A Lower Bound for the Crossing Number of $C_m \times C_n$ *

Gelasio Salazar

IICO–UASLP

Av. Karakorum 1470. Lomas 4ta. Seccion. San Luis Potosi, SLP Mexico 78210

25 November 1999

* Final Revised Version for the Journal of Graph Theory

Abstract. We prove that the crossing number of $C_m \times C_n$ is at least $(m-2)n/3$, for all m, n such that $n \geq m$. This is the best general lower bound known for the crossing number of $C_m \times C_n$, whose exact value has been long conjectured to be $(m-2)n$, for $3 \leq m \leq n$.

1. Introduction

The *crossing number* $cr(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane. Long–standing conjectures in graph theory involve the crossing numbers of various families of graphs, such as the complete graphs K_n [5], the complete bipartite graphs $K_{m,n}$ [13], and the Cartesian products of cycles [6]. In [6], Harary et al. conjectured that the crossing number of the Cartesian product $C_m \times C_n$ is $(m-2)n$, if $n \geq m \geq 3$. This has been proved only for m, n satisfying $n \geq m$, $3 \leq m \leq 6$ [9], [4], [3], [8], [7], [1], [10], and for the special case $m = n = 7$ [2]. Our aim in this paper is to give a general lower bound for the crossing number of $C_m \times C_n$.

Theorem 1. *If $n \geq m \geq 3$, then $cr(C_m \times C_n) \geq (m-2)n/3$.*

Prior to this paper, the best general lower bound known for $cr(C_m \times C_n)$ is reported in [12], where the dual of Dilworth's Theorem is used to show that $cr(C_m \times C_n)$ is at least $mn/9$. Our proof of Theorem 1 is by induction on $m+n$, using as an inductive base the fact that $cr(C_3 \times C_l) = l$ for each $l \geq 3$ [9].

The *Cartesian product* $C_m \times C_n$ is a 4–regular graph with mn vertices $v_{i,j}$, where $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$. The vertices are labeled in such a way that the four vertices adjacent to $v_{i,j}$ are $v_{i-1,j}, v_{i+1,j}, v_{i,j-1}$, and $v_{i,j+1}$, where indices i and j are read modulo m and n, respectively. Thus, the edge set of $C_m \times C_n$ can be partitioned into m edge sets of n–cycles and n edge sets of m–cycles.

To simplify the discussion, we color all n–cycles blue and all m–cycles red. We label the blue cycles $(v_{i,j}), 0 \leq j \leq n-1$, by B_i, $0 \leq i \leq m-1$, and the red cycles $(v_{i,j})$, $0 \leq i \leq m-1$, by R_j, $0 \leq j \leq n-1$. We denote by $P_{i,j}$ the blue 2–path that joins $v_{i,j-1}$ to $v_{i,j+1}$.

A crossing in a drawing of $C_m \times C_n$ is *proper* if no monochromatic cycle contains both edges involved in the crossing. If H is a subgraph of $C_m \times C_n$, we denote by $pcr(H)$ the number of proper crossings that involve at least one edge in H.

A drawing of a graph is *good* if every intersection of edges is a crossing rather than tangential, no edge crosses itself, no two adjacent edges cross, and no two edges cross more than once. It is a routine exercise to show that if $\mathcal{D}$ is a drawing of G with the minimum number of crossings, then $\mathcal{D}$ is good.

As shown in [6], it is possible to draw $C_m \times C_n$ in the plane with exactly $(m-2)n$ crossings, which is thus an upper bound. General lower bounds are a lot more difficult to compute. The only previous general lower bound appears in [12], where lower bounds for the number of crossings in drawings of $C_m \times C_n$ in the projective plane and in the Klein bottle are also given.

The strategy of the proof of Theorem 1 is as follows. We use induction on $m+n$, with the inductive base that every drawing of $C_3 \times C_l$ has at least l crossings, for each $l \geq 3$ (see [9]). If a given drawing $\mathcal{D}$ of $C_m \times C_n$ has a red cycle with at least $(m-2)/3$ crossings, then by the induction hypothesis it follows that $\mathcal{D}$ has at least $(m-2)n/3$ crossings, as required. On the other hand, as established in Section 2, if no red cycle has $(m-2)/3$ crossings, then there are a lot of proper crossings that involve blue edges. In Section 3 we show that the number of such crossings is large enough to guarantee that some blue cycle has at least $n/3$ crossings. Thus, in this case the induction hypothesis also implies that $\mathcal{D}$ has at least $(m-2)n/3$ crossings.

