On the crossing numbers of loop networks and Generalized Petersen Graphs

Gelasio Salazar[1]

IICO–UASLP
Av. Karakorum 1470, Lomas 4ta. Seccion
San Luis Potosi, SLP
78210 Mexico

December 13, 2002

Abstract. Bhatt and Leighton proved that the crossing number of a network (graph) is closely related to the minimum layout area required for the implementation of a VLSI circuit for that network. With this important application in mind, it makes most sense to analyze the crossing numbers of graphs with good interconnection properties, such as the circulant graphs $G(n; \pm s_1, \ldots, \pm s_m)$. In this work we find tight bounds for the crossing numbers of the (double fixed step) circulant graphs $G(n; \pm 1, \pm k)$. Specifically, we show that for values of n sufficiently large compared to k, the crossing number of $G(n; \pm 1, \pm k)$ is bounded by above and by below by linear functions of n, both of which have coefficients that approach 1 as k goes to infinity. As an additional application of these bounds, we show that the crossing numbers of the Generalized Petersen Graphs $P(n, k)$ are bounded by below and by above by linear functions of n, whose coefficients approach $2/5$ and 2, respectively, as k goes to infinity.

2000 Mathematics Subject Classification: 05C10, 05C62

Keywords : Crossing numbers, circulant, loop networks, interconnection networks, VLSI layout, Generalized Petersen Graphs.

---

[1]E–mail address: `gsalazar@cactus.iico.uaslp.mx`.
Supported by CONACYT (Grant J32168E) and by FAI–UASLP

1 Introduction

One of the major motivations for the study of crossing numbers of graphs arises from the relationship between the crossing number of a network and the layout area required for the fabrication of a VLSI circuit for that network [4, 17].

From this perspective, the analysis of crossing numbers of graphs makes most sense when one focuses on graphs that have good properties as interconnection networks, such as meshes of trees [6, 7] or circulant graphs. The (undirected) *circulant graph* $G(n; \pm s_1, \pm s_2, \ldots, \pm s_m)$ has n vertices $v(0), v(1), \ldots, v(n-1)$, so that $v(i)$ is adjacent to the $2m$ vertices $v(i \pm s_1), v(i \pm s_2), \ldots, v(i \pm s_m)$ (indices are read modulo n). In this paper we focus on *double fixed step graphs* $G(n; \pm s_1, \pm s_2)$, and in particular on graphs of the type $G(n; \pm 1, \pm k)$ (see [3, 8, 14, 15, 23]). The comprehensive survey [2] is a standard reference for circulant graphs (and more general distributed loop networks).

See Figure 1 for an embedding of $G(n; \pm 1, \pm k)$ in the torus.

In this paper we prove that, for fixed k, the crossing number of $G(n; \pm 1, \pm k)$ is tightly bounded by above and by below by linear functions of n, whose coefficients approach 1 as k goes to infinity.

Theorem 1 *Let k, n be integers such that $k \geq 5$ and $n \geq k^4$. Then*

$$\left(1 - \frac{4}{k}\right)n + \left(4k^2 + 1 - k^3\right) \leq \mathrm{cr}(G(n; \pm 1, \pm k)) \leq \left(1 - \frac{2}{k}\right)n + \left(k^2/2 + k/2 + 1\right).$$

We also give an argument to show that these bounds imply the following bounds for the crossing numbers of Generalized Petersen Graphs $P(n, k)$ (see definition below).

Theorem 2 *Let k, n be integers such that $k \geq 5$ and $n \geq k$. Then*

$$\frac{2}{5}\left[\left(1 - \frac{4}{k}\right)\left(n - k^4\right)\right] + (4k^2 + 1 - k^3) \leq \mathrm{cr}(P(n, k)) \;\leq\; \left(2 - \frac{2}{k}\right)n + (k^2/2 + k/2 + 1).$$

The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane. Crossing numbers are in general very difficult to compute. The crossing numbers of few interesting families of graphs are known. For instance, $\mathrm{cr}(K_n)$ is known only for $n \leq 10$ [9], and $\mathrm{cr}(K_{m,n})$ is known only for $\min\{m, n\} \leq 6$ [16], and for a few special cases [24].

Upper bounds for crossing numbers of graphs are easy to obtain: the number of pairwise intersections of edges in any drawing of a graph G is an upper bound for the crossing number of G. Lower bounds are a lot more difficult to obtain. Since the exact determination of the crossing number of a graph (or of a family of graphs) is most of the time an unrealistic goal, a lot of research is focused on establishing nontrivial lower bounds for the crossing numbers of interesting families of graphs.

A long–standing conjecture is that the crossing number of the Cartesian product $C_m \times C_n$ is $(m-2)n$, for all integers m, n such that $n \geq m \geq 3$ [13]. The largest value of m for which this conjecture has been verified is 7 [1]. Recently, it was proved that the conjecture is true for all but finitely many values of n, for each m [12].

