Improved lower bounds on book crossing numbers of complete graphs.

E. de Klerk* D.V. Pasechnik† G. Salazar‡

August 1, 2012

Abstract

A *book with k pages* consists of a straight line (the *spine*) and k half-planes (the *pages*), such that the boundary of each page is the spine. If a graph is drawn on a book with k pages in such a way that the vertices lie on the spine, and each edge is contained in a page, the result is a *k-page book drawing* (or simply a *k-page drawing*). The *k-page crossing number* $\nu_k(G)$ of a graph G is the minimum number of crossings in a k-page drawing of G. In this paper we investigate the k-page crossing numbers of complete graphs. We use semidefinite programming techniques to give improved lower bounds on $\nu_k(K_n)$ for various values of k. We also use a maximum satisfiability reformulation to calculate the exact value of $\nu_k(K_n)$ for several values of k and n. Finally, we investigate the best construction known for drawing K_n in k pages, calculate the resulting number of crossings, and discuss this upper bound in the light of the new results reported in this paper.

Keywords: 2-page crossing number, book crossing number, semidefinite programming, maximum k-cut, Frieze-Jerrum maximum-k-cut bound, maximum satisfiability problem

AMS Subject Classification: 90C22, 90C25, 05C10, 05C62, 57M15, 68R10

1 Introduction

Motivated by applications to VLSI design, Chung, Leighton and Rosenberg [7] studied embeddings of graphs in books. A *book* consists of a line (the *spine*) and $k \geq 1$ half-planes

*Department of Econometrics and OR, Tilburg University, The Netherlands.

†School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore. Supported by Singapore Ministry of Education ARF Tier 2 Grant MOE2011-T2-1-090.

‡Instituto de Fisica, Universidad Autonoma de San Luis Potosi, San Luis Potosi, SLP Mexico 78000. Supported by CONACYT Grant 106432.

(the *pages*), such that the boundary of each page is the spine. In a *book embedding*, each edge is drawn on a single page, and no edge crossings are allowed. In a *book drawing* (or *k-page drawing*, if the book has k pages), each edge is drawn on a single page, but edge crossings are allowed.

Obviously every fixed graph can be embedded in a book with sufficiently many pages. On the other hand, it is desirable to do so using as few pages as possible. Given a graph G, the minimum k such that G can be embedded in a k-page book is the *pagenumber* (or *book thickness*) of G [2, 7, 21]. Determining the pagenumber of an arbitrary graph is NP-Complete [7], but some results are known for particular families of graphs. For instance, it is not difficult to prove that the pagenumber of the complete graph K_n is $\lceil n/2 \rceil$. On the other hand, with few exceptions, the pagenumbers of the complete bipartite graphs $K_{m,n}$ are unknown (see [11, 26]). Yannanakis proved [40] that four pages are always sufficient, and sometimes required, to embed a planar graph.

1.1 The k-page crossing number $\nu_k(G)$ of a graph G

When the number k of pages is fixed, the goal is to minimize the number of crossings in a k-page drawing of an input graph. The *k-page crossing number* $\nu_k(G)$ of a graph G is the minimum number of crossings in a k-page drawing of G.

Clearly, a graph G has $\nu_1(G) = 0$ if and only if it is outerplanar. Equivalent to 1-page drawings are *circular drawings*, in which the vertices are placed on a circle and all edges are drawn in its interior. In a similar vein, k-page drawings of $G = (V, E)$ can be alternatively viewed as a set of k circular drawings of graphs $G^{(i)} = (V, E^{(i)})$ $(i = 1, \ldots, k)$, where the edge sets $E^{(i)}$ form a k-partition of E. In other words, we assign each edge in E to exactly one of the k circular drawings. In Figure 1 we illustrate a 3-page drawing of K_7.

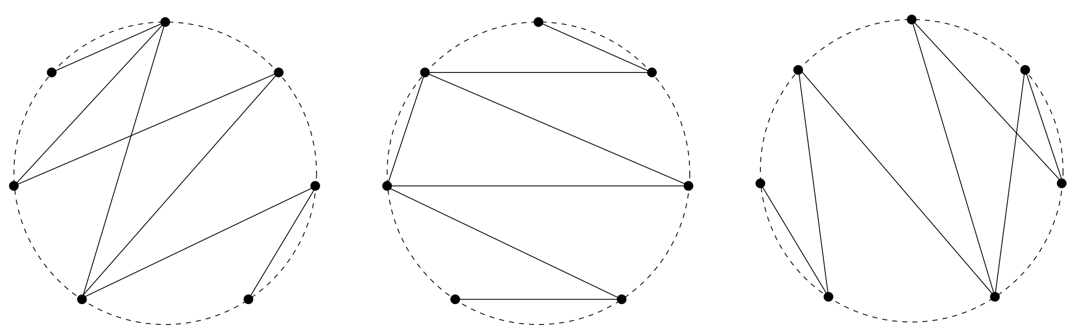

Figure 1: A 3-page drawing of K_7 with 2 crossings.

Several computational approaches and heuristics for estimating (or determining) $\nu_1(G)$ and $\nu_2(G)$ have been devised (see for instance [18, 19, 20, 24, 25]). Few exact results or nontrivial bounds are known, and very little is known about $\nu_k(G)$ for $k > 2$.

Although the special cases $k = 1$ and 2 have received considerable attention, the only thorough investigation of $\nu_k(G)$ for arbitrary k is the work by Shahrokhi, Sýkora, Székely, and Vrt'o [32]. In this paper, Shahrokhi et al. give general lower bounds for $\nu_k(G)$, for any graph G. They also give lower and upper bounds for $\nu_k(K_n)$, and use their upper bounds for $\nu_k(K_n)$ to give general upper bounds for $\nu_k(G)$ for arbitrary graphs G.

As with every graph-theoretical parameter, there is a natural interest in computing (or at least estimating) the k-page crossing number of the complete graph K_n. Besides, estimates on $\nu_k(K_n)$ are an essential tool to derive bounds for $\nu_k(G)$ for other graphs G, via the embedding method. This is the approach followed by Sharokhi et al. in [32], where the constructions that yield their upper bounds for $\nu_k(K_n)$ are used to generate k-page drawings of dense graphs, whose number of crossings is within a constant factor of their k-page crossing number.

1.2 Structure of the rest of the paper

Our main contributions in this paper are improved lower bounds for the k-page crossing numbers of K_n. We also compute the exact value of $\nu_k(K_n)$ for several k and n (no exact values were previously known for any n, for any $k > 2$).

In Section 2 we survey the bounds and exact results known for $\nu_k(K_n)$. In Section 3 we show how $\nu_k(K_n)$ may be obtained from the solution of a maximum-k-cut problem on a suitable graph, or via the solution of a suitable weighted maximum satisfiability problem. These reformulations are used in Section 4 to obtain previously unknown exact values and improved lower bounds on $\nu_k(K_n)$ for various values of k and n via computation. In Section 5 we review the construction that gives the best upper bounds available for $\nu_k(K_n)$, calculate the resulting number of crossings, and analyze this upper bound in the light of the new results obtained in the previous sections. Finally, in Section 6 we present some concluding remarks and open questions.

2 k-page drawings of K_n: exact results and bounds

2.1 1-page drawings of K_n

Calculating the 1-page crossing number of K_n is straightforward. Indeed, it is obvious that, in every 1-page drawing of K_n, every four vertices define a crossing, and therefore $\nu_1(K_n) \geq \binom{n}{4}$. It is easy to give 1-page drawings with exactly $\binom{n}{4}$ crossings, and so the reverse inequality $\nu_1(K_n) \leq \binom{n}{4}$ follows.

It follows that the problem of calculating or estimating $\nu_k(K_n)$ is only of interest for $k \geq 2$.

2.2 2-page drawings of K_n

We recall that the *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of crossings in a drawing of G in the plane. Harary and Hill [16] described how to draw K_n in the plane with $Z_2(n)$ crossings, where

$$Z_2(n) := \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor. \tag{1}$$

No drawings of K_n with fewer than $Z_2(n)$ crossings are known, and to this date the Harary-Hill Conjecture $\mathrm{cr}(K_n) = Z_2(n)$ is still open (it has been settled only for $n \leq 12$; see [27]).

The drawings given in [16] are not 2-page drawings, but it was later noticed that 2-page drawings with $Z_2(n)$ crossings do exist [3] (see also [15, 17]). This observation gave rise to the conjecture $\nu_2(K_n) = Z_2(n)$, popularized by Vrt'o [38].

Buchheim and Zheng [4] proved that $\nu_2(K_n) = Z_2(n)$ for $n \leq 14$. Recently, De Klerk and Pasechnik [10] verified that $\nu_2(K_n) = Z_2(n)$ for $n \leq 18$ and $n = 20$ and 24, and used semidefinite programming techniques to give asymptotic estimates on $\nu_2(K_n)/Z_2(n)$. More recently, Ábrego et al. proved that $\nu_2(K_n) = Z_2(n)$ for every n [1].

2.3 k-page drawings of K_n for $k \geq 3$: upper bounds

Much less is known of the k-page crossing number $\nu_k(K_n)$ for $k > 2$.

As we mentioned above, Blažek and Koman [3] seemed to have been the first to construct 2-page drawings of K_n with $Z_2(n)$ crossings. In the same paper they briefly observed that their construction could be extended to k pages. Although no details were given, they gave the number of crossings obtained for the case $k = 3$ (see Section 5.4).

Damiani, D'Antona and Salemi proposed a way to draw K_n on k pages [8], using the adjacency matrix representation (we call this the *DDS construction*). They included a table with the resulting number of crossings for $n \leq 18$, and all $k \leq \lceil n/2 \rceil$ (recall that $\nu_k(K_n) = 0$ if $k > \lceil n/2 \rceil$). The exact number of crossings resulting from their construction was not explicitly given (we have calculated this number; see Proposition 5). The (table) results given in [8] coincide for the case $k = 3$ with the expression given by Blažek and Koman. Although Blažek and Koman did not explain in detail their proposed construction for $k > 2$, it is not difficult to fill out the details; by doing so, one can confirm that the (or, at least, one possible) natural way to generalize their construction for $k > 2$ yields precisely the DDS construction.

