested in meshes that are submeshes of optimal meshes, it follows from the Jordan Curve Theorem that each red curve intersects each blue curve either exactly once or an even number of times).

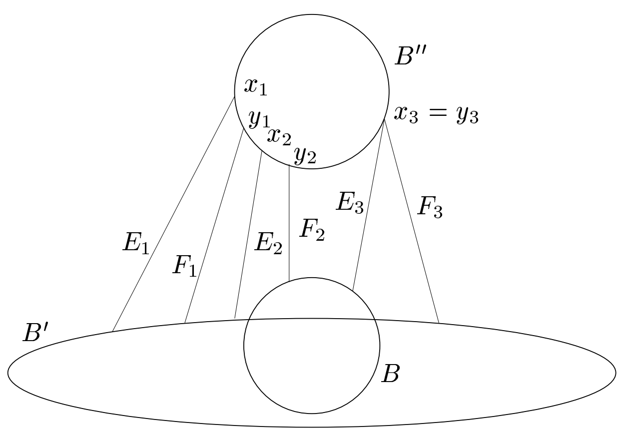

FIGURE 3. Illustration of the proof of (ii) in Proposition 2. For each $i \in \{1,2,3\}$, E_i and F_i are subarcs of the red curve R_i. It is not possible to draw a fourth blue curve B_0, disjoint from the other blue curves, and not separating two blue curves, in such a way that B_0 intersects each $E_i \cup F_i$ at least once and no more than twice. Note that x_i and y_i might coincide for some i (such as $x_3 = y_3$ in this figure). In any case, we can assume without any loss of generality that a curve drawn very close to B'' (and disjoint from B'') crosses $E_1, F_1, E_2, F_2, E_3, F_3$ in the given cyclic order.

First we note that no R_i intersects B'' more than twice (if R_i intersected B'' more than twice, it would do so in at least 4 points, and so it would have at least 6 intersections with the curves in $\mathcal{B}$). Therefore, for each i, the intersection $R_i \cap B''$ equals $\{x_i, y_i\}$ for some (not necessarily different) points x_i, y_i.

Moreover, it is straightforward to check that, since (a) and (b) hold, it follows that we can assume without any loss of generality that the points in $(R_1 \cup R_2 \cup R_3) \cap B''$ appear in B'' in the cyclic order $x_1, y_1, x_2, y_2, x_3, y_3$ (here x_i, y_i must be read as a single point if $x_i = y_i$).

For each $i \in \{1,2,3\}$, there are exactly two simple red arcs E_i, F_i (each of them a subarc of R_i) that satisfy the following conditions: (i) each of E_i and F_i has one end point in $B \cup B'$, and the other end point in $\{x_i, y_i\}$; (ii) each of E_i and F_i is disjoint from $B \cup B' \cup B''$, except for its endpoints; (iii) if $x_i = y_i$, then $E_i \cap F_i = \{x_i\}$, and if $x_i \neq y_i$, then $E_i \cap F_i = \emptyset$; and (iv) if $i \neq j$, then $(E_i \cup F_i) \cap (E_j \cup F_j) = \emptyset$.

We can assume without any loss of generality that a curve in C drawn very close to B'' (and disjoint from B'') intersects the arcs $E_1, F_1, E_2, F_2, E_3, F_3$ in the given cyclic order.

Since B_0 intersects no blue curve, it follows that B_0 intersects R_i (only) in $E_i \cup F_i$. Recall that B_0 intersects each R_i at most twice. If B_0 intersects both E_i and F_i, then both intersections have to be crossings rather than tangential (as otherwise one of the intersections could be smoothly removed without altering the mesh properties of $(\mathcal{B} \cup \{B_0\}, \mathcal{R})$). On the other hand, it is readily checked that in this case B_0 would separate B'' from each of B and B', contradicting our hypothesis. Thus B_0 intersects (moreover, crosses exactly twice), for each i, exactly one of E_i and F_i. Also, by assumption, B_0 intersects no curve in $\mathcal{B}$. It is straightforward to check that the statement in the previous paragraph implies that these conditions cannot hold simultaneously. ∎

3.3 Meshes with exactly four blue–blue intersections

Our last geometric result can be proved using the same techniques as in Propositions 1 and 2. Alternatively, it follows immediately from the fact that $i^*(3,5) = 20$ [6].