Another family of graphs whose crossing numbers have been investigated are the Generalized Petersen Graphs. Let $C = (v(0), e(0), \ldots, e(n-2), v(n-1), e(n-1), v(0))$ be an n–cycle, where each $v(i)$ is a vertex and each $e(i)$ is an edge, and let $\overline{v}(0), \ldots, \overline{v}(n-1)$ be vertices not in C. Let $k < n$ be a positive integer. For each i, join $v(i)$ to $\overline{v}(i)$ by an edge $s(i)$ (a *spoke*), and join $\overline{v}(i)$ to $\overline{v}(i+k)$ (indices are read modulo n) by an edge $f(i)$. The graph thus obtained is the *Generalized Petersen Graph* $P(n,k)$.

Exoo et al. [10] computed $\mathrm{cr}(P(n,2))$ for every n. McQuillan and Richter computed $\mathrm{cr}(P(8,3))$ and found a nontrivial lower bound for $\mathrm{cr}(P(10,3))$ [19]. Fiorini [11] claimed to have computed $\mathrm{cr}(P(n,3))$. Richter and Salazar [22] found a gap in Fiorini's proof, and refined the techniques in [11] to establish $\mathrm{cr}(P(n,3))$. Recently, Lovrečič Saražin showed that $\mathrm{cr}(P(10,4)) = 4$ [18].

This paper is organized as follows. In the next section, we break Theorem 1 into two inequalities, namely Theorem 3 and Theorem 4. These theorems are proved in Sections 3 and 4, respectively. In Section 5 we show that Theorem 2 follows from Theorem 1 and some elementary combinatorial and optimization arguments. Section 6 contains some concluding remarks.

2 Proof of Theorem 1

In this paper we exploit a close relationship between the crossing numbers of Cartesian products of cycles and of Generalized Petersen Graphs. As we see below, this relationship allows us to prove the following lower bound for $\mathrm{cr}(G(n; \pm 1, \pm k))$.

Theorem 3 (Lower Bound) *Let k, n be integers such that $k \geq 5$ and $n \geq k^4$. Then* $\mathrm{cr}(G(n; \pm 1, \pm k)) \geq (1 - (4/k))n + (4k^2 + 1 - k^3)$.

We also describe drawings of $G(n; \pm 1, \pm k)$ that yield the following upper bound.

Theorem 4 (Upper bound) *Let k, n be integers such that $k \geq 3$ and $n \geq 2k$. Then* $\mathrm{cr}(G(n; \pm 1, \pm k)) \leq (1 - (2/k))n + (k^2/2 + k/2 + 1)$.

Proof of Theorem 1. It is an immediate consequence of Theorems 3 and 4. ∎

3 The Lower Bound

The strategy to bound $\mathrm{cr}(G(n;\pm1,\pm k))$ is to show that $G(n;\pm1,\pm k)$ contains a subgraph H that is a subdivision of a graph whose crossing number is not much smaller than the crossing number of the Cartesian product $C_{k-1}\times C_{\lfloor n/k\rfloor-k^2+1}$. Since we know the exact crossing number of $C_{k-1}\times C_{\lfloor n/k\rfloor-k^2+1}$, this provides the required bound for $\mathrm{cr}(G(n;\pm1,\pm k))$.

In what follows, the edges that join $v(i)$ to $v(i+1)$ and $v(i-1)$ are denoted $e(i)$, whereas the edges that join $v(i)$ to $v(i+k)$ and $v(i-k)$ are denoted $f(i)$.

A *collapsing* on a graph G consists of subdividing two distinct edges in G, and identifying the subdividing vertices (into a *collapsed* degree four vertex). If H is obtained by performing k such operations on G, then H is a k–*collapsing* of G.

Proposition 5 *If H is a k–collapsing of G, then* $\mathrm{cr}(G)\le\mathrm{cr}(H)+k$.

Proof. Consider a drawing of H with $\mathrm{cr}(H)$ crossings. Each collapsed vertex can be splitted into two degree two vertices, in such a way that each such splitting introduces at most one additional crossing. The result is a drawing of a subdivision of G (whose crossing number is equal to $\mathrm{cr}(G)$), with at most $\mathrm{cr}(H)+k$ crossings. ∎

Proof of Theorem 3. It suffices to show that $G(n;\pm1,\pm k)$ has a subgraph H which is an $(\lfloor n/k\rfloor-k^2)$–collapsing of a graph K isomorphic to a subdivision of $C_{k-1}\times C_{\lfloor n/k\rfloor-k^2+1}$. Indeed, the existence of such an H implies $\mathrm{cr}(G(n;\pm1,\pm k))\ge\mathrm{cr}(H)\ge\mathrm{cr}(K)-(\lfloor n/k\rfloor-k^2)=\mathrm{cr}(C_{k-1}\times C_{\lfloor n/k\rfloor-k^2+1})-(\lfloor n/k\rfloor-k^2)=(k-3)(\lfloor n/k\rfloor-k^2+1)-(\lfloor n/k\rfloor-k^2)>(1-(4/k))n+(4k^2+1-k^3)$. Here we use $\mathrm{cr}(C_m\times C_n)=(m-2)n$, which holds for $n\ge(m/2)((m+3)^2/2+1)$ (see [12]). This equality applies here, since the assumption $n\ge k^4$ implies $\lfloor n/k\rfloor-k^2+1>((k-1)/2)(((k-1)+3)^2/2+1)$.