In [32], Shahrokhi et al. described a construction that draws K_n on k pages. This is also a natural generalization of the Blažek-Koman construction to $k > 2$ pages. Moreover, it agrees with the DDS construction whenever k divides n (the DDS construction yields slightly better results for other values of k and n). Based on their construction, Shahrokhi et al. gave the following general upper bound:

$$\nu_k(K_n) \leq \frac{2}{k^2}\left(1 - \frac{1}{2k}\right)\binom{n}{4} + \frac{n^3}{2k}. \tag{2}$$

In Section 5 we include a detailed discussion on the DDS construction, including the calculation of the number of crossings that result by drawing K_n on k pages using this paradigm.

2.4 k-page drawings of K_n for $k \geq 3$: lower bounds

Shahrokhi et al. proved in [32] that for every graph G and every positive integer k, one has $\nu_k(G) \geq m^3/37k^2n^2 - 27kn/37$. Following the derivation of this bound, it is easy to see that the factor $1/37$ in this expression can be improved, but only marginally so. Applying this bound to K_n, we obtain

$$\nu_k(G) \geq \frac{n(n-1)^3}{296k^2} - \frac{27kn}{37} = \frac{3}{37k^2}\binom{n}{4} + O(n^3).$$

This lower bound can be improved if n is sufficiently large compared to k, as follows. We recall that a k-planar drawing is similar to a k-page drawing, but involves k unrestricted planar drawings. Formally, let $G = (V, E)$ be a graph. A *k-planar drawing* of G is a set of k planar drawings of graphs $G^{(i)} = (V, E^{(i)})$ $(i = 1, \ldots, k)$, where the edge sets $E^{(i)}$ form a k-partition of E. Loosely speaking, to obtain the k-planar drawing, we take the drawings of the graphs $G^{(i)}$, and (topologically) identify the k copies of each vertex. The *k-planar crossing number* $\mathrm{cr}_k(G)$ of G is the minimum number of crossings in a k-planar drawing of G.

If k is even, then it is easy to obtain, from a k-page drawing of a graph G, a $k/2$-planar drawing of G with the same number of crossings. Therefore $\nu_k(G) \geq \mathrm{cr}_{k/2}(G)$ for every graph G and any positive even integer k. In [33], Shahrokhi et al. proved that for all $n \geq 2r^2 + 6r - 1$ and all $r \geq 1$,

$$\mathrm{cr}_r(K_n) \geq \frac{1}{2(3r-1)^2}\binom{n}{4}.$$

From our previous observation, it follows that for all $n \geq 2(k/2)^2 + 6(k/2) - 1 = k^2/2 + 3k - 1$ and all even $k \geq 2$, $\nu_k(K_n) \geq \frac{2}{(3k-2)^2}\binom{n}{4}$. Obviously $\nu_{k-1}(G) \geq \nu_k(G)$ for any graph G and any integer $k \geq 2$, and so for any odd $k \geq 3$ and any $n \geq (k-1)^2/2 + 3(k-1) - 1 = k^2/2 + 2k - 7/2$ we have $\nu_k(K_n) \geq \nu_{k+1}(K_n) \geq \frac{2}{(3(k+1)-2)^2}\binom{n}{4} = \frac{2}{(3k+1)^2}\binom{n}{4}$. Now in their exhaustive investigation of biplanar (2-planar) crossing numbers [5, 6], Czabarka, Sýkora, Székely, and Vrt'o prove the slightly better bound (for the 2-planar, or *biplanar* crossing number) $\mathrm{cr}_2(K_n) \geq n^4/952$. From this it follows that $\nu_4(K_n) \geq n^4/952$. Putting all these results together, we obtain the following lower bounds:

$$\nu_k(K_n) \geq \begin{cases} \frac{3}{119}\binom{n}{4} + O(n^3), & \text{if } k = 4; \\ \frac{2}{(3k-2)^2}\binom{n}{4}, & \text{if } k \text{ is even, } k > 4 \text{, and } n \geq k^2/2 + 3k - 1 \text{ ;} \\ \frac{2}{(3k+1)^2}\binom{n}{4}, & \text{if } k \text{ is odd, and } n \geq k^2 + 2k - 7/2. \end{cases} \tag{3}$$

2.5 k-page drawings of K_n: asymptotic lower and upper bounds

The following type of result is well-known and easily shown; see e.g. [29] or [31, Theorem 2].

Claim 1. For any integers $k > 0$ and $n > m \geq 4$,

$$\frac{\nu_k(K_n)}{\binom{n}{4}} \geq \frac{\nu_k(K_m)}{\binom{m}{4}}.$$

As a consequence, the sequence $\frac{\nu_k(K_n)}{\binom{n}{4}}$ is monotonically non-decreasing in n. Since it is also bounded from above by (2), the limit exists, and using (2) and (3) one has:

$$\frac{3}{119} \leq \lim_{n\to\infty} \frac{\nu_4(K_n)}{\binom{n}{4}} \leq \frac{7}{64}; \tag{4}$$

$$\frac{2}{(3k-2)^2} \leq \lim_{n\to\infty} \frac{\nu_k(K_n)}{\binom{n}{4}} \leq \frac{2}{k^2}\left(1 - \frac{1}{2k}\right), \text{ if } k \text{ is even, } k > 4. \tag{5}$$

$$\frac{2}{(3k+1)^2} \leq \lim_{n\to\infty} \frac{\nu_k(K_n)}{\binom{n}{4}} \leq \frac{2}{k^2}\left(1 - \frac{1}{2k}\right), \text{ if } k \text{ is odd, } k \geq 3. \tag{6}$$

3 Formulating $\nu_k(K_n)$ as a maximum k-cut or maximum satisfiability problem

We will show that $\nu_k(K_n)$ can be obtained by computing the maximum k-cut size in a certain graph $G_n = (V_n, E_n)$, say, which is a certain subgraph of the complement of the line graph of K_n. The same graph was used in [10] to investigate $\nu_2(K_n)$, and the general construction of graphs of this type is due to Buchheim and Zheng [4].

To define the graph $G_n = (V_n, E_n)$, we consider a Hamiltonian cycle C_n with vertices $v_1, v_2, \ldots, v_n$. Let V_n be the set of *chords* of the cycle, that is, the edges v_iv_j with v_i and

v_j at cyclic distance at least 2. Let us say that the chords v_iv_j and v_kv_ℓ *overlap* if i, k, j, ℓ occur in this cyclic order as we traverse C_n, either in its natural or in its reverse direction. Finally, to define E_n, we let two chords v_iv_j and v_kv_ℓ be adjacent if they overlap.

Thus one has $|V_n| = \binom{n}{2} - n$, and it is easy to verify that $|E_n| = \binom{n}{4}$. The automorphism group of G_n is isomorphic to the dihedral group D_n on n elements, and there are $d-1$ orbits of vertices, where $d = \lfloor n/2 \rfloor$. The equivalency classes of vertices (i.e. orbits) may be described as follows: since vertices correspond to chords in C_n, the chords that connect vertices of C_n at the same cyclic distance belong to the same equivalency class. If n is odd, then the vertices corresponding to chords with cyclic distance i have valency $i(i-1)+2(i-1)(d-i)$, as is easy to check.

For later use, we will label the vertices of G_n so that its adjacency matrix is partitioned into symmetric circulant blocks. To this end, consider the cycle C_n with vertices numbered $\{0, 1, \dots, n-1\}$ in the usual way. The vertices of G_n that correspond to chords connecting points at cyclic distance i are now labeled successively, starting with the chord

$$\{\lfloor n/2 \rfloor \times i \mod n, (\lfloor n/2 \rfloor + 1) \times i \mod n\},$$

and then obtaining the next chords in the ordering via clockwise cyclic shifts.

Thus the adjacency matrix of G_n is partitioned into a block structure, where each row of blocks is indexed by a cyclic distance $i \in \{2, \dots, d\}$, and each block has size $n \times n$.

Moreover, one may readily verify that block (i,j) $(i, j \in \{2, \dots, d\},\ i \le j)$ is given by the symmetric $n \times n$ circulant matrix with first row

$$[0\ \mathbf{0}_{\ell_{ij}}^T\ \mathbf{1}_{i-1}^T\ \mathbf{0}_{n-2(i-1)-1-2\ell_{ij}}^T\ \mathbf{1}_{i-1}^T\ \mathbf{0}_{\ell_{ij}}^T], \tag{7}$$

where $\mathbf{1}_k$ and $\mathbf{0}_k$ denote the all-ones and all-zeroes vectors in $\mathbb{R}^k$, respectively, and

$$\ell_{ij} = \begin{cases} d(i-j) \mod n & \text{if } i \text{ and } j \text{ have the same parity} \\ d(i-j) - j \mod n & \text{otherwise.} \end{cases} \tag{8}$$

We may now relate the maximum k-cut problem for G_n to $\nu_k(K_n)$.

Lemma 2. One has

$$\nu_k(K_n) = |E_n| - \text{max-}k\text{-cut}(G_n),$$

where max-k-cut(G_n) denotes the cardinality of a maximum k-cut in G_n.

Proof. First of all, recall that the maximum k-cut problem for $G = (V, E)$ may be seen as a vertex coloring problem, where the vertices V are colored with k colors in such a way that the number of edges with differently colored endpoints is maximized. Consider a fixed k-page drawing of K_n, viewed as k circular drawings. Fix a set of k colors. Assign the edges on page i of the drawing the ith color $(1 \le i \le k)$. This defines a k-partition (or k-coloring) of the vertices V_n of G_n. Moreover, the number of edges in E_n with endpoints of the same color equals the number of crossings in the drawing, by construction. □

As a consequence of this lemma, one may calculate $\nu_k(K_n)$ for fixed (in practice, sufficiently small) values of n by solving a maximum cut problem. This was done by Buchheim and Zheng [4] for $k = 2$ and $n \le 13$, by solving the maximum cut problem with a branch-and-bound algorithm. Using the `BiqMac` solver [28], De Klerk and Pasechnik [10] computed the exact value of $\nu_2(K_n)$ for $n \le 18$ and for $n \in \{20, 24\}$.