Proposition 3 *Let $(\mathcal{B}, \mathcal{R})$ be a $(3,5)$–mesh with exactly four blue–blue intersections. Suppose that each red curve has exactly three bichromatic intersections. Then the red curves are not pairwise disjoint.* ∎

4 Analysis of types of intersections in $(3, m)$–meshes

Our aim in this section is to establish a balance between the bichromatic and monochromatic intersections in certain $(3,m)$–meshes. We recall from Section 2 that if $(\mathcal{B}, \mathcal{R})$ is an (m,m)–mesh, then $i^*(\mathcal{B}, \mathcal{R})$ is bounded by below by a sum of quantities of the form $\mu(\mathcal{B}_i, \mathcal{R}) / \left(m\binom{m}{3}\right) + \beta(\mathcal{B}_i, \mathcal{R}) / \binom{m-1}{2}^2$, where each $(\mathcal{B}_i, \mathcal{R})$ is a $(3,m)$–submesh of $(\mathcal{B}, \mathcal{R})$.

Each of these terms is greater than $(1/(m^2(m-1)(m-2)))(6\mu(\mathcal{B}_i, \mathcal{R}) + 4\beta(\mathcal{B}_i, \mathcal{R}))$. Our goal is to bound $6\mu(\mathcal{P}, \mathcal{R}) + 4\beta(\mathcal{P}, \mathcal{R})$ for certain $(3,m)$–meshes $(\mathcal{P}, \mathcal{R})$. It will suffice to find bounds for $(3,m)$–meshes with a very small (either 0, 2, 4, or 6) number of blue–blue intersections. These cases are analyzed separately in Lemmas 4, 5, 6, and 7, respectively.

Lemma 4 *Let $(\mathcal{P}, \mathcal{R})$ be a $(3,m)$–mesh with no blue–blue intersections. Suppose that no curve in $\mathcal{P} \cup \mathcal{R}$ separates two blue curves. Then for every $\epsilon > 0$ there exists an $M_1(\epsilon)$ such that $m \geq M_1(\epsilon)$ implies $6\mu(\mathcal{P}, \mathcal{R}) + 4\beta(\mathcal{P}, \mathcal{R}) > (31/3 - \epsilon)m^3$.*

Proof. Since $(\mathcal{P}, \mathcal{R})$ is a mesh, it follows that each curve in $\mathcal{R}$ is of one of the types described just before Proposition 1. Let $\mathcal{R}_i$ be the subset of $\mathcal{R}$ consisting of those curves of Type i, for $i \in \{1, 2, 3\}$. Define $r_i = |\mathcal{R}_i|/m$. Clearly, $r_1 + r_2 + r_3 = 1$.

Since the curves involved in a bichromatic intersection are contained in $\binom{m-1}{2}$ different $(3,3)$–submeshes of $(\mathcal{P}, \mathcal{R})$, it follows that $\beta(\mathcal{P}, \mathcal{R}) \geq \binom{m-1}{2}(6r_1m + 4r_2m + 3r_3m)$.

The curves involved in each monochromatic (necessarily red–red) intersection are contained in $m-2$ different $(3,3)$–submeshes of $(\mathcal{P}, \mathcal{R})$. Thus, the number of monochromatic intersections times $m-2$ gives a lower bound on $\mu(\mathcal{P}, \mathcal{R})$.

Proposition 1 gives a bound on the number of monochromatic intersections for each type of 3–set of $\mathcal{R}_2 \cup \mathcal{R}_3$. In particular, no such 3–set consists of pairwise disjoint curves. Let G denote the graph with vertex set $\mathcal{R}_2 \cup \mathcal{R}_3$, where two vertices (curves) are joined by an edge iff they are disjoint. Let H be the subgraph of G induced by $\mathcal{R}_3$. Note that G does not contain a clique on three vertices. Thus, an application of Turán's Theorem to both G and H (cf. Theorem 7.1.1 in [3]) and Proposition 1 yield that the minimum number of monochromatic intersections is attained when (i) the curves in $\mathcal{R}_2 \cup \mathcal{R}_3$ are partitioned into two groups $\mathcal{G}, \mathcal{G}'$ of sizes as equal as possible, such that no curve in $\mathcal{G}$ intersects a curve in $\mathcal{G}'$, and (ii) the curves in $\mathcal{R}_3$ are divided into two groups of sizes as equal as possible, one contained in $\mathcal{G}$ and the other one contained in $\mathcal{G}'$. A straightforward counting procedure then shows that $\mu(\mathcal{P}, \mathcal{R}) \geq (m-2)(2(\binom{\lfloor (r_2+r_3)m/2 \rfloor}{2} + \binom{\lceil (r_2+r_3)m/2 \rceil}{2}) + 2(\binom{\lfloor r_3m/2 \rfloor}{2} + \binom{\lceil r_3m/2 \rceil}{2}))$.