We will use the following terminology. Suppose that we currently stand at vertex $v(j)$. To *traverse one e–edge in the negative direction* means to follow the edge $e(j-1)$, thus arriving to the vertex $v(j-1)$. To *traverse one f–edge in the positive direction* means to follow the edge $f(j)$, thus arriving to the vertex $v(j+k)$.

To obtain H, first we find pairwise disjoint cycles $B(i)$, for $i=1,\dots,k-1$, and then we find cycles $R(j)$, for $j=0,\dots,\lfloor n/k\rfloor-k^2$, that pairwise intersect in at most one vertex.

To construct $B(i)$, we proceed as follows (an illustration of the construction of $B(i)$ is given in Figure 2).

If k divides n, then $B(i)$ is simply the cycle induced by the edges $f(rk+i)$, for all r such that $0\le r\le n/k-1$.

Suppose now that k does not divide n. In this case, we follow these steps to construct $B(i)$.

(i) Start at vertex $v((\lfloor n/k \rfloor - k^2)k + i)$, and traverse $i+1$ f–edges in the positive direction, thus arriving at vertex $v((\lfloor n/k \rfloor - k^2 + i + 1)k + i)$.

(ii) Traverse an e–edge in the negative direction, thus arriving to vertex $v((\lfloor n/k \rfloor - k^2 + i + 1)k + i - 1)$.

(iii) Iterate the following procedure $k - 1 - n \bmod k$ times (in particular, if $n \bmod k = k-1$ then do not perform the procedure at all): traverse k f–edges in the positive direction, and then traverse one e–edge in the negative direction. Note that if one such iteration is started when we stand at vertex $v(j)$, then at the end of the iteration we stand on vertex $v(j + k^2 - 1)$. Therefore, after performing $k - 1 - n \bmod k$ iterations, we stand on vertex $v((\lfloor n/k \rfloor - k - k\; n \bmod k + i)k + i + n \bmod k)$.

Since $n \bmod k \geq 1$, and $i \leq k-1$, it follows that we stand on a vertex of the form $v(tk + i + n \bmod k)$, where t is an integer that satisfies $t < \lfloor n/k \rfloor$.

Note that so far we have not attained any vertex in $\{v(n-k), v(n-(k-1)), \ldots, v(n-1)\}$.

(iv) Now traverse as many f–edges in the positive direction as needed, until reaching vertex $v((\lfloor n/k \rfloor - 1)k + i + n \bmod k) = v(n - k + i)$.

(v) Traverse one f–edge in the positive direction, thus arriving to $v(n+i) = v(i)$.

(vi) Finally, traverse $\lfloor n/k \rfloor - k^2 + i + 1$ f–edges in the positive direction, thus getting back to the initial vertex $v((\lfloor n/k \rfloor - k^2 + i + 1)k + i)$.

It is straightforward to check that the $k-1$ cycles $B(1), B(2), \ldots, B(k-1)$ thus constructed are indeed pairwise disjoint. This is illustrated in Figure 3.

For each $j = 0, \ldots, \lfloor n/k \rfloor - k^2$, let $R(j)$ denote the cycle $v(jk), e(jk), v(jk+1), e(jk+1), \ldots, v(jk+(k-1)), e(jk+(k-1)), v((j+1)k), f(jk), v(jk)$.

Finally, let H denote the subgraph of $G(n; \pm 1, \pm k)$ defined by the union of all cycles $R(j)$ and all cycles $B(i)$. See Figure 4.

Let K denote the graph obtained by splitting in H each vertex $v(jk)$ with $j = 1, \ldots, \lfloor n/k \rfloor - k^2$ into two degree two vertices, so that one of the two vertices remains incident with the edges $e(jk)$ and $f(jk)$ (thus, the other vertex remains incident with $e(jk-1)$ and $f((j-1)k)$). Thus, H is an $(\lfloor n/k \rfloor - k^2)$–collapsing of K. Moreover, K is isomorphic to a subdivision of $C_{k-1} \times C_{\lfloor n/k \rfloor - k^2 + 1}$. Thus, H is an $(\lfloor n/k \rfloor - k^2)$–collapsing of a graph isomorphic to a subdivision of $C_{k-1} \times C_{\lfloor n/k \rfloor - k^2 + 1}$, as required. ∎

4 The Upper Bound

Proof of Theorem 4. We shall explain how to draw $G(n;\pm1,\pm k)$ with at most $\lfloor n/k\rfloor(k-2)+1+(k+1)k/2$ crossings. This is clearly enough, since this number is less than or equal to $(1-2/k)n+(k^2/2+k/2+1)$.