3.1 The Frieze-Jerrum max-k-cut bound

We follow the standard practice to use $\mathbb{R}^{p\times q}$ (respectively, $\mathbb{C}^{p\times q}$) to denote the space of $p \times q$ matrices over $\mathbb{R}$ (respectively, $\mathbb{C}$). For $\mathbf{A} \in \mathbb{R}^{p\times p}$, the notation $\mathbf{A} \succeq 0$ means that $\mathbf{A}$ is symmetric positive semidefinite, whereas for $\mathbf{A} \in \mathbb{C}^{p\times p}$, it means that $\mathbf{A}$ is Hermitian positive semidefinite.

Let G be a graph with p vertices, and let $\mathbf{L}$ be its Laplacian matrix. Frieze and Jerrum introduced the following semidefinite programming-based upper bound on max-k-cut(G):

$$\mathcal{FJ}_k(G) := \max\left\{ \frac{k-1}{k}\operatorname{trace}(\mathbf{LX}) \;\middle|\; \mathbf{X} \succeq 0,\ X_{ii} = 1\ (1 \le i \le p),\ \mathbf{X} \ge \frac{-1}{k-1}\mathbf{J} \right\}, \tag{9}$$

where $\mathbf{J}$ denotes the all-ones matrix of order p.

For $k = 2$ this bound coincides with the maximum-cut bound of Goemans and Williamson [13].

The associated dual semidefinite program takes the form:

$$\mathcal{FJ}_k(G) = \min_{\mathbf{w}\in\mathbb{R}^p, \mathbf{S}\ge 0} \left\{ \sum_{i=1}^p w_i + \frac{1}{k-1}\operatorname{trace}(\mathbf{JS}) \;\middle|\; \operatorname{Diag}(\mathbf{w}) - \frac{k-1}{2k}\mathbf{L} - \mathbf{S} \succeq 0 \right\}, \tag{10}$$

where Diag is the operator that maps a p-vector to a $p \times p$ diagonal matrix in the obvious way.

3.2 The Frieze-Jerrum bound for G_n

Using the technique of symmetry reduction for semidefinite programming (see e.g. [12]), one can simplify the dual problem (10) for the graphs G_n defined in Section 3, by using the dihedral automorphism group of G_n. We state the final expression as the following lemma. The proof is very similar to that of [10, Lemma 4], and we therefore only give an outline.

Lemma 3. Let $n > 0$ be an odd integer and $d = \lfloor n/2 \rfloor$. One has

$$\mathcal{FJ}_k(G_n) = \min_{y\in\mathbb{R}^{d-1}} n \sum_{i=2}^d y_i + \frac{n}{k-1}\operatorname{trace}(JX^{(0)})$$

subject to

$$\operatorname{Diag}\left(y - \frac{k-1}{2k} val\right) + \Lambda^{(m)} \succeq 0 \ (0 \le m \le d), \tag{11}$$

where

$$\begin{aligned}
val_i &= i(i-1) + 2(i-1)(d-i), \quad 2 \le i \le d, \\
\Lambda^{(m)}_{ij} &= \frac{k-1}{k} \sum_{t=\ell_{ij}+1}^{\ell_{ij}+i} e^{\frac{-2\pi m t\sqrt{-1}}{n}} - X^{(0)}_{ij} - 2\sum_{t=1}^{d} X^{(t)}_{ij} e^{\frac{-2\pi m t\sqrt{-1}}{n}} \quad 2 \le i \le j \le d, \\
\ell_{ij} &= \begin{cases} d(i-j) \mod n & \text{if } i \text{ and } j \text{ have the same parity} \\ d(i-j) - j \mod n & \text{otherwise} \end{cases} \\
X^{(m)} &= (X^{(m)})^T \ge 0, \quad \text{for all } 0 \le m \le d.
\end{aligned}$$

Proof. Assume that $\mathbf{w}, \mathbf{S}$ are optimal in (10) for $G = G_n$ and denote the Laplacian matrix of G_n by $\mathbf{L}$. We now project the posititive semidefinite matrix

$$\operatorname{Diag}(\mathbf{w}) - \frac{k-1}{2k}\mathbf{L} - \mathbf{S} \succeq 0$$

onto the centralizer ring of $\operatorname{Aut}(G_n)$, via the Reynolds projection operator (or group average), say $\mathcal{R}_{G_n}$:

$$\mathcal{R}_{G_n}(X) := \frac{1}{|\operatorname{Aut}(G_n)|} \sum_{P \in \operatorname{Aut}(G_n)} P^T X P \qquad (X \in \mathbb{R}^{|V_n| \times |V_n|}),$$

where the matrices P are given by the permutation matrix representation of $\operatorname{Aut}(G_n)$. Note that this projection preserves positive semidefiniteness as well as entrywise nonnegativity. Moreover, as explained in Section 3, we may assume that $\mathcal{R}_{G_n}(X)$ is a block matrix consisting of symmetric circulant blocks of order n.

Also note that the projection $\mathcal{R}_{G_n}(\operatorname{Diag}(\mathbf{w}))$ simply averages the components of $\mathbf{w}$ over the $d-1$ orbits of $\operatorname{Aut}(G_n)$. Denoting the average of the $\mathbf{w}$ components in orbit i by y_i ($2 \le i \le d$), and $\mathbf{Z} = \mathcal{R}_{G_n}(\mathbf{S}) \ge 0$, we obtain

$$\operatorname{Diag}(\mathbf{y} \otimes \mathbf{1}_n) - \frac{k-1}{2k}\mathbf{L} - \mathbf{Z} \succeq 0, \tag{12}$$

since $\mathcal{R}_{G_n}(\mathbf{L}) = \mathbf{L}$.

Thus we have obtained the reformulation

$$\mathcal{FJ}_k(G_n) = \min_{y \in \mathbb{R}^{d-1}, \mathbf{0} \le \mathbf{Z} \in \mathcal{A}} \left\{ n \sum_{i=2}^{d} y_i + \frac{1}{k-1} \operatorname{trace}(\mathbf{J}\mathbf{Z}) \mid \text{s.t. } (12) \right\},$$

where $\mathcal{A} \subset \mathbb{R}^{|V_n|\times|V_n|}$ denotes the centralizer ring of $\mathrm{Aut}(G_n)$, i.e. the matrix $*$-algebra consisting of matrices of order $|V_n|$ that are partitioned into symmetric circulant blocks of order n.

We may now reduce this formulation further by using the discrete Fourier transform matrix to simultaneously diagonalize the circulant blocks of $\mathbf{Z}$ and $\mathbf{L}$.

To this end, let $\mathbf{Q}$ denote the (unitary) discrete Fourier transform matrix of order n. Condition (12) is equivalent to

$$(\mathbf{I}_{d-1} \otimes \mathbf{Q}) \left(\mathrm{Diag}(\mathbf{y} \otimes \mathbf{1}_n) - \frac{k-1}{2k}\mathbf{L} - \mathbf{Z} \right) (\mathbf{I}_{d-1} \otimes \mathbf{Q})^* \succeq 0. \tag{13}$$

Since the unitary transform involving $\mathbf{Q}$ diagonalizes any circulant matrix (see e.g. [14]), the matrix $(\mathbf{I}_{d-1} \otimes \mathbf{Q})\mathbf{L}(\mathbf{I}_{d-1} \otimes \mathbf{Q})^*$ becomes a block matrix where each $n \times n$ block is diagonal, with diagonal entries of block (i, j) given by the eigenvalues of the circulant matrix with first row given by

$$\begin{cases} [0 \; \mathbf{0}^T_{\ell_{ij}} \; -\mathbf{1}^T_{i-1} \; \mathbf{0}^T_{n-2(i-1)-1-2\ell_{ij}} \; -\mathbf{1}^T_{i-1} \; \mathbf{0}^T_{\ell_{ij}}] & \text{if } i \neq j \\ [val_i \; -\mathbf{1}^T_{i-1} \; \mathbf{0}^T_{n-2(i-1)-1} \; -\mathbf{1}^T_{i-1}] & \text{if } i = j, \end{cases}$$

due to (7). Also, clearly one has

$$(\mathbf{I}_{d-1} \otimes \mathbf{Q}) \left(\mathrm{Diag}(\mathbf{y} \otimes \mathbf{1}_n)\right) (\mathbf{I}_{d-1} \otimes \mathbf{Q})^* = \mathrm{Diag}(\mathbf{y} \otimes \mathbf{1}_n).$$

Finally, the rows and columns of the left hand side of (13) may now be re-ordered to form a block diagonal matrix with n diagonal blocks, each of size $d-1 \times d-1$. Only $d+1$ of these n blocks are distinct, and these correspond to the left-hand-side matrices in (11). The matrices $\Lambda^{(i)}$ $(0 \leq i \leq d)$ in (11) correspond to the distinct blocks obtained from the reordering of

$$-(\mathbf{I}_{d-1} \otimes \mathbf{Q})\mathbf{Z}(\mathbf{I}_{d-1} \otimes \mathbf{Q})^*$$

into block-diagonal form. In particular, we use $X^{(m)}_{ij}$ to denote element m of the first row of the symmetric circulant block (i, j) of $\mathbf{Z}$. □

A few remarks on Lemma 3:

1. Note that we obtain a reduced semidefinite program with $d = \lfloor n/2 \rfloor$ linear matrix inequalities involving matrices of order $d-1$, as well as $d+1$ nonnegative matrix variables of order $d-1$. This should be compared to the original formulation (10) to obtain $\mathcal{FJ}_k(G_n)$, that involved a linear matrix inequality of order $n(d-1)$, as well as a nonnegative matrix variable of the same order.