Using the bounds obtained for $\beta(\mathcal{P}, \mathcal{R})$ and $\mu(\mathcal{P}, \mathcal{R})$, we obtain a bound for $6\mu(\mathcal{P}, \mathcal{R}) + 4\beta(\mathcal{P}, \mathcal{R})$. This is a cubic polynomial on m, whose coefficients depend on r_1, r_2, and r_3.

A routine elementary calculus argument shows that the cubic coefficient is minimized at $31/3$, when $r_1 = r_2 = r_3 = 1/3$. On the other hand, the coefficients of the non–cubic terms are bounded functions of the variables r_i, when taking into account the constraints $r_i \geq 0$, $r_1 + r_2 + r_3 = 1$. Thus, for each fixed $\epsilon > 0$, if m is sufficiently large, then $6\mu(\mathcal{P}, \mathcal{R}) + 4\beta(\mathcal{P}, \mathcal{R}) > (31/3 - \epsilon)m^3$. ∎

Lemma 5 *Let $(\mathcal{P}, \mathcal{R})$ be a $(3,m)$–mesh with exactly two blue–blue intersections. Suppose that no curve separates two blue curves. Then for every $\epsilon > 0$ there exists an $M_2(\epsilon)$ such that, for each $m \geq M_2(\epsilon)$, the following statements hold.*

(i) Suppose that the total number of bichromatic intersections in $(\mathcal{P}, \mathcal{R})$ is at most $(13/4)m$. Then $6\mu(\mathcal{P}, \mathcal{R}) + 4\beta(\mathcal{P}, \mathcal{R}) > (95/8 - \epsilon)m^3$.

(ii) Suppose that there is a (blue) curve B_0, disjoint from the curves in $\mathcal{P}$, such that $(\mathcal{P} \cup \{B_0\}, \mathcal{R})$ is a $(4,m)$–mesh, and such that B_0 has at most $(31/18)m$ intersections with the curves in $\mathcal{R}$. Then $6\mu(\mathcal{P}, \mathcal{R}) + 4\beta(\mathcal{P}, \mathcal{R}) > (98/9 - \epsilon)m^3$.

Proof of (i). Let $\mathcal{R}_1$ denote the subset of $\mathcal{R}$ consisting of those curves with at least 4 bichromatic intersections (with the curves in $\mathcal{P}$), and let $\mathcal{R}_2 = \mathcal{R} \setminus \mathcal{R}_1$. Define r_i by the relation $r_im = |\mathcal{R}_i|$, for each $i \in \{1, 2\}$. As in the proof of Lemma 4, it follows easily that $\beta(\mathcal{P}, \mathcal{R}) \geq \binom{m-1}{2}(4r_1m + 3r_2m)$.

Note that, by assumption, $r_1 \leq m/4$.

By (i) in Proposition 2, given any three curves in $\mathcal{R}_2$, at least two of them intersect in at least four points. Let G denote the graph with vertex set $\mathcal{R}_2$, where two vertices (curves) are joined by an edge iff they are disjoint. Note that G does not contain a clique on three vertices. As in the proof of Lemma 4, an application of Turán's Theorem (to G) yields that the minimum number of monochromatic intersections is attained when the curves in $\mathcal{R}_2$ are divided into two groups of sizes as equal as possible, such that any two curves in distinct groups are disjoint. Since each $(3,3)$–submesh of $(\mathcal{P},\mathcal{R})$ has two blue–blue intersections, it follows that $\mu(\mathcal{P},\mathcal{R}) \geq 2\binom{m}{3} + (m-2)4(\binom{\lfloor r_2 m/2 \rfloor}{2} + \binom{\lceil r_2 m/2 \rceil}{2})$.