Let E_0 denote the set of edges $\{e(n-1), f(n-1), f(n-2), \ldots, f(n-k)\}$. We will show that there is a drawing $\mathcal{D}_0$ of $G(n;\pm1,\pm k) - E_0$ with $\lfloor n/k\rfloor(k-2)+1$ crossings, in such a way that the vertices $v(0), v(1), \ldots, v(k-1), v(n-k), v(n-k+1), \ldots, v(n-1)$ all lie in the boundary of the infinite region of $\mathcal{D}_0$ (Figure 5). The existence of such a drawing $\mathcal{D}_0$ completes the proof: in order to extend $\mathcal{D}_0$ to a drawing $\mathcal{D}$ of $G(n;\pm1,\pm k)$, it suffices to add the edges in E_0. But each edge in E_0 has both endpoints in $\{v(0), v(1), \ldots, v(k-1), v(n-k), v(n-k+1), \ldots, v(n-1)\}$, and so the edges in E_0 can be drawn in the infinite region of $\mathcal{D}_0$, with a consequent cost of at most $\binom{E_0}{2} = (k+1)k/2$ crossings. Thus, we will conclude that $G(n;\pm1,\pm k)$ can be drawn in the plane with at most $\lfloor n/k\rfloor(k-2)+1+(k^2/2+k/2)$ crossings, as claimed.

To obtain the drawing $\mathcal{D}_0$ of $G(n;\pm1,\pm k) - E_0$ with the required properties, first consider the following embedding $\mathcal{E}$ of $G(n;\pm1,\pm k) - E_0 - \{e(k-1), e(2k-1), \ldots, e(\lfloor n/k\rfloor k-1)\}$. For each $j = 0, 1, \ldots, n-1$, draw $v(j)$ in the point with Cartesian coordinates $(\lfloor j/k\rfloor, j-\lfloor j/k\rfloor)$, and draw edges among the vertices that are a unit distance apart from each other. It is readily checked that this is an embedding of $G(n;\pm1,\pm k) - E_0 - \{e(k-1), e(2k-1), \ldots, e(\lfloor n/k\rfloor k-1)\}$, in which the vertices $v(0), v(1), \ldots, v(k-1), v(n-k), v(n-k+1), \ldots, v(n-1)$ all lie in the boundary of the infinite region of $\mathcal{E}$.

Now, add each edge $e(k-1), e(2k-1), \ldots, e((\lfloor n/k\rfloor-1)k-1)$ by joining its endpoints with a straight segment. In this way, each such edge is drawn, without intersecting the infinite region of $\mathcal{E}$, with exactly $k-2$ crossings.

If n/k is an integer, then $e(\lfloor n/k\rfloor k-1) = e(n-1) \in E_0$, and so this completes the drawing $\mathcal{D}_0$ (recall that $\mathcal{D}_0$ is a drawing of $G(n;\pm1,\pm k) - E_0$) with $(\lfloor n/k\rfloor-1)(k-2) < \lfloor n/k\rfloor(k-2)+1$ crossings, as required. If n/k is not integer, then it is readily seen that $e(\lfloor n/k\rfloor k-1)$ can be drawn with $k-1$ crossings without intersecting the infinite region of $\mathcal{E}$. Thus, in this case, the described drawings of the edges $e(k-1), e(2k-1), \ldots, e((\lfloor n/k\rfloor-1)k-1), e(\lfloor n/k\rfloor k-1)$ added to $\mathcal{E}$ produce a drawing $\mathcal{D}_0$ with $(\lfloor n/k\rfloor-1)(k-2)+k-1 = \lfloor n/k\rfloor(k-2)+1$ crossings, as required. ∎

5 Proof of Theorem 2

Proof of $\mathrm{cr}(P(n,k)) \ \le\ \Big(2-2/k\Big)n + \Big(k^2/2+k/2+1\Big)$.

By Theorem 1, there is a drawing of $G(n; \pm 1, \pm k)$ with at most $(1 - 2/k)n + (k^2/2 + k/2 + 1)$ crossings. In order to obtain from this drawing a drawing of $P(n,k)$, we split each vertex, to create a corresponding spoke. Since the splitting of each vertex can be arranged to introduce at most one crossing, and there are n vertices in total (in $G(n; \pm 1, \pm k)$), it follows that there is a drawing of $P(n,k)$ with at most $n + (1-2/k)n + (k^2/2 + k/2 + 1) = (2-2/k)n + (k^2/2 + k/2 + 1)$ crossings, as required.

Proof of $\mathrm{cr}(P(n,k)) \geq (2/5)\left[\Big(1 - (4/k)\Big)(n - k^4)\right] + \Big(4k + 1 - k^3\Big)$.

Let $k \geq 5$ be fixed. We prove this inequality by induction on n, using as base case those n such that $k \leq n \leq k^4$ (for those n there is nothing to prove, since $(1 - (4/k))\,(n - k^4)$ and $4k^2 + 1 - k^3$ are both zero or less; note that we are using the assumption $k \geq 5$).

Thus, we let $n_0 > k^4$, and suppose that the inequality holds for $n < n_0$.

Let $\mathcal{D}$ be an optimal drawing of $P(n_0, k)$. Let x_1 denote the number of crossings in $\mathcal{D}$ that involve two spokes; let x_2 denote the number of crossings in $\mathcal{D}$ that involve a spoke and a non–spoke edge; and let x_3 denote the number of crossings in $\mathcal{D}$ that involve two non–spoke edges.