2. A Lemma

The following result is the heart of the proof of Theorem 1.

Lemma 2. *In any good drawing of $C_m \times C_n$, at least one of the following statements holds:*

(I) For each fixed j, $\sum_{i=0}^{m-1} pcr(P_{i,j}) \geq \lceil 2m/3 \rceil$.

(II) There is a red cycle with at least $\lceil (m-2)/3 \rceil$ crossings.

Proof. We assume that neither (I) nor (II) holds, and derive a contradiction. Thus, we assume that (i) there is a j such that $\sum_{i=0}^{m-1} pcr(P_{i,j}) < \lceil 2m/3 \rceil$, and (ii) every red cycle has fewer than $\lceil (m-2)/3 \rceil$ crossings.

Let $\mathcal{P}$ be any maximal subset of pairwise noncrossing 2–paths in $\cup_{i=0}^{m-1}\{P_{i,j}\}$. Let $\mathcal{P}_{rot}$ be the subset of $\mathcal{P}$ that consists of those 2–paths whose middle vertex has rotation scheme red–blue–red–blue. Let $\mathcal{R}_j$ be the subset of $\mathcal{P} \setminus \mathcal{P}_{rot}$ that consists of those 2–paths that are crossed by an edge in R_j, and let $\mathcal{R}_{j-1,j+1}$ be the subset of $\mathcal{P} \setminus (\mathcal{P}_{rot} \cup \mathcal{R}_j)$ that consists of those 2–paths that are crossed by an edge in either R_{j-1} or R_{j+1}.

We now show that if $\mathcal{Q} = \mathcal{P} \setminus (\mathcal{P}_{rot} \cup \mathcal{R}_j \cup \mathcal{R}_{j-1,j+1})$ is not empty, then there are at least $|\mathcal{Q}|-2$ proper crossings that involve only edges in R_{j-1}, R_j, or R_{j+1}.

The end vertices of each 2–path in $\mathcal{Q}$ can be joined by a new green edge very close to the 2–path, in such a way that no green edge crosses an element in $\mathcal{Q}$, no green edge crosses an edge in R_{j-1}, R_j, or R_{j+1}, and no two green edges cross. As a result we obtain a drawing $\mathcal{D}'$ of $C_3 \times C_{|\mathcal{Q}|}$ that consists of $|\mathcal{Q}|$ blue–green cycles and three red cycles. Since no blue–green cycle is crossed in $\mathcal{D}'$, it follows from the Main Theorem in [11] that there are at least $|\mathcal{Q}|-2$ proper crossings that involve only edges in R_{j-1}, R_j, or R_{j+1}. Let q_j be the number of such crossings that involve an edge in R_j.

It is easy to check that the following inequalities hold: (a) $\sum_{i=0}^{m-1} pcr(P_{i,j}) \geq 2(m-|\mathcal{P}|)+(|\mathcal{R}_j|+|\mathcal{R}_{j-1,j+1}|)$; (b) $pcr(R_{j-1})+pcr(R_{j+1}) \geq 2(|\mathcal{Q}|-2)-q_j+|\mathcal{R}_{j-1,j+1}| = 2|\mathcal{P}|-2|\mathcal{P}_{rot}|-2|\mathcal{R}_j|-|\mathcal{R}_{j-1,j+1}|-q_j-4$; and (c) $pcr(R_j) \geq |\mathcal{P}_{rot}|+|\mathcal{R}_j|+q_j$. We remark that to derive (c) we use the fact that if a red cycle and a blue cycle have a vertex in common with rotation scheme red–blue–red–blue, then these cycles cross at least once. This follows from the Jordan Curve Theorem, and the observation that each red cycle has exactly one vertex in common with each blue cycle.