These bounds for $\beta(\mathcal{P},\mathcal{R})$ and $\mu(\mathcal{P},\mathcal{R})$ yield a bound for $6\mu(\mathcal{P},\mathcal{R}) + 4\beta(\mathcal{P},\mathcal{R})$, a cubic polynomial on m whose coefficients depend on r_1 and r_2. The cubic coefficient (recall the constraint $r_1 \leq m/4$) is minimized at $95/8$, when $r_2 = 3/4$ (and so $r_1 = 1/4$, its maximum feasible value). The coefficients of the non–cubic terms are bounded functions of r_1 and r_2, since $0 \leq r_1, r_2 \leq 1$. Thus, for each fixed $\epsilon > 0$, if m is sufficiently large, then $6\mu(\mathcal{P},\mathcal{R}) + 4\beta(\mathcal{P},\mathcal{R}) > (95/8 - \epsilon)m^3$.

Proof of (ii). Let $\mathcal{R}_0$ denote the subset of $\mathcal{R}$ consisting of those curves that intersect B_0 in at least 4 points. Since by assumption B_0 has at most $(31/18)m$ intersections (m of which are forced bichromatic intersections, one with each curve in $\mathcal{R}$), it follows that $|\mathcal{R}_0|$ is at most $(1/3)((31/18)m - m) = (13/54)m$.

Let $\mathcal{R}_1$ denote the subset of $\mathcal{R} \setminus \mathcal{R}_0$ consisting of those curves with at least 6 bichromatic intersections with the curves in $\mathcal{P}$, let $\mathcal{R}_2$ denote the subset of $\mathcal{R} \setminus (\mathcal{R}_0 \cup \mathcal{R}_1)$ consisting of those curves with at least 4 bichromatic intersections with the curves in $\mathcal{P}$, and let $\mathcal{R}_3 = \mathcal{R} \setminus (\mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2)$. Define r_i by the relation $r_i m = |\mathcal{R}_i|$, for each $i \in \{0,1,2,3\}$. Thus, $r_0 \leq 13/54$, and so $r_1 + r_2 + r_3 \geq 41/54$.

As in the proof of Lemma 4, it follows easily that $\beta(\mathcal{P},\mathcal{R}) \geq \binom{m-1}{2}(6r_1 m + 4r_2 m + 3(r_0 + r_3)m) = \binom{m-1}{2}(3m) + \binom{m-1}{2}(3r_1 m + r_2 m)$ (note that, in principle, each curve in $\mathcal{R}_0$ could have only 3 bichromatic intersections with the curves in $\mathcal{P}$).

Statements (i) and (ii) in Proposition 2 give a bound on the number of red–red intersections for each type of 3–set of $\mathcal{R}_2 \cup \mathcal{R}_3$. In particular, no such 3–set consists of pairwise disjoint curves. Let G denote the graph with vertex set $\mathcal{R}_2 \cup \mathcal{R}_3$, where two vertices (curves) are joined by an edge iff they are not disjoint. Let H be the subgraph of G induced by $\mathcal{R}_3$. As in Lemma 4, an application of Turán's Theorem (to both G and H) and Proposition 1 yield that the minimum number of monochromatic intersections is attained when (i) the curves in $\mathcal{R}_2 \cup \mathcal{R}_3$ are partitioned into two groups $\mathcal{G}, \mathcal{G}'$ of sizes as equal as possible, such that no curve in $\mathcal{G}$ intersects a curve in $\mathcal{G}'$, and (ii) the curves in $\mathcal{R}_3$ are divided into two groups of sizes as equal as possible, one contained in $\mathcal{G}$ and the other one contained in $\mathcal{G}'$. A straightforward counting procedure then shows that $\mu(\mathcal{P},\mathcal{R}) \geq 2\binom{m}{3} + (m-2)(2(\binom{\lfloor (r_2+r_3)m/2 \rfloor}{2} + \binom{\lceil (r_2+r_3)m/2 \rceil}{2}) + 2(\binom{\lfloor r_3 m/2 \rfloor}{2} + \binom{\lceil r_3 m/2 \rceil}{2}))$ (we remark that the first term on the right hand side corresponds to the two blue–blue intersections in $(\mathcal{P},\mathcal{R})$).

Using the bounds obtained for $\beta(\mathcal{P},\mathcal{R})$ and $\mu(\mathcal{P},\mathcal{R})$, we obtain a bound for $6\mu(\mathcal{P},\mathcal{R}) + 4\beta(\mathcal{P},\mathcal{R})$. This is a cubic polynomial on m, whose coefficients depend on r_1, r_2, and r_3.