Suppose first that $2x_1 + x_2 > (2/5)(1 - (4/k))n_0$. It follows from the pigeon–hole principle that there are k consecutive spokes (that is, spokes $s(i), s(i+1), \ldots, s(i+(k-1))$) whose deletion removes at least $(2/5)(1-(4/k))k$ crossings from $\mathcal{D}$. The graph that results by deleting these spokes is isomorphic to $P(n_0 - k, k)$, that has crossing number at least $(2/5)(1 - (4/k))((n_0 - k) - k^4) + (4k^2 + 1 - k^3)$, by the induction hypothesis. This implies that $\mathcal{D}$ has at least $(2/5)(1 - (4/k))((n_0 - k) - k^4) + (4k^2 + 1 - k^3) + (2/5)(1 - (4/k))k = (2/5)(1 - (4/k))(n_0 - k^4) + (4k^2 + 1 - k^3)$ crossings, as required.

To complete the proof, we analyze the case $2x_1 + x_2 \leq (2/5)(1 - (4/k))n_0$.

It is straightforward to check that the spokes in $\mathcal{D}$ can be contracted to yield a drawing of $G(n_0; \pm 1, \pm k)$ with (exactly) $4x_1 + 2x_2 + x_3$ crossings (every spoke–spoke crossing becomes four crossings, every spoke–no spoke crossing becomes two crossings, and every no spoke–no spoke crossing remains one crossing). Since (by Theorem 3) this drawing has at least $(1 - (4/k))n_0 + (4k^2 + 1 - k^3)$ crossings, it follows that $4x_1 + 2x_2 + x_3 \geq (1 - (4/k))n_0 + (4k^2 + 1 - k^3)$.

Note that $\mathrm{cr}(\mathcal{D}) = x_1 + x_2 + x_3$. Thus, the problem of bounding $\mathrm{cr}(\mathcal{D})$ reduces to minimizing $x_1 + x_2 + x_3$ subject to the constraints $2x_1 + x_2 \leq (2/5)(1 - (4/k))n_0$ and $4x_1 + 2x_2 + x_3 \geq (1 - (4/k))n_0 + (4k^2 + 1 - k^3)$.

A routine linear optimization argument yields that the minimum is attained when $x_1 = (1/5)(1 - (4/k))n_0$, $x_2 = 0$, and $x_3 = (1 - (4/k))n_0 + (4k^2 + 1 - k^3) - (4/5)(1 -$

$(4/k))n_0 = (1/5)(1-(4/k))n_0 + (4k^2+1-k^3)$. The value of the minimum is then $(2/5)(1-(4/k))n_0 + (4k^2+1-k^3)$. Thus we are done, since this last expression is greater than $(2/5)(1-(4/k))(n_0 - k^4) + (4k^2+1-k^3)$. ∎

6 Concluding Remarks

We expect that the kind of arguments we used above would yield some insight for the crossing numbers of graphs $G(n; \pm s_1, \pm s_2)$ (at least for the case when one of s_1 and s_2 is sufficiently large compared to the other). For those cases, we would expect that the crossing number of $G(n; \pm s_1, \pm s_2)$ is also roughly n.

A natural direction for future work is the investigation of the crossing number of the circulant $G(n; \pm s_1, \pm s_2, \ldots, \pm s_m)$. It is not clear to us even which kinds of drawings seem natural candidates to be (nearly) optimal for such a general family of graphs.

The drawings described in the proof of Theorem 2 are essentially (that is, up to some marginal improvements) the best drawings of $P(n,k)$ that we can generate for large values of n compared to k. Naturally, we conjecture that these drawings are not far from optimal, at least when k is sufficiently large and n is large compared to k.

Conjecture *There is a function* $g(k)$ *with* $\lim_{k\to\infty} g(k) = 0$, *such that* $\mathrm{cr}(P(n,k)) > (2 - g(k))n + O(k^2)$.

The estimates for the crossing numbers of $G(n; \pm 1, \pm k)$ and $P(n,k)$ in this work, and in the various papers that deal with the crossing number of $C_m \times C_n$, take advantage of the special structure of these families of graphs (namely, a chain of copies of a fixed structure pasted in a specific way). A general framework for the analysis of such structures was recently introduced by Pinontoan and Richter [20], [21].

Acknowledgements

I thank Bruce Richter and Imrich Vrto for helpful comments.

References

[1] J. Adamsson and R.B. Richter, Arrangements and the Crossing Number of $C_7 \times C_n$, manuscript.

[2] J.C. Bermond, F. Comellas, and D.F. Hsu, Distributed Loop Computer Networks: A Survey. *Journal of Parallel and Distributed Computing* **24** (1995), 2–10.

[3] J.–C. Bermond and D. Tzvieli, Minimal diameter double–loop networks: Dense optimal families. *Networks* **21** (1991), 1–9.

[4] S.N. Bhatt and F.T. Leighton, A framework for solving VLSI graph layout problems, *J. Comput. & Systems Sciences* **28** (1984), 300–343.

[5] F.T. Boesch and R. Tindell, Circulants and their connectivity. *J. Graph Theory* **8** (1984), 487–499.