2. Lemma 3 generalizes [10, Lemma 4] to include the case $k > 2$, but also refines it in the sense that the dihedral symmetry of the graph G_n is fully exploited. Indeed in [10,

Lemma 4], only the cyclic part of $\mathrm{Aut}(G_n)$ was used, leading to (complex) Hermitian linear matrix inequalities, as opposed to the real symmetric linear matrix inequalities of Lemma 3.

3. The computation of $\mathcal{FJ}_k(G_n)$ is simpler in the case $k = 2$, since the bound then becomes the Goemans-Williamson maximum cut bound. Indeed, in [10], values of $\mathcal{FJ}_2(G_n)$ were reported for n close to $1,000$. For $k > 2$, one is limited to more modest values: the largest value of n for which we will report computational results will be $n = 69$; see Section 4. The difference in size of n that may be handled is primarily due to the nonnegative matrix variables $X^{(m)}$ $(0 \le m \le d)$. These variables may be eliminated if $k = 2$, but not if $k > 2$.

3.3 A maximum satisfiability reformulation

It is well-known that the maximum k-cut problem may be reformulated as a maximum satisfiability problem, and we will use this reformulation later on for computational purposes.

Consider a graph $G = (V, E)$ and a set of k colors (used to color the vertices V). We define the following logical variables:

$$x_i^j = \begin{cases} \text{TRUE} & \text{if vertex } i \text{ has color } j \\ \text{FALSE} & \text{otherwise.} \end{cases}$$

Consider the clause:

$$\neg x_i^p \vee \neg x_j^p \text{ if } (i,j) \in E \tag{14}$$

for each color $p = 1, \ldots, k$. For a given edge, and a given color, this clause is satisfied if and only if the endpoints of the edge are not both colored using this color.

Moreover, each vertex should be assigned a color:

$$x_i^1 \vee \ldots \vee x_i^k \quad (i \in V). \tag{15}$$

In order to obtain the maximum k-cut in G, we therefore need values of the logical variables that satisfy all the clauses (15), and as many of the clauses (14) as possible. This may be done by solving a weighted maximum satisfiability problem, where the weights of the satisfied clauses is maximized. In order to guarantee that the clauses (15) are all satisfied, we assign these clauses weight $k|E|$, while the clauses (14) are assigned weight 1.

Thus the cardinality of a maximum k-cut in $G = (V, E)$ coincides with the maximum weight of satisfied clauses in a truth assignment for the weighted logical formula:

$$\begin{array}{ll} \neg x_i^p \vee \neg x_j^p & ((i,j) \in E,\ 1 \le p \le k) \\ k|E| \left(x_i^1 \vee \ldots \vee x_i^k \right) & (i \in V). \end{array} \tag{16}$$

We may now apply this idea to obtain $\nu_k(K_n)$ as follows.

Lemma 4. Consider the set of weighted clauses (16) for the graph $G = G_n = (V_n, E_n)$. Then $\nu_k(K_n)$ is the minimum weight of the unsatisfied clauses, taken over all possible truth assignments.

Proof. The proof follows directly from Lemma 2. □

4 Numerical results

4.1 Exact computations

It is possible to compute $\nu_k(K_n)$ exactly using software for the weighted maximum satisfiability problem in Lemma 4. In Table 1 we show results obtained using the solver `Akmaxsat` by Kügel [22].

k \ n	7	8	9	10	11	12	13	14	15
3	2	5	9	20	34	51	83	121	165*
4	0	0	3	7	12	18	34		
5	0	0	0	0	4	9			

Table 1: Exact values of $\nu_k(K_n)$ for small values of k and n, as computed using the maximum satisfiability solver `Akmaxsat`. *The value $\nu_3(K_{15}) = 165$ was not determined using `Akmaxsat`; see Proposition 12.

Each entry in Table 1 required at most 48 hours of computation on a laptop with 2.5GHz dual core processor and 4GB RAM; the values that are missing from the table could not be computed using `Akmaxsat` within this time.

We finally note that to our knowledge, prior to this work, the exact value of $\nu_k(K_n)$ was not known for any n, k with $2 < k < \lceil n/2 \rceil$ (we recall that K_n can be drawn without crossings in $\lceil n/2 \rceil$ pages; thus $\nu_k(K_n) = 0$ for $k \geq \lceil n/2 \rceil$ and $\nu_k(K_n) > 0$ for $k < \lceil n/2 \rceil$).

Some preliminary computation work was done using Sage [30].

4.2 Asymptotic bounds

As an example of the numerical results presented here, let $m = 69$ and $k = 10$. We computed $\mathcal{FJ}_{10}(G_{69}) \approx 856,520$, and using this value we get

$$\frac{\nu_{10}(K_{69})}{\binom{69}{4}} \geq \frac{\binom{69}{4} - \mathcal{FJ}_{10}(G_{69})}{\frac{69}{4}} \approx 9.2313 \times 10^{-3}.$$

Recall that, for all $n > m \geq 4$,

$$\frac{\nu_k(K_n)}{\binom{n}{4}} \geq \frac{\nu_k(K_m)}{\binom{m}{4}}.$$

Thus it follows that

$$\frac{\nu_{10}(K_n)}{\binom{n}{4}} \geq \frac{\nu_{10}(K_{69})}{\frac{69}{4}} \geq 9.2313 \times 10^{-3} \quad (n > 69).$$

For $n > 69$, this is an improvement on the best previously known lower bound (from (3)), namely

$$\frac{\nu_{10}(K_n)}{\binom{n}{4}} \geq \frac{2}{(3(10)-2)^2} \approx 2.5510 \times 10^{-3}.$$

In Table 2 we give a systematic list of such improved bounds. Computation was done on a Dell Precision T7500 workstation with 92GB of RAM memory, using the semidefinite programming solver SDPT3 [36, 37] under Matlab 7 together with the Matlab package YALMIP [23].

k \ m	39	49	59	69	Lower bounds from (3)
3	1.4266×10^{-1}	1.4827×10^{-1}	1.5194×10^{-1}	1.5452×10^{-1}	2.0000×10^{-2}
4	7.4205×10^{-2}	7.9473×10^{-2}	8.2837×10^{-2}	8.5127×10^{-2}	2.5210×10^{-2}
5	4.2208×10^{-2}	4.6916×10^{-2}	5.0019×10^{-2}	5.2141×10^{-2}	7.8125×10^{-3}
6	2.5728×10^{-2}	2.9633×10^{-2}	3.2258×10^{-2}	3.4151×10^{-2}	7.8125×10^{-3}
7	1.6260×10^{-2}	1.9605×10^{-2}	2.1895×10^{-2}	2.3524×10^{-2}	4.1322×10^{-3}
8	1.0544×10^{-2}	1.3390×10^{-2}	1.5356×10^{-2}	1.6812×10^{-2}	4.1322×10^{-3}
9	6.9603×10^{-3}	9.3377×10^{-3}	1.1062×10^{-2}	1.2333×10^{-2}	2.5510×10^{-3}
10	4.6086×10^{-3}	6.6189×10^{-3}	8.1148×10^{-3}	9.2314×10^{-3}	2.5510×10^{-3}
11	3.0659×10^{-3}	4.7436×10^{-3}	6.0329×10^{-3}	7.0285×10^{-3}	1.7301×10^{-3}
12	2.0007×10^{-3}	3.4078×10^{-3}	4.5294×10^{-3}	5.3894×10^{-3}	1.7301×10^{-3}
13	1.2987×10^{-3}	2.4613×10^{-3}	3.4307×10^{-3}	4.2025×10^{-3}	1.2500×10^{-3}
14	8.2096×10^{-4}	1.7736×10^{-3}	2.6077×10^{-3}	3.2930×10^{-3}	1.2500×10^{-3}
15	4.7807×10^{-4}	1.2613×10^{-3}	1.9718×10^{-3}	2.5870×10^{-3}	9.4518×10^{-4}
16	2.6556×10^{-4}	8.9554×10^{-4}	1.5141×10^{-3}	2.0348×10^{-3}	9.4518×10^{-4}
17	1.3191×10^{-4}	6.2938×10^{-4}	1.1514×10^{-3}	1.6023×10^{-3}	7.3964×10^{-4}
18	5.2726×10^{-5}	4.2802×10^{-4}	8.5199×10^{-4}	1.2562×10^{-3}	7.3964×10^{-4}
19	8.8699×10^{-6}	2.7320×10^{-4}	6.3294×10^{-4}	9.8258×10^{-4}	5.9453×10^{-4}
20	0	1.7127×10^{-4}	4.7985×10^{-4}	7.7482×10^{-4}	5.9453×10^{-4}

Table 2: Lower bounds for $\frac{\nu_k(K_n)}{\binom{n}{4}} \geq \frac{\binom{m}{4} - \mathcal{FJ}_k(G_m)}{\binom{m}{4}}$, for all $n > m$, $m \in \{39, 49, 59, 69\}$ and $k = 3, 4, \ldots, 20$, and comparison with the previous best lower bounds on $\frac{\nu_k(K_n)}{\binom{n}{4}}$ (from (3)).

.