A routine elementary calculus argument shows that the cubic coefficient is minimized (taking into account the constraints $r_1 + r_2 + r_3 \geq (41/54)m$, $r_1, r_2, r_3 \geq 0$) at $98/9$, when $r_1 = 5/54, r_2 = r_3 = 1/3$. On the other hand, the coefficients of the non–cubic terms are bounded functions of the variables r_i, when taking into account the constraints

$r_i \geq 0$, $r_1 + r_2 + r_3 \leq 1$. Thus, for each fixed $\epsilon > 0$, if m is sufficiently large, then $6\mu(\mathcal{P},\mathcal{R}) + 4\beta(\mathcal{P},\mathcal{R}) > (98/9 - \epsilon)m^3$. ∎

Lemma 6 *Let $(\mathcal{P},\mathcal{R})$ be a $(3,m)$–mesh with exactly four blue–blue intersections. Then for every $\epsilon > 0$ there is an $M_3(\epsilon)$ such that $m \geq M_3(\epsilon)$ implies $6\mu(\mathcal{P},\mathcal{R}) + 4\beta(\mathcal{P},\mathcal{R}) > (34/3 - \epsilon)m^3$.*

Proof. Let $\mathcal{R}_1$ denote the set of curves in $\mathcal{R}$ that have at least four bichromatic intersections, and let $\mathcal{R}_2 = \mathcal{R} \setminus \mathcal{R}_1$. Define r_i by the relation $|\mathcal{R}_i| = r_i m$, for each $i \in \{1,2\}$.

Using Proposition 3 and the same arguments used in the proofs of Lemmas 4 and 5, we get $6\mu(\mathcal{P},\mathcal{R}) + 4\beta(\mathcal{P},\mathcal{R}) \geq 4\binom{m-1}{2}(4r_1 + 3r_2)m + 6(4\binom{m}{3} + (m-2)2(4\binom{\lfloor r_2 m/4 \rfloor}{2})))$. The leading coefficient of this cubic polynomial in m is minimized at $34/3$, when $r_2 = 2/3$ (and so $r_1 = 1/3$). Since $0 \leq r_1, r_2 \leq 1$, the non–cubic coefficients are bounded functions of r_1 and r_2. So, for each $\epsilon > 0$, if m is sufficiently large, then $6\mu(\mathcal{P},\mathcal{R}) + 4\beta(\mathcal{P},\mathcal{R}) > (34/3 - \epsilon)m^3$. ∎

Finally, we analyze the case in which there are at least six blue–blue intersections.

Lemma 7 *Let $(\mathcal{P},\mathcal{R})$ be a $(3,m)$–mesh with at least six blue–blue intersections. Then, for every $\epsilon > 0$ there is an $M_4(\epsilon)$ such that $m \geq M_4(\epsilon)$ implies $6\mu(\mathcal{P},\mathcal{R}) + 4\beta(\mathcal{P},\mathcal{R}) > (12 - \epsilon)m^3$.*

Proof. Since each $(3,3)$–submesh of $(\mathcal{P},\mathcal{R})$ has at least 6 monochromatic intersections and 9 bichromatic intersections, then $6\mu(\mathcal{P},\mathcal{R}) + 4\beta(\mathcal{P},\mathcal{R}) \geq (6 \cdot 6 + 4 \cdot 9)\binom{m}{3} = 72\binom{m}{3}$. Thus, if m is sufficiently large, then $6\mu(\mathcal{P},\mathcal{R}) + 4\beta(\mathcal{P},\mathcal{R}) > (72/6 - \epsilon)m^3 = (12 - \epsilon)m^3$. ∎

5 Proof of the Main Theorem

The Main Theorem is an immediate consequence of the following result.

Theorem 8 *Let $\epsilon > 0$ be given. There exists an $M'(\epsilon)$ such that, for every (m,m)–mesh $(\mathcal{B},\mathcal{R})$ with $m \geq M'(\epsilon)$, $i^*(\mathcal{B},\mathcal{R}) > (31/18 - \epsilon)(m - M'(\epsilon))^2$.*

Proof. Let $M'(\epsilon)$ be an integer larger than the maximum of $72, M_1(\epsilon), M_2(\epsilon), M_3(\epsilon)$, and $M_4(\epsilon)$, in the statements of Lemmas 4, 5, 6, and 7, respectively.

We proceed by induction on m. The statement is obviously true for $m = M'(\epsilon)$. We assume it holds for $m = k - 1 \geq M'(\epsilon)$, and consider the case $m = k$.