Using (a), (b), (c), and (i) and (ii) at the beginning of the proof, we obtain that the following inequalities must hold: $2(m-|\mathcal{P}|)+|\mathcal{R}_j|+|\mathcal{R}_{j-1,j+1}| < \lceil 2m/3\rceil$; $2|\mathcal{P}|-2|\mathcal{P}_{rot}|-2|\mathcal{R}_j|-|\mathcal{R}_{j-1,j+1}|-q_j-4 < \lceil (m-2)/3\rceil+\lfloor (m-2)/3\rfloor$; and $|\mathcal{P}_{rot}|+|R_j|+q_j < \lceil (m-2)/3\rceil$. Thus the following inequalities hold:

(1) $2(m-|\mathcal{P}|)+|\mathcal{R}_j|+|\mathcal{R}_{j-1,j+1}| \leq \lceil 2m/3\rceil - 1$.

(2) $2|\mathcal{P}|-2|\mathcal{P}_{rot}|-2|\mathcal{R}_j|-|\mathcal{R}_{j-1,j+1}|-q_j-4 \leq \lceil (m-2)/3\rceil+\lfloor (m-2)/3\rfloor - 1$.

(3) $|\mathcal{P}_{rot}|+|\mathcal{R}_j|+q_j \leq \lceil (m-2)/3\rceil - 1$.

Combining (2) and (3) we get $2|\mathcal{P}|+q_j-|\mathcal{R}_{j-1,j+1}| \leq 3\lceil (m-2)/3\rceil+\lfloor (m-2)/3\rfloor+1$. Since $q_j \geq 0$, this implies $2|\mathcal{P}|-|\mathcal{R}_{j-1,j+1}|-1 \leq 3\lceil (m-2)/3\rceil+\lfloor (m-2)/3\rfloor$. On the other hand, by the nonnegativity of $|\mathcal{R}_j|$, (1) implies $2m-\lceil 2m/3\rceil \leq 2|\mathcal{P}|-|\mathcal{R}_{j-1,j+1}|-1$. Therefore $2m-\lceil 2m/3\rceil \leq 3\lceil (m-2)/3\rceil+\lfloor (m-2)/3\rfloor$. A straightforward manipulation shows that this inequality never holds, and so we obtain a contradiction. ∎

3. Proof of Theorem 1

The proof is by induction on $m+n$.

As an inductive base we use the fact that every drawing of $C_3 \times C_l$ has at least l crossings [9]. Let m, n be given, where $n \geq m \geq 4$. Suppose that the statement is true for all graphs $C_r \times C_s$ such that $r + s < m + n$.

Let $\mathcal{D}$ be an arbitrary good drawing of $C_m \times C_n$. If there is a red cycle in $\mathcal{D}$ with at least $\lceil (m-2)/3 \rceil$ crossings, then by the induction hypothesis $\mathcal{D}$ has at least $cr(C_m \times C_{n-1}) + \lceil (m-2)/3 \rceil \geq (m-2)(n-1)/3 + \lceil (m-2)/3 \rceil \geq (m-2)n/3$ crossings, as required. Thus we suppose that no red cycle has $\lceil (m-2)/3 \rceil$ (or more) crossings. It follows from Lemma 2 that, for each j, $\sum_{i=0}^{m-1} pcr(P_{i,j}) \geq \lceil 2m/3 \rceil$. We analyze the cases n even and n odd separately.

Suppose that n is even. If j and j' are odd, and $j \neq j'$, then $P_{i,j}$ and $P_{i',j'}$ are edge–disjoint, for every i, i'. Therefore, since $\sum_{j\ odd} \sum_{i=0}^{m-1} pcr(P_{i,j}) \geq (n/2)\lceil 2m/3 \rceil$, it follows that there is an i such that $\sum_{j\ odd} pcr(P_{i,j}) \geq ((n/2)\lceil 2m/3 \rceil)/m \geq n/3$. Therefore some blue cycle has at least $n/3$ crossings. Thus, by the induction hypothesis, $\mathcal{D}$ has at least $cr(C_{m-1} \times C_n) + n/3 \geq (m-3)n/3 + n/3 = (m-2)n/3$ crossings.