Note that the values in the column "$m = 69$" improve on the known lower bounds (3) in all cases, for $n > 69$.

k	Previous lower bound on $\lim_{n\to\infty} \frac{\nu_k(K_n)}{\binom{n}{4}}$	Improved lower bound on $\lim_{n\to\infty} \frac{\nu_k(K_n)}{\binom{n}{4}}$	Best upper bound on $\lim_{n\to\infty} \frac{\nu_k(K_n)}{\binom{n}{4}}$	Quotient between lower and upper bound
3	2.0000×10^{-2}	1.5452×10^{-1}	1.8518×10^{-1}	0.8344
4	2.5210×10^{-2}	8.5127×10^{-2}	1.0937×10^{-1}	0.7783
5	7.8125×10^{-2}	5.2141×10^{-2}	7.2000×10^{-2}	0.7241
6	7.8125×10^{-3}	3.4151×10^{-2}	5.0925×10^{-2}	0.6706
7	4.1322×10^{-3}	2.3524×10^{-2}	3.7900×10^{-2}	0.6706
8	4.1322×10^{-3}	1.6812×10^{-2}	2.9296×10^{-2}	0.5738
9	2.5510×10^{-3}	1.2333×10^{-2}	2.3319×10^{-2}	0.5287
10	2.5510×10^{-3}	9.2314×10^{-3}	1.9000×10^{-2}	0.4858
11	1.7301×10^{-3}	7.0285×10^{-3}	1.5777×10^{-2}	0.4454
12	1.7301×10^{-3}	5.3894×10^{-3}	1.3310×10^{-2}	0.4049
13	1.2500×10^{-3}	4.2025×10^{-3}	1.1379×10^{-2}	0.3693
14	1.2500×10^{-3}	3.2930×10^{-3}	9.8396×10^{-3}	0.3346
15	9.4518×10^{-3}	2.5870×10^{-3}	8.5925×10^{-3}	0.3010
16	9.4518×10^{-4}	2.0348×10^{-3}	7.5683×10^{-3}	0.2688
17	7.3964×10^{-4}	1.6023×10^{-3}	6.7168×10^{-3}	0.2385
18	7.3964×10^{-4}	1.2562×10^{-3}	6.0013×10^{-3}	0.2093
19	5.9453×10^{-4}	9.8258×10^{-4}	5.3943×10^{-3}	0.1821
20	5.9453×10^{-4}	7.7482×10^{-4}	4.8750×10^{-3}	0.1589

Table 3: Summary of lower and upper bounds for $\lim_{n\to\infty} \nu_k(K_n)/\binom{n}{4}$. The second column gives the previously best lower bounds, as given in (4), (5), and (6). The third column presents the lower bounds we obtained by computing $\mathcal{FJ}_k(G_{69})$ for $k = 3, 4, \ldots, 20$ (this is the fifth column of Table 2). In the fourth column we show the best upper bounds known, given by (2) (alternatively, using Observation 10 and that $\nu_k(K_n) \leq Z_k(n)$). Finally, in the fifth column we show the ratio between the values given in the third and fourth columns.

In Table 3 we summarize the best lower and upper bounds known for $\lim_{n\to\infty} \nu_k(K_n)/\binom{n}{4}$.

5 Drawing K_n in k pages: conjectures and results

In this section we calculate the number $Z_k(n)$ of crossings that result by drawing K_n on k pages using the construction by Damiani et al. [8] (we recall that we call this the *DDS construction*), and a generating function $G_k(z) := \sum_{n\geq 0} Z_k(n)z^n$ for it. This construction is a natural generalization of the construction by Blažek and Koman [3] (who fully described it for 2 pages, and briefly mentioned that it could be generalized to $k > 2$ pages), and a slight refinement of the construction by Shahrokhi et al. [32]. The description of the construction and the calculation of $Z_k(n)$ and $G_k(z)$ are in Section 5.1.

We calculate the exact value of $Z_k(n)$ for two reasons. First, the value $Z_k(n)$ was determined in neither [3] nor [8]; since no better (crossing-wise) construction to draw K_n in k pages is known, this is a calculation worth doing. Second, for all values of k and n for which we now (that is, with the results reported in this paper) know the exact value of $\nu_k(K_n)$, we have $\nu_k(K_n) = Z_k(n)$. These confirmations, as well as an additional feature that we shall explain below (namely Proposition 8, from which we will compute $\nu_3(K_{15})$), lend credibility to the conjecture $\nu_k(K_n) = Z_k(n)$, which we formally put forward in Section 5.2. As we shall see, the value of $Z_k(n)$ depends on $n \bmod k$, and for each fixed k and each fixed $q \in \{0, 1, \ldots, k-1\}$, there is a degree 4 polynomial $G_{q,k}(n)$ such that $Z_k(n) = G_{q,k}(n)$ for all n such that $n \bmod k = q$. In Section 5.3 we explicitly give this polynomial for the case $n \bmod k = 0$. We finally present, in Section 5.4, a slightly more detailed discussion and further results for the case $k = 3$.

5.1 The DDS construction: an upper bound $Z_k(n)$ for the k-page crossing number of K_n

The DDS construction was described in [8] in terms of the adjacency matrix. We have found it both more lively and more convenient (for our calculations) to follow the more geometrical viewpoint of Shahrokhi et al. to describe this construction, and this is the approach we follow below.

We draw K_n in k pages using the circular model. Label the vertices $0, 1, 2, \ldots, n-1$ in the clockwise order in which they occur in the boundary of the circle. For $i = 0, 1, \ldots, n-1$, let M_i be the set of edges whose endpoints have sum i (modulo n). Thus, M_i is a matching for each $i \in \{0, 1, \ldots, n-1\}$. We note that each edge belongs to exactly one matching M_i. For $s, t \in \{0, 1, \ldots, n-1\}$, $s < t$, let $\mathcal{M}_{s,t} := M_s \cup M_{s+1} \cup \cdots M_t$. Loosely speaking, $\mathcal{M}_{s,t}$ consists of the edges of $t - s + 1$ "consecutive" matchings. In Figure 2 we illustrate the sets $\mathcal{M}_{0,3}$ (left), $\mathcal{M}_{4,6}$ (center), and $\mathcal{M}_{7,9}$ (right), for the case $n = 10$.

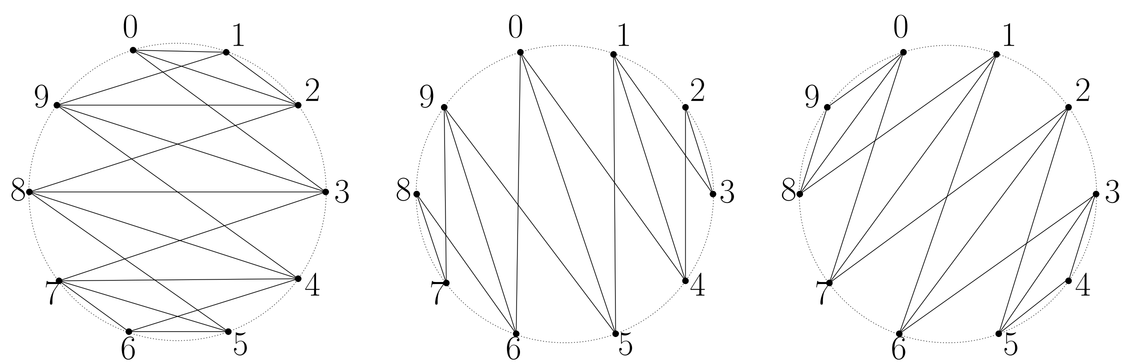

Figure 2: To draw K_{10} in 3 pages, we place the edges in $\mathcal{M}_{0,3} = M_0 \cup M_1 \cup M_2 \cup M_3$ in page 0 (left), the edges in $\mathcal{M}_{4,6} = M_4 \cup M_5 \cup M_6$ in page 1 (center), and the edges in $M_{7,9} = M_7 \cup M_8 \cup M_9$ in page 2 (right).

Let $p := \lfloor n/k \rfloor$ and $q := n \bmod k$ (thus $n = pk + q$). The DDS construction consists simply on distributing the edges of K_n into k pages $0, 1, \ldots, k-1$ as follows:

1. for $0 \leq \ell < q$, place in page ℓ the edges in $\mathcal{M}_{\ell(p+1),\ell(p+1)+p}$; and
2. for $q \leq \ell < k$, place in page ℓ the edges in $\mathcal{M}_{\ell p+q,\ell p+q+(p-1)}$.

Thus, if $0 \leq \ell < q$, then page ℓ contains the edges of $p+1$ matchings, and if $q \leq \ell < k$, then page ℓ contains the edges of p matchings. Note that if k divides n (that is, $q = 0$), then there is no ℓ such that $0 \leq \ell < q$, and so each page contains the edges of p matchings. In Figure 2 we illustrate the DDS construction for the case $k = 3, n = 10$.

We shall give $Z_k(n)$ in terms of a function F that we now define. First, let

$$f(r) := \frac{rn}{2} - \frac{r^2}{2} - \frac{n}{2} + \frac{1}{2},$$

and then let

$$F(r,n) := \sum_{\ell=0}^{r-1}(r-\ell)f(\ell) = -\frac{r^4}{24} + \frac{nr^3}{12} - \frac{nr^2}{4} + \frac{7r^2}{24} + \frac{nr}{6} - \frac{r}{4}. \tag{17}$$

Proposition 5. The number of crossings that result from drawing K_n in $k \geq 1$ pages using the DDS construction is

$$Z_k(n) := (n \bmod k) \cdot F\left(\left\lfloor \frac{n}{k} \right\rfloor + 1, n\right) + \big(k - (n \bmod k)\big) \cdot F\left(\left\lfloor \frac{n}{k} \right\rfloor, n\right).$$

Thus K_n can be drawn in k pages with $Z_k(n)$ crossings, and so

$$\nu_k(K_n) \leq Z_k(n).$$

Note that $Z_k(n)$ is a quasi-polynomial of period k in n (cf. e.g. R. Stanley [35, Sect. 4.4]). This implies that its generating function $G_k(z)$ is rational, i.e. the ratio of two polynomials in z, with denominator having only k-th roots of unity as roots. We will calculate the generating function $G_k(z)$ for $Z_k(n)$ below (cf. Proposition 6).

Proof of Proposition 5. Suppose first that $k = 1$. All the edges are then drawn in the same page. Thus every four points define a crossing, and so we have $\binom{n}{4}$ crossings in total. Since

$$Z_1(n) = F(n,n) = -\frac{n^4}{24} + \frac{n^4}{12} - \frac{n^3}{4} + \frac{7n^2}{24} + \frac{n^2}{6} - \frac{n}{4} = \binom{n}{4},$$

it follows that the statement is true for $k = 1$.

Thus we suppose for the rest of the proof that $k \geq 2$.