[6] R. Cimikowski, Topological properties of some interconnection network graphs. *Proc. of the 27th. Southeastern Intl. Conference on Combinatorics, Graph Theory and Computing* (Baton Rouge, LA, 1996). Congr. Numer. **121** (1996), 19–32.

[7] R. Cimikowski, Crossing number bounds for the mesh of trees. *Proc. of the 29th. Southeastern Intl. Conference on Combinatorics, Graph Theory and Computing* (Boca Raton FL, 1998). Congr. Numer. **134** (1998), 107-116.

[8] D.Z. Du, D.F. Hsu, Q. Li, and J.A. Xu, A combinatorial problem related to distributed loop networks. *Networks* **20** (1990), 173–180.

[9] P. Erdös and R.K. Guy, Crossing number problems. *Amer. Math. Monthly* **80** (1973), 52–58.

[10] G. Exoo, F. Harary, and J. Kabell, The crossing numbers of some generalized Petersen graphs, *Math. Scand.* **48** (1981), no. 2, 184–188.

[11] S. Fiorini, On the crossing number of generalized Petersen graphs, Combinatorics '84 (Bari, 1984), 225–241, North-Holland Math. Stud., 123, North-Holland, Amsterdam-New York, 1986.

[12] L. Glebsky and G. Salazar, The conjecture $\mathrm{cr}(C_m \times C_n) = (m-2)n$ is true for all but finitely many n, for each m, submitted.

[13] F. Harary, P.C. Kainen, and A.J. Schwenk, Toroidal graphs with arbitrarily high crossing numbers, *Nanta Math.* **6** (1973), 58–67.

[14] D.F. Hsu and J.A. Shapiro, A census of tight one–optimal double loop networks. In *Graph Theory, Combinatorics, Algorithms, and Applications*, Y. Alavi et al. (Eds.). SIAM (1991), 254–265.

[15] D.F. Hsu and J.A. Shapiro, Bounds for the minimal number of transmission delays in double loop networks, *J. of Combinatorics, Information and System Sciences*, **16** (1991), 55–62.

[16] D.J. Kleitman, The crossing number of $K_{5,n}$, *J. Comb. Theory* **9** (1970), 315–323.

[17] F.T. Leighton, *Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes.* Morgan Kaurmann, San Mateo, California (1992).

[18] M. Lovrečič Saražin, The crossing number of the generalized Petersen graph $P(10,4)$ is four, *Math. Slovaca* **47** (1997), no. 2, 189–192.

[19] D. McQuillan and R.B. Richter, On the crossing numbers of certain generalized Petersen graphs, *Discrete Math.* **104** (1992), no. 3, 311–320.

[20] B. Pinontoan and R.B. Richter, Crossing numbers of sequences of graphs I: general tiles, submitted.

[21] B. Pinontoan and R.B. Richter, Crossing numbers of sequences of graphs II: planar tiles, submitted.

[22] R.B. Richter and G. Salazar, The crossing number of $P(N,3)$, *Graphs and Combinatorics* **18**, no. 2 (2002), 381–394.

[23] D. Tzvieli, Double loop networks: optimal and suboptimal families. Morgan Kaufmann, 1992.

[24] D.R. Woodall, Cyclic–order graphs and Zarankiewicz's crossing–number conjecture, *J. Graph Theory* **17** (1993), no. 6, 657–671.