A similar argument works if n is odd. The crucial observation in this case is that, as a consequence of Lemma 2, there is a j_0 such that $\sum_{i=0}^{m-1} pcr(b_{i,j_0}) \geq \lceil (\lceil 2m/3 \rceil)/2 \rceil$ ($b_{i,j}$ is the blue edge that joins $v_{i,j-1}$ to $v_{i,j}$). Now for each fixed $j \in \{j_0+1, j_0+3, \ldots, j_0-2\}$ we have $\sum_{i=0}^{m-1} pcr(P_{i,j}) \geq \lceil 2m/3 \rceil$. Therefore, by a similar argument as in the even case, it follows that there is an i such that $\sum_{j \in \{j_0+1, j_0+3, \ldots, j_0-2\}} pcr(P_{i,j}) + pcr(b_{i,j_0}) \geq (((n-1)/2)\lceil 2m/3 \rceil + \lceil (\lceil 2m/3 \rceil)/2 \rceil)/m \geq n/3$. Thus some blue cycle has at least $n/3$ crossings, and so by the induction hypothesis $\mathcal{D}$ has at least $cr(C_{m-1} \times C_n) + n/3 \geq (m-3)n/3 + n/3 = (m-2)n/3$ crossings. ∎

4. Concluding Remarks

In order to prove the lower bound in Theorem 1, we make extensive use of our knowledge of drawings of $C_3 \times C_n$. It is natural to ask whether the bound in Theorem 1 can be improved by using the fact that $cr(C_m \times C_n)$ is also known for $m = 4, 5$ and 6. Unfortunately, this is not the case: the best balance between the number of blue crossings and the number of red crossings that can be guaranteed is achieved by considering drawings of $C_3 \times C_n$.

As pointed out in [12], if n is sufficiently close to m, then a simple counting argument gives good lower bounds for $C_m \times C_n$. These bounds are based on the knowledge of $cr(C_k \times C_k)$. For example, if $cr(C_k \times C_k)$ is as conjectured, then $cr(C_m \times C_n)$ is at least $((k-2)/k)mn$, for $n \leq (k/(k-1))(m-1)$. The largest k for which it is known that $cr(C_k \times C_k)$ is as conjectured is 7 [2]. Therefore, if $n \geq m \geq 7$ and $n \leq 7(m-1)/6$, then $cr(C_m \times C_n) \geq (5/7)mn$. Unfortunately, these simple counting arguments yield very poor lower bounds for $cr(C_m \times C_n)$ for arbitrary values of m and n.

Acknowledgements

I thank Bruce Richter for discussions. I also thank two anonymous referees for their valuable suggestions. This work was partially supported by FAI–UASLP and by CONACYT under contract I26994–E.

References

[1] M. Anderson, R.B. Richter, and P. Rodney, The Crossing Number of $C_6 \times C_6$, *Cong. Numerantium* **118** (1996), 97–107.

[2] M. Anderson, R.B. Richter, and P. Rodney, The Crossing Number of $C_7 \times C_7$, *Cong. Numerantium* **125** (1997), 97–117.

[3] L.W. Beineke and R.D. Ringeisen, On the crossing numbers of products of cycles and graphs of order four, *J. Graph Theory* **4** (1980), 145–155.

[4] A.M. Dean and R.B. Richter, The Crossing Number of $C_4 \times C_4$, *J. Graph Theory* **19** (1995), 125–129.

[5] P. Erdös and R.K. Guy, Crossing number problems, *Amer. Math. Monthly* **80** (1973), 52–58.

[6] F. Harary, P.C. Kainen, and A.J. Schwenk, Toroidal graphs with arbitrarily high crossing numbers, *Nanta Math.* **6** (1973), 58–67.

[7] M. Klešč, R.B. Richter and I. Stobert, The crossing number of $C_5 \times C_n$, *J. Graph Theory* **22** (1996), 239–243.

[8] R.B. Richter and C. Thomassen, Intersections of Curve Systems and the Crossing Number of $C_5 \times C_5$, *Disc. Comp. Geom.* **13** (1995), 149–159.

[9] R.D. Ringeisen and L.W. Beineke, The crossing number of $C_3 \times C_n$, *J. Combinat. Theory* **24** (1978), 134–136.

[10] G. Salazar, Ph.D. thesis. Carleton University (1997).

[11] G. Salazar, Drawings of $C_m \times C_n$ with one disjoint family II, submitted.

[12] F. Shahrokhi, O. Sýkora, L. A. Székely and I. Vrťo, Intersection of Curves and Crossing Number of $C_m \times C_n$ on Surfaces, *Discrete Comput. Geom.* **19** (1998), no. 2, 237–247.

[13] P. Turán, A note of welcome, *J. Graph Theory* **1** (1977), 7–9.