By interchanging $\mathcal{B}$ and $\mathcal{R}$ if necessary, we assume that the blue curve with the largest number of intersections has no more intersections than the red curve with the largest number of intersections. It is readily checked that if either (a) a curve in $\mathcal{B} \cup \mathcal{R}$ separates two blue curves, or (b) some blue curve has at least $(31/18)k$ intersections, then the result follows from the induction hypothesis. Thus, we assume that neither (a) nor (b) holds.

We now apply (1) with $m = k$. Each term on the right hand side of (1) is greater than $(1/(k^2(k-1)(k-2)))\{6\mu(\mathcal{B}_i,\mathcal{R}) + 4\beta(\mathcal{B}_i,\mathcal{R})\}$, and so

$$i^*(\mathcal{B},\mathcal{R}) \geq (1/(k^2(k-1)(k-2))) \sum_{i=1}^{\binom{k}{3}} \{6\mu(\mathcal{B}_i,\mathcal{R}) + 4\beta(\mathcal{B}_i,\mathcal{R})\}. \tag{2}$$

We now show that $6\mu(\mathcal{B}_i,\mathcal{R})+4\beta(\mathcal{B}_i,\mathcal{R}) > (31/3-\epsilon)k^3$ for each $(3,k)$–submesh $(\mathcal{B}_i,\mathcal{R})$ of $(\mathcal{B},\mathcal{R})$.

Let $(\mathcal{B}_i,\mathcal{R})$ be a $(3,k)$–submesh of $(\mathcal{B},\mathcal{R})$.

First we note that it follows from Lemmas 4, 6, and 7 that if $(\mathcal{B}_i,\mathcal{R})$ has either 0 or more than 2 blue–blue intersections, then $6\mu(\mathcal{B}_i,\mathcal{R})+4\beta(\mathcal{B}_i,\mathcal{R}) > (31/3-\epsilon)k^3$, as claimed. Thus it suffices to analyze the case where $(\mathcal{B}_i,\mathcal{R})$ has exactly two blue–blue intersections.

Suppose then that $(\mathcal{B}_i,\mathcal{R})$ has exactly two blue–blue intersections. If the total number of bichromatic intersections that involve curves in $\mathcal{B}_i$ is at most $(13/4)k$, then we are done by (i) in Lemma 5. Thus we suppose that there are more than $(13/4)k$ bichromatic intersections that involve curves in $\mathcal{B}_i$.

Since by assumption no curve in $\mathcal{B}_i$ has $(31/18)k$ (or more) intersections in total, it follows that the curves in $\mathcal{B}_i$ have fewer than $(93/18)k$ intersections in total. More than $(13/4)k$ of these are bichromatic, and so the curves in $\mathcal{B}_i$ have fewer than $(93/18-13/4)k = (23/12)k$ monochromatic intersections in total.

If each curve in $\mathcal{B}\setminus\mathcal{B}_i$ intersected a curve in $\mathcal{B}_i$, then the curves in $\mathcal{B}_i$ would have at least $2(k-3)$ monochromatic intersections (recall that blue curves that intersect do so in at least two points). Since $2(k-3) > (23/12)k$ (as $k > 72$), it follows that some curve in $\mathcal{B}\setminus\mathcal{B}_i$ is disjoint from every curve in $\mathcal{B}_i$. In this case the hypotheses of (ii) in Lemma 5 hold, and so $6\mu(\mathcal{B}_i,\mathcal{R})+4\beta(\mathcal{B}_i,\mathcal{R}) > (98/9-\epsilon)k^3 > (31/3-\epsilon)k^3$, as required.

Therefore, for each $(3,k)$–submesh $(\mathcal{B}_i,\mathcal{R})$ of $(\mathcal{B},\mathcal{R})$ the inequality $6\mu(\mathcal{B}_i,\mathcal{R})+4\beta(\mathcal{B}_i,\mathcal{R}) > (31/3-\epsilon)k^3$ holds. Hence it follows by (2) that $i^*(\mathcal{B},\mathcal{R}) \geq (1/(k^2(k-1)(k-2)))\sum_{i=1}^{\binom{k}{3}}(31/3-\epsilon)k^3 = (31/18-\epsilon/6)k^2 > (31/18-\epsilon)k^2 > (31/18-\epsilon)(k-M'(\epsilon))^2$. ∎

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