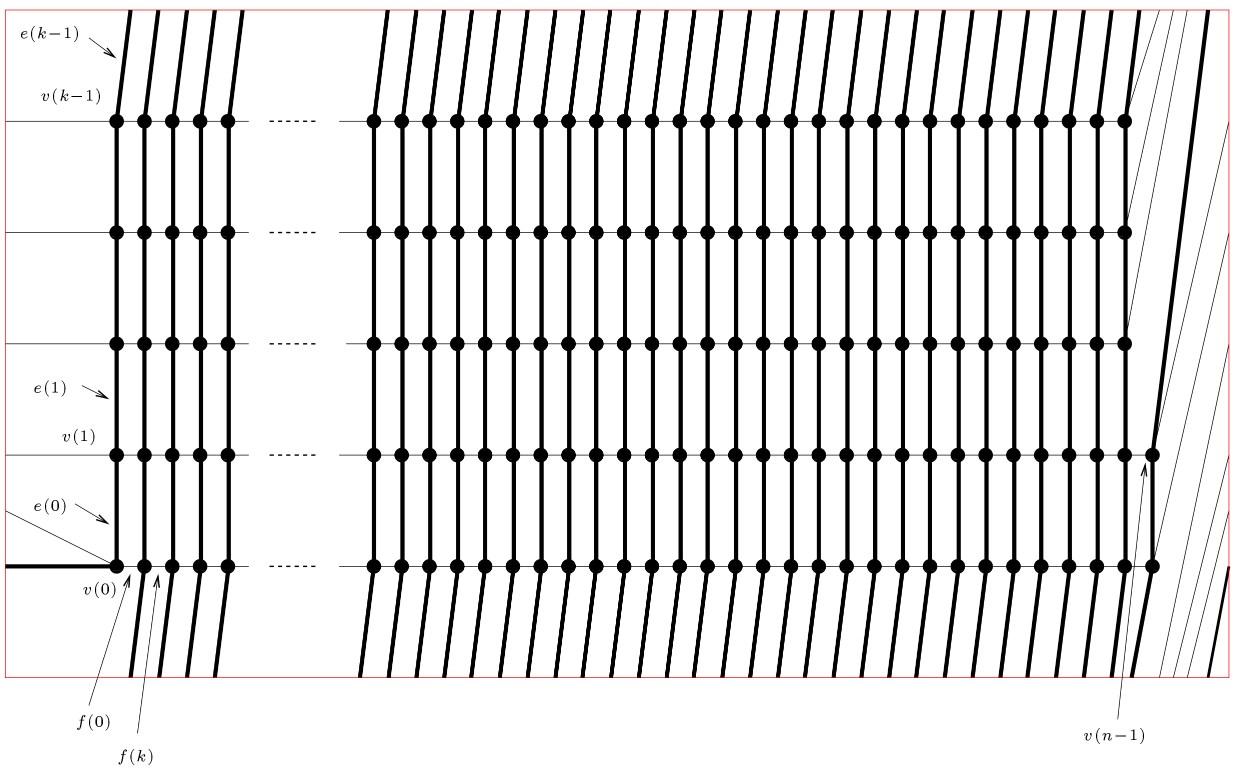

FIGURE 1. An embedding of $G(n; \pm 1, \pm k)$ in the torus (here $k = 5$, and $n \bmod k = 2$). The edges $e(i)$ are drawn with thick segments, and the edges $f(i)$ are drawn with thin segments. This is easily generalized to an embedding of $G(n; \pm 1, \pm k)$ for arbitrary n and k.