To calculate the number of crossings in each page, we first need to calculate the crossings between edges in distinct matchings, when these matchings are placed in the same page. If M_i and M_j are in the same page, then the number $\mathrm{cr}(M_i, M_j)$ of crossings involving an edge in M_i and an edge in M_j depends on the parity of i, j, and n, as well as on $j-i$. It is an easy exercise to show that for all i, j such that $0 \le i < j \le n-1$ and $j - i \le n/2$,

$$\mathrm{cr}(M_i, M_j) = \begin{cases} f(j-i), & \text{if } n \text{ is odd;} \\ f(j-i) - \frac{1}{2}, & \text{if } i \text{ and } j \text{ are odd and } n \text{ is even;} \\ f(j-i), & \text{if } i \text{ and } j \text{ have distinct parity and } n \text{ is even;} \\ f(j-i) + \frac{1}{2}, & \text{if } i, j, \text{ and } n \text{ are even.} \end{cases} \tag{18}$$

With this information at hand, we may proceed to calculate the number $\mathrm{cr}(\mathcal{M}_{s,t})$ of crossings with both edges in $\mathcal{M}_{s,t}$, when all the edges in $\mathcal{M}_{s,t}$ are in the same page. Formally, for s, t such that $0 \le s < t \le n-1$, let $\mathrm{cr}(\mathcal{M}_{s,t}) := \sum_{s \le i < j \le t} \mathrm{cr}(M_i, M_j)$. Our aim (as this is all we shall need) is to calculate $\mathrm{cr}(\mathcal{M}_{s,t})$ for values of s and t such that $0 \le s < t \le n-1$ and $t - s \le n/2$.

Note that there are two types of collections $\mathcal{M}_{s,t}$ that appear in the construction: those of the form $\mathcal{M}_{\ell(p+1),\ell(p+1)+p}$ for $\ell \in \{0, 1, \ldots, q-1\}$ (these contain the edges in p matchings, and we call them *large* collections), and those of the form $\mathcal{M}_{\ell p+q, \ell p+q+(p-1)}$ for $\ell \in \{q, q+1, \ldots, k-1\}$ (these contain the edges in $p-1$ matchings, and we call them *small* collections). Thus there are q large collections and $k-q$ small collections.

We observe that it follows immediately from the construction that

$$Z_k(n) = \sum_{\mathcal{M}_{s,t} \text{ large}} \mathrm{cr}(\mathcal{M}_{s,t}) + \sum_{\mathcal{M}_{s,t} \text{ small}} \mathrm{cr}(\mathcal{M}_{s,t}), \tag{19}$$

where the first summation is over all collections $\mathcal{M}_{s,t}$ in the construction that are large, and the second summation is over all collections $\mathcal{M}_{s,t}$ in the construction that are small.

We will analyze separately the two possibilities for the parity of n.

Case 1. *n is odd*

Let n be odd, and let s, t satisfy $0 \le s < t \le n-1$ and $t - s \le n/2$.

Then

$$\begin{aligned} \mathrm{cr}(\mathcal{M}_{s,t}) &= \sum_{s \le i < j \le t} \mathrm{cr}(M_i, M_j) = \sum_{s \le i < j \le t} f(j-i) = \sum_{1 \le \ell \le t-s} ((t-s+1) - \ell) f(\ell) \\ &= \sum_{1 \le \ell \le t-s} ((t-s+1) - \ell) \left(\frac{\ell n}{2} - \frac{\ell^2}{2} - \frac{n}{2} + \frac{1}{2} \right) = F(t-s+1, n). \end{aligned}$$

Thus it follows that if $\mathcal{M}_{s,t}$ is a large collection, then $\mathrm{cr}(\mathcal{M}_{s,t}) = F(t-s+1, n) = F(p+1, n)$, and if it is small, then $\mathrm{cr}(\mathcal{M}_{s,t}) = F(t-s+1, n) = F(p, n)$. Using this and (19), and recalling that there are $q = n \bmod k$ large collections and $k-q$ small collections, and that $p = \lfloor n/k \rfloor$, we obtain

$$\begin{aligned} Z_k(n) &= q \cdot F(p+1, n) + (k-q) \cdot F(p, n) \\ &= (n \bmod k) \cdot F\left(\left\lfloor \frac{n}{k} \right\rfloor + 1, n\right) + (k - (n \bmod k)) \cdot F\left(\left\lfloor \frac{n}{k} \right\rfloor, n\right). \end{aligned}$$

Case 2. *n is even*

Let n be even, and let s, t satisfy $0 \leq s < t \leq n-1$ and $t - s \leq n/2$. For this case (n even), the determination of $\mathrm{cr}(\mathcal{M}_{s,t})$ is more involved, since it depends both on the parity of $t-s$ and on the parity of s. for $i < r$; and $\mathcal{R}_i = \{R_{im+r}, R_{im+r+1}, \ldots, R_{(i+1)m+(r-1)}\}$ for $r \leq i < k$.

To simplify the expressions it is convenient to define

$$\begin{aligned} \mathrm{oo}_{s,t} &:= \big|\{(i,j) \mid s \leq i < j \leq t, i \text{ odd}, j \text{ odd}\}\big|, \\ \mathrm{oe}_{s,t} &:= \big|\{(i,j) \mid s \leq i < j \leq t, i \text{ odd}, j \text{ even}\}\big|, \\ \mathrm{eo}_{s,t} &:= \big|\{(i,j) \mid s \leq i < j \leq t, i \text{ even}, j \text{ odd}\}\big|, \\ \mathrm{ee}_{s,t} &:= \big|\{(i,j) \mid s \leq i < j \leq t, i \text{ even}, j \text{ even}\}\big|. \end{aligned}$$

An elementary argument shows that

$$\mathrm{ee}_{s,t} - \mathrm{oo}_{s,t} = \begin{cases} \frac{t-s}{4}, & \text{if both } s \text{ and } t \text{ are even;} \\ 0, & \text{if } s \text{ and } t \text{ have distinct parity; and} \\ -\frac{t-s}{4}, & \text{if both } s \text{ and } t \text{ are odd.} \end{cases} \tag{20}$$

We have

$$\begin{aligned} \mathrm{cr}(\mathcal{M}_{s,t}) &= \sum_{s \leq i < j \leq t} \mathrm{cr}(M_i, M_j) = \sum_{\substack{s \leq i < j \leq t \\ i \text{ even}}} \mathrm{cr}(M_i, M_j) + \sum_{\substack{s \leq i < j \leq t \\ i \text{ odd}}} \mathrm{cr}(M_i, M_j) \\ &= \sum_{\substack{s \leq i < j \leq t \\ i \text{ even}, j \text{ even}}} \mathrm{cr}(M_i, M_j) + \sum_{\substack{s \leq i < j \leq t \\ i \text{ even}, j \text{ odd}}} \mathrm{cr}(M_i, M_j) \\ &+ \sum_{\substack{s \leq i < j \leq t \\ i \text{ odd}, j \text{ even}}} \mathrm{cr}(M_i, M_j) + \sum_{\substack{s \leq i < j \leq t \\ i \text{ odd}, j \text{ odd}}} \mathrm{cr}(M_i, M_j) \\ &= \sum_{\substack{s \leq i < j \leq t \\ i \text{ even}, j \text{ even}}} \big(f(j-i) + \frac{1}{2}\big) + \sum_{\substack{s \leq i < j \leq t \\ i \text{ even}, j \text{ odd}}} f(j-i) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{s\le i<j\le t\\ i\,\text{odd}, j\,\text{even}}} f(j-i) + \sum_{\substack{s\le i<j\le t\\ i\,\text{odd}, j\,\text{odd}}} \big(f(j-i) - \frac{1}{2}\big) \\
&= \sum_{s\le i<j\le t} f(j-i) + \sum_{\substack{s\le i<j\le t\\ i\,\text{even}, j\,\text{even}}} \frac{1}{2} \; - \sum_{\substack{s\le i<j\le t\\ i\,\text{odd}, j\,\text{odd}}} \frac{1}{2} \\
&= \sum_{s\le i<j\le t} f(j-i) + \frac{1}{2}\text{ee}_{s,t} - \frac{1}{2}\text{oo}_{s,t} \\
&= \sum_{1\le \ell\le t-s} ((t-s+1)-\ell) f(\ell) + \frac{1}{2}\text{ee}_{s,t} - \frac{1}{2}\text{oo}_{s,t} \\
&= F(t-s+1,n) + \frac{1}{2}\text{ee}_{s,t} - \frac{1}{2}\text{oo}_{s,t}.
\end{aligned}
$$

Using this last expression and (20), it follows that

$$
\text{cr}(\mathcal{M}_{s,t}) = \begin{cases} F(t-s+1,n) + \frac{t-s}{4}, & \text{if } s \text{ and } t \text{ are even;} \\ F(t-s+1,n), & \text{if } s \text{ and } t \text{ have distinct parity;} \\ F(t-s+1,n) - \frac{t-s}{4}, & \text{if } s \text{ and } t \text{ are odd.} \end{cases} \tag{21}
$$

Let us say that a collection $\mathcal{M}_{s,t}$ is *even-odd* if s is even and t is odd; it is *even-even* if s and t are even; it is *odd-even* if s is odd and t is even; and it is *odd-odd* if s and t are odd.

We now proceed to compute $Z_k(n)$, analyzing separately the cases when p is odd and when p is even.