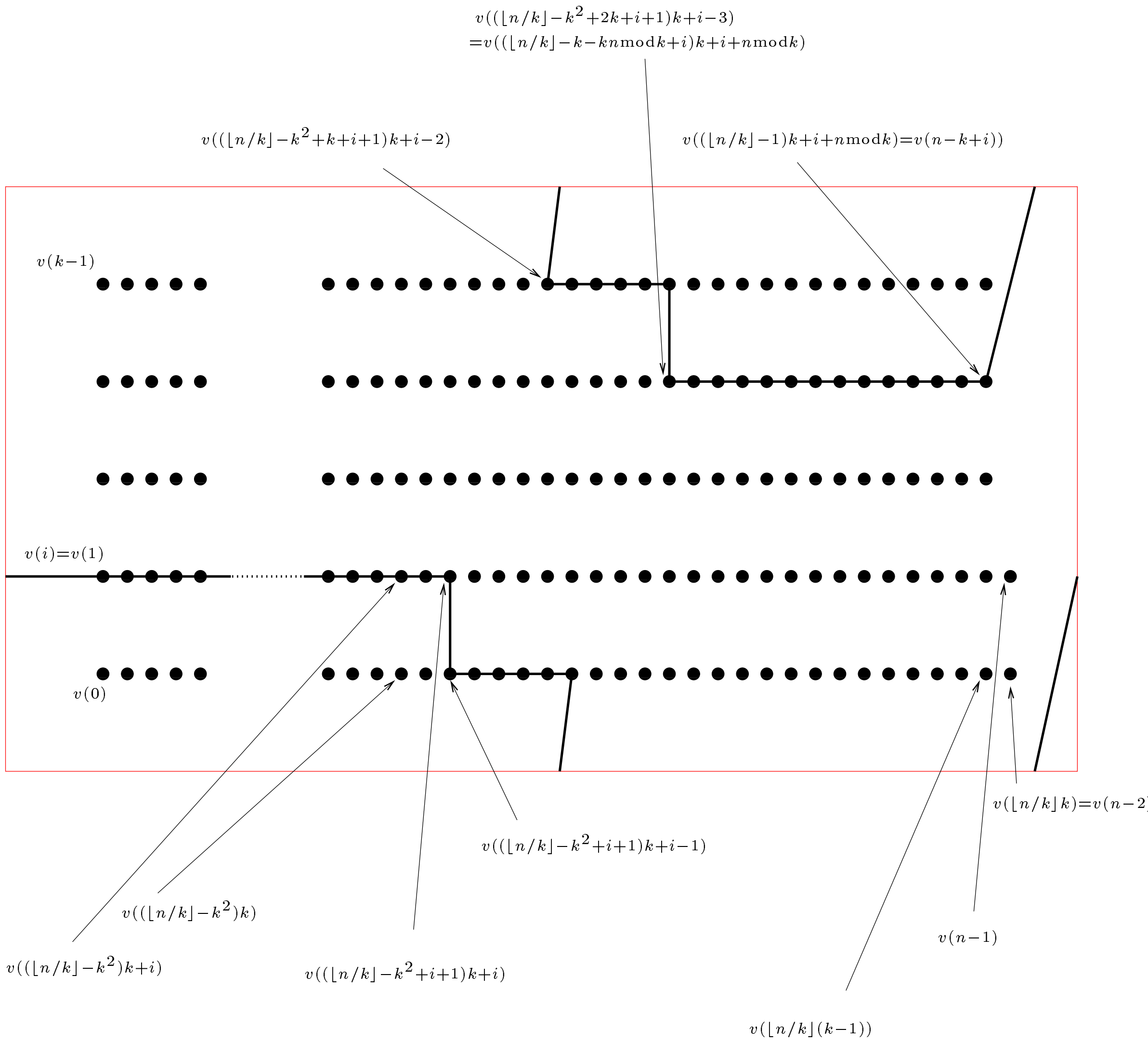

FIGURE 2. The construction of $B(i)$ for the case $i=1$. In this case, $n\mathrm{mod}k=2$, and so the iteration in Step (iii) of the construction is performed $k-1-n\mathrm{mod}k=2$ times, starting at $v((\lfloor n/k\rfloor-k^2+i+1)k+i-1)$ and ending at $v((\lfloor n/k\rfloor-k^2+2k+i+1)k+i-3)=v((\lfloor n/k\rfloor-k-k\ n\mathrm{mod}k+i)k+i+n\mathrm{mod}k)$.

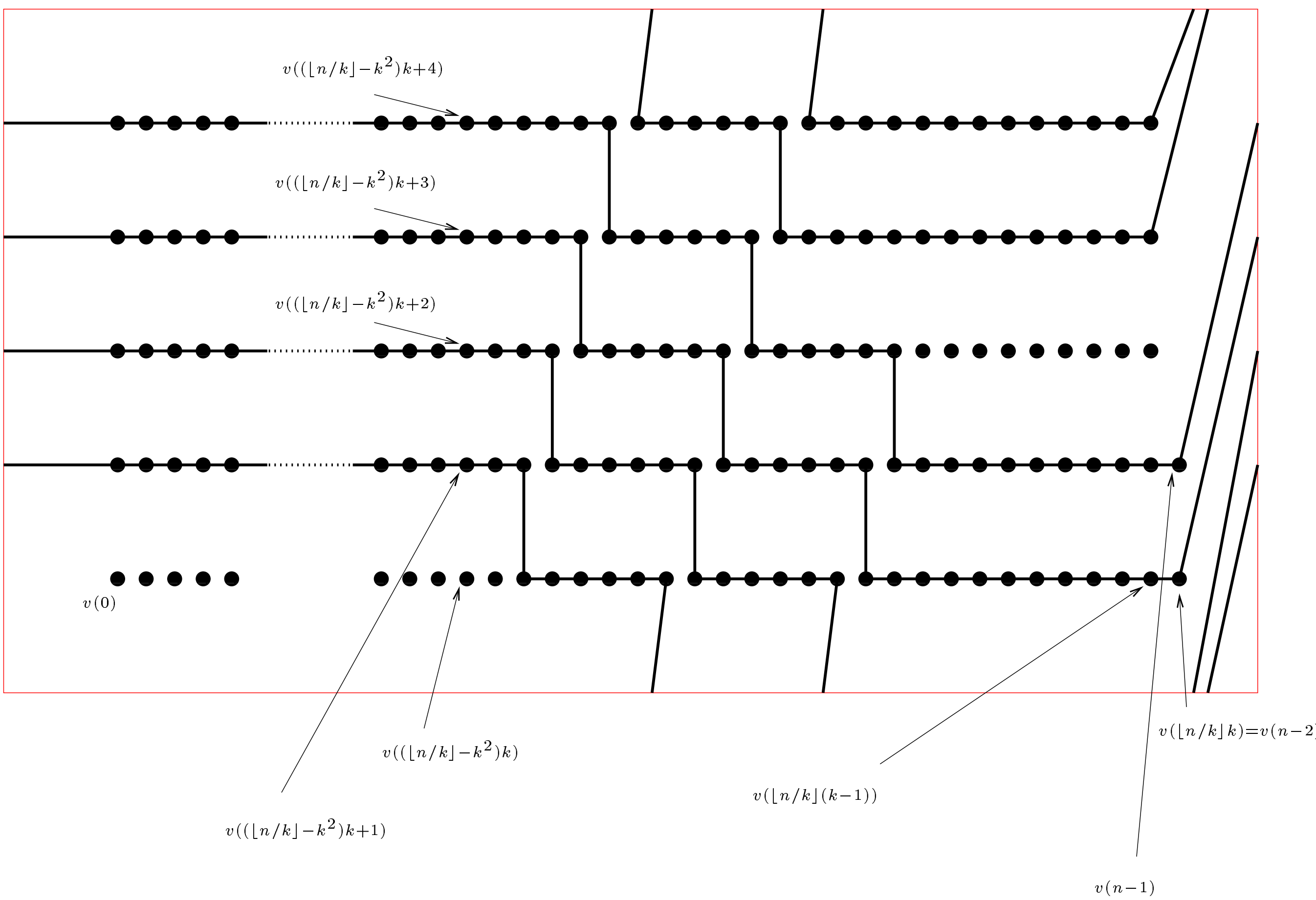

Figure 3. The pairwise disjoint cycles $B(1), B(2), \ldots, B(k-1)$.