Suppose first that p is odd. It is readily verified that in this case (i) each of the q large collections is either even-odd or odd-even; and (ii) out of the $k-q$ small collections, $(k-q)/2$ are even-even and $(k-q)/2$ are odd-odd. Recalling that if a collection $\mathcal{M}_{s,t}$ is large then $t-s=p$ and that if it is small then $t-s=p-1$, and using (19) and (21), it follows that

$$
\begin{aligned}
Z_k(n) &= q\cdot F(p+1,n) + \frac{(k-q)}{2}\cdot\left(F(p,n) + \frac{(p-1)}{4}\right) + \frac{(k-q)}{2}\cdot\left(F(p,n) - \frac{(p-1)}{4}\right) \\
&= q\cdot F(p+1,n) + (k-q)\cdot F(p,n) \\
&= (n \bmod k)\cdot F\left(\left\lfloor \frac{n}{k}\right\rfloor + 1, n\right) + (k-(n\bmod k))\cdot F\left(\left\lfloor\frac{n}{k}\right\rfloor, n\right).
\end{aligned}
$$

Suppose finally that p is even. It is easily checked that in this case (i) out of the q large collections, $q/2$ are even-even and $q/2$ are odd-odd; and (ii) each of the $k-q$ small collections is either even-odd or odd-even. Recalling again that if a collection $\mathcal{M}_{s,t}$ is large then $t-s=p$ and that if it is small then $t-s=p-1$, and using (19) and (21), it follows that

$$
Z_k(n) = \frac{q}{2}\cdot\left(F(p+1,n)+\frac{p}{4}\right) + \frac{q}{2}\cdot\left(F(p+1,n)-\frac{p}{4}\right) + (k-q)\cdot F(p,n)
$$

$$= q \cdot F(p+1,n) + (k-q) \cdot F(p,n)$$
$$= (n \bmod k) \cdot F\left(\left\lfloor \frac{n}{k} \right\rfloor + 1, n\right) + (k - (n \bmod k)) \cdot F\left(\left\lfloor \frac{n}{k} \right\rfloor, n\right). \qquad \square$$

Proposition 6. For a fixed k, the generating function $G_k(z)$ for $Z_k(n)$ is

$$G_k(z) := \sum_{n \geq 0} Z_k(n) z^n = z^{2k+1} \frac{(k-2)(1-z) + 1 - z^{k+1}}{(1-z)^3(1-z^k)^3}.$$

As a first application of this formula, one sees at once that $Z_k(n) = 0$ for $n \leq 2k$, as the first nonzero coefficient in the expansion of G_k into powers of z comes up for the $2k+1$-th power.

Sketch of proof of Proposition 6. Note that

$$G_k(z) = \sum_{s \geq 0} z^{sk} \sum_{\rho=0}^{k-1} Z_k(sk + \rho) z^\rho, \tag{22}$$

and in this form one does not have to worry about $n \mod k$ and $\lfloor \frac{n}{k} \rfloor$, as $Z_k(sk+\rho)$ is a polynomial in s and ρ. One computes the inner sum in (22) to see that it is equal to an explicit degree 4 polynomial in s divided by $(z-1)^3$, namely,

$$\begin{aligned}\frac{24(z-1)^3}{s(s-1)} \sum_{\rho=0}^{k-1} Z_k(sk+\rho) z^\rho &= (2k^2s^2 + 2k^2 - 13ks - 12k + 4s + 16 - ks^2 + 4k^2s) z^{2+k} \\ &+ (-4k^2s^2 - 8k^2s + 2ks^2 - 4k^2 + 18ks - 4s - 4 + 16k) z^{1+k} \\ &+ (2ks^2 + 4ks - s^2 + 2k - 5s - 4) k z^k + (4k^2s - 2k^2s^2 + 9ks + ks^2 - 10k - 4s - 16) z^2 \\ &+ (4k^2s^2 - 8k^2s - 2ks^2 - 10ks + 16k + 4s + 4) z - 2k^2s^2 + 4k^2s + ks^2 + ks - 6k.\end{aligned}$$

It remains to observe that the outer sum in (22) becomes a finite sum of terms of the form $C \sum_{s \geq 0} s^\ell z^s$, with C independent of s. A direct computation then gives the claimed formula. $\square$

5.2 A conjecture for the k-page crossing number of K_n

The DDS construction described in Section 5.1 draws K_n in k pages with $Z_k(n)$ crossings. We conjecture the optimality of this construction:

Conjecture 7. For all positive integers k and n,

$$\nu_k(K_n) = Z_k(n).$$

The naturality and aesthetical appeal of the construction, plus the fact that no construction to draw K_n in k pages with fewer crossings is known, seem good enough reasons to put forward this conjecture. Still, there is further evidence supporting the conjecture:

- The statement is true for $k \le 2$. For $k = 1$ this is readily checked, and for $k = 2$ it follows since it has been recently verified that $\nu_2(K_n) = Z_2(n)$ [1].
- For all k, n for which we now know the exact value of $\nu_k(K_n)$ (Table 1, plus all k, n such that $k > \lceil n/2 \rceil$, for which it is known that $\nu_k(K_n) = 0$), we have $\nu_k(K_n) = Z_k(n)$.

There is yet another argument that supports Conjecture 7, at a somewhat (but not completely; see Section 5.4) more speculative level. Recall that $Z_2(n) := \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$. A well-known counting argument shows that for every positive integer r, $\nu_2(K_{2r-1}) = Z_2(2r-1)$ implies $\nu_2(K_{2r}) = Z_2(2r)$. This "odd implies even" phenomenon is used, for instance, to determine that the (usual) crossing number of K_{12} is $Z_2(12)$: this follows at once since the crossing number of K_{11} is $Z_2(11)$ [27]. An appealing feature of Conjecture 11 is that it implies a similar phenomenon for every k:

Proposition 8. For every positive integers k and r, one has $krZ_k(kr-1) = (kr-4)Z_k(kr)$, and

$$\nu_k(K_{kr-1}) = Z_k(kr-1) \quad \Longrightarrow \quad \nu_k(K_{kr}) = Z_k(kr).$$

Proof. The first claim follows from

$$Z_k(kr-1) = (k-1)F(r, kr-1) + F(r-1, kr-1) = \frac{kr-4}{r}F(r, kr) = \frac{kr-4}{kr}Z_k(kr),$$

after a long but routine manipulation.

Since $\nu_k(K_{kr}) \le Z_k(kr)$ (cf. Proposition 5), we only need to prove the reverse inequality $\nu_k(K_{kr}) \ge Z_k(kr)$. Suppose that $\nu_k(K_{kr-1}) = Z_k(kr-1)$. Consider a k-page drawing D of K_{kr} with $\nu_k(K_{kr})$ crossings. This drawing contains kr drawings of K_{kr-1}, each of which has at least $\nu_k(K_{kr-1}) = Z_k(kr-1)$ crossings. It is easy to see that each crossing gets counted exactly $kr-4$ times, and so

$$\nu_k(K_{kr}) \ge \frac{krZ_k(kr-1)}{kr-4} = kF(r, kr) = Z_k(kr). \qquad \square$$

In Section 5.4 we will use Proposition 8 to prove that $\nu_3(K_{15}) = 165$ (cf. Proposition 12).

5.3 Explicit estimates for $Z_k(n)$

It is clear that for each fixed k and $q \in \{0, 1, \ldots, k-1\}$, there exists a polynomial $G_{k,q}(n)$ such that $G_{k,q}(n) = Z_k(n)$ for all n such that $n \bmod k = q$. For the case $q = 0$, a routine manipulation yields the following.

Observation 9. If k divides n, then

$$Z_k(n) = \left(\left(\frac{1}{12k^2}\right)\left(1-\frac{1}{2k}\right)\right)n^4 + \left(-\frac{1}{4k}\right)n^3 + \left(\frac{7}{24k}+\frac{1}{6}\right)n^2 + \left(-\frac{1}{4}\right)n.$$

We recall (see Section 2.5) that $\nu_k(K_n)/\binom{n}{4} \geq \nu_k(K_m)/\binom{m}{4}$, whenever $n > m \geq 4$. Using this and Observation 9, we obtain the following asymptotic general estimate for $Z_k(n)$:

Observation 10. For each positive integer k,

$$Z_k(n) = \left(\left(\frac{1}{12k^2}\right)\left(1-\frac{1}{2k}\right)\right)n^4 + O(n^3).$$

We note that the upper bound (2), given by Shahrokhi et al. [32], is in line with this last observation (this was expected, since the DDS construction and the construction in [32] agree whenever k divides n).

5.4 Drawing K_n in 3 pages: further results

A long but straightforward manipulation shows that

$$Z_3(n) = \begin{cases} \frac{(n-6)(n-3)n(5n-9)}{648}, & \text{if } n \equiv 0 \pmod 3; \\ \frac{(n-4)(n-1)(5n^2-29n+30)}{648}, & \text{if } n \equiv 1 \pmod 3; \\ \frac{(n-2)(n-3)(n-5)(5n-4)}{648}, & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{23}$$

We remark that this coincides with the number of crossings given by Blažek and Koman in [3], where they briefly mentioned that their construction for 2 pages could be generalized to $k > 2$ pages, and reported the expression in (23) for the number of crossings obtained by drawing K_n in 3 pages (no further details were given).

As we have already observed, $\nu_3(K_n) = Z_3(n)$ for all 8 values of $n \geq 7$ for which we have calculated $\nu_3(K_n)$ (Table 1). This evidence gives special credence to the case $k = 3$ of Conjecture 7:

Conjecture 11 (3-page crossing number of K_n)**.** For every positive integer n,

$$\nu_3(K_n) = Z_3(n).$$

As an additional support for this conjecture, we note that our calculations reported in Section 4 confirm that $\nu_3(K_n)$ is reasonably close to $Z_3(n)$, at least asymptotically. Indeed, from

Table 3 we have $\lim_{n\to\infty} \nu_3(K_n)/\binom{n}{4} \geq 0.15452$. This implies that $\lim_{n\to\infty} \nu_3(K_n)/n^4 \geq 0.006438$. Now from (23) we have $\lim_{n\to\infty} Z_3(n)/n^4 = 5/648$. These results yield

$$\lim_{n\to\infty} \frac{\nu_3(K_n)}{Z_3(n)} > \frac{0.006438}{5/648} \approx 0.8344.$$

We finally show that, as hinted above, Proposition 8 (the generalization to $k > 2$ pages of the "odd implies even" phenomenon for $k = 2$) is not only a speculative curiosity: we use this statement to determine the exact value of $\nu_3(K_{15})$:

Proposition 12. $\nu_3(K_{15}) = 165$.

Proof. It follows from Proposition 8, using (from our calculation, reported in Table 1) that $\nu_3(K_{14}) = Z_3(14)$. □

6 Concluding remarks and open questions

De Klerk, Pasechnik and Warners [9] proved the lower bounds α_k on the ratio $\frac{\text{max-}k\text{-cut}(G)}{\mathcal{FJ}(G)}$ given in Table 4. These lower bounds may be used to obtain upper bounds on $\nu_k(K_n)$ $(3 \leq k \leq 10)$, namely,

$$\nu_k(K_n) \leq |E_n| - \alpha_k \mathcal{FJ}(G_n),$$

but these bounds seem weaker than the upper bounds given by the best known drawings, based on computations for $3 \leq k \leq 10$ and $n \leq 69$.

k:	3	4	5	6	7	8	9	10
α_k	0.836008	0.857487	0.876610	0.891543	0.903259	0.912664	0.920367	0.926788

Table 4: MAX-k-CUT approximation guarantees for $3 \leq k \leq 10$

Limits of the type (6) are of independent interest if one replaces the k-page crossing number by the rectilinear crossing number. (The rectilinear crossing number of a graph is the minimum number of edge crossings in a drawing of the graph in the plane if all edges are drawn by straight lines.) Indeed, for the rectilinear crossing number $\overline{\text{cr}}(K_n)$, the limit

$$\lim_{n\to\infty} \frac{\overline{\text{cr}}(K_n)}{\binom{n}{4}}$$

is related to the Sylvester four point problem in geometric probability as follows. Consider an open set $R \in \mathbb{R}^2$ with finite area. Denote by $q(R)$ the probability that the convex hull of four points in R, drawn uniformly at random, is a convex quadrilateral (as opposed to a line or triangle). Scheinerman and Wilf [31] showed that

$$\inf_R q(R) = \lim_{n\to\infty} \frac{\overline{\mathrm{cr}}(K_n)}{\binom{n}{4}},$$

where the infimum is taken over all open sets R in the plane with finite area.

It remains an interesting question whether these limits also have alternative interpretations if one replaces the rectilinear crossing number by other notions of crossing numbers, like the k-page crossing number.

In the second part of this work we will investigate the k-page crossing numbers of certain complete bipartite graphs. We will once again use optimization techniques, but the details are somewhat different from those presented here, and are therefore best presented seperately.

Acknowledgements. The authors are grateful to Imrich Vrt'o for helpful comments.

References

[1] B.M. Ábrego, O. Aichholzer, S. Fernández-Merchant, P. Ramos, and G. Salazar, The 2-page crossing number of K_n. *Proceedings of the 28th ACM Symposium on Computational Geometry (SoCG'12)*, to appear.

[2] F. Bernhart and P. C. Kainen, The book thickness of a graph, *J. Combin. Theory Ser. B* **27**(3) (1979), 320–331.

[3] J. Blažek and M. Koman, A minimal problem concerning complete plane graphs. *1964 Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963)* pp. 113–117. Publ. House Czechoslovak Acad. Sci., Prague.

[4] C. Buchheim and L. Zheng, Fixed Linear Crossing Minimization by Reduction to the Maximum Cut Problem, *Lecture Notes in Computer Science* **4112** (2006), 507–516.

[5] E. Czabarka, O. Sýkora, O.; L.A. Székely, and I. Vrt'o, Biplanar crossing numbers. I. A survey of results and problems. *More sets, graphs and numbers*, Bolyai Soc. Math. Stud., 15, Springer, Berlin, 2006, pp. 57–77.

[6] E. Czabarka, O. Sýkora, O.; L.A. Székely, and I. Vrt'o, Biplanar crossing numbers. II. Comparing crossing numbers and biplanar crossing numbers using the probabilistic method. *Random Structures Algorithms* **33** (2008), 480–496.

[7] F.R.K. Chung, F.T. Leighton, and A.L. Rosenberg, Embedding graphs in books: a layout problem with applications to VLSI design, *SIAM J. Algebraic Discrete Methods* **8** (1987), 33–58.

[8] E. Damiani, O. D'Antona, and P. Salemi, An upper bound to the crossing number of the complete graph drawn on the pages of a book. *J. Combin. Inform. System Sci.* **19** (1994), 75–84.

[9] E. de Klerk, D.V. Pasechnik, and J.P Warners, On approximate graph colouring and MAX-k-CUT algorithms based on the ϑ-function. *J. Comb. Optim.* **8** (2004), 267–294.

[10] E. de Klerk and D.V. Pasechnik, Improved lower bounds for the 2-page crossing numbers of $K_{m,n}$ and K_n via semidefinite programming, *SIAM J. Opt.*, **22** (2012), 581–595.

[11] H. Enomoto, T. Nakamigawa, and K. Ota, On the Pagenumber of Complete Bipartite Graphs. *J. Comb. Theory Ser. B* **71** (1997), 111–120.

[12] K. Gatermann and P.A. Parrilo, Symmetry groups, semidefinite programs, and sums of squares, *J. Pure Appl. Algebra*, **192** (2004), 95–128.

[13] M.X. Goemans and D.P. Williamson, Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming, *Journal of the ACM*, **42**(6) (1995), 1115–1145.

[14] R.M. Gray, Toeplitz and Circulant Matrices: A review, *Foundations and Trends in Communications and Information Theory*, **2**(3), (2006), 155–239.

[15] R.K. Guy, T. Jenkyns, and J. Schaer, The toroidal crossing number of the complete graph, *J. Combinatorial Theory* **4** (1968), 376–390.

[16] F. Harary and A. Hill, On the number of crossings in a complete graph, *Proc. Edinburgh Math. Soc. (2)* **13** (1962/1963), 333–338.

[17] H. Harborth, Special numbers of crossings for complete graphs. *Discrete Mathematics* **244** (2002), 95–102.

[18] H. He, A. Sălăgean, and E. Mäkinen, One- and two-page crossing numbers for some types of graphs, *International Journal of Computer Mathematics* **87** (2010), 1667–1679.

[19] H. He, O. Sýkora, and E. Mäkinen, Genetic algorithms for the 2-page drawing problem of graphs, *Journal of Heuristics* **13** (2007), 77–93.

[20] H. He, O. Sýkora, and I. Vrt'o, Minimisation Heuristics for 2-page Drawings, *Electronic Notes in Discrete Mathematics* **22** (2005), 527–534.

[21] P.C. Kainen, An introduction to topological graph theory, in *Graphs and Combinatorics*, R. Bari and F. Harary, Eds. Springer, New York, 1974, pp. 76–108.

[22] A. Kügel, Improved Exact Solver for the Weighted Max-SAT Problem. In: *Proceedings of POS-10. Pragmatics of SAT* (Daniel Le Berre, Ed.). EasyChair Proceedings in Computing **8** (2012), pp. 15–27.

[23] J. Löfberg, YALMIP: A Toolbox for Modeling and Optimization in MATLAB. *Proceedings of the CACSD Conference, Taipei, Taiwan, 2004*, http://control.ee.ethz.ch/~joloef/yalmip.php

[24] S. Masuda, K. Nakajima, T. Kashiwabara, and T. Fujisawa, On the NP-completeness of a computer network layout problem, *Proceedings of the 1987 IEEE International Symposium on Circuits and Systems*, IEEE Press, New York, NY (1987), 292–295.

[25] S. Masuda, K. Nakajima, T. Kashiwabara, and T. Fujisawa, Crossing minimization in linear embeddings of graphs, *IEEE Trans. Comput.* **39** (1990), 124–127.

[26] D.J. Muder, M.L. Weaver, and D.B. West, Pagenumber of complete bipartite graphs. *J. Graph Theory* **12** (1988) 469–489.

[27] S. Pan and R.B. Richter, The crossing number of K_{11} is 100, *J. Graph Theory* **56** (2007), 128–134.

[28] F. Rendl, G. Rinaldi, and A. Wiegele, Solving Max-Cut to Optimality by Intersecting Semidefinite and Polyhedral Relaxations, *Math. Programming*, **121** (2010), 307–335.

[29] R.B. Richter and C. Thomassen, Relations between crossing numbers of complete and complete bipartite graphs, *Amer. Math. Monthly* **104** (1997), 131–137.

[30] W. Stein et. al., Sage: Open Source Mathematics Software (version 5.0). The Sage Group, June 2012, `http://www.sagemath.org`.

[31] E.R. Scheinerman and H.S. Wilf, The rectilinear crossing number of a complete graph and Sylvester's "four point problem", *Amer. Math. Monthly*, **101**:10 (1994), 939–943.

[32] F. Shahrokhi, O. Sýkora, L.A. Székely, and I. Vrt'o, The book crossing number of a graph, *J. Graph Theory* **21** (1996), 413–424.

[33] F. Shahrokhi, O. Sykora, L.A. Szekely, I. Vrt'o, On k-planar crossing numbers, *Discrete Applied Mathematics* 155 (2007), 1106-1115.

[34] J. Spencer, The biplanar crossing number of the random graph. *Towards a theory of geometric graphs* Contemp. Math., 342, Amer. Math. Soc., Providence, RI, 2004, pp. 269– 271.

[35] R. Stanley, *Enumerative Combinatorics*, volume 1, 2nd edition, Cambridge University Press, 2011, `http://www-math.mit.edu/~rstan/ec/ec1/`

[36] K.C. Toh, M.J. Todd, and R.H. Tutuncu, SDPT3 — a Matlab software package for semidefinite programming, *Optimization Methods and Software* **11** (1999), 545–581.

[37] R.H. Tutuncu, K.C. Toh, and M.J. Todd, Solving semidefinite-quadratic-linear programs using SDPT3, *Mathematical Programming Ser. B* **95** (2003), 189–217.

[38] I. Vrt'o, Two special crossing number problems for complete graphs, *Abstracts of the 6th Slovenian International Conference on Graph Theory Bled'07.*

[39] D.R. Woodall, Cyclic-Order Graphs and Zarankiewicz's Crossing-Number Conjecture, *J. Graph Theory* **17** (1993), 657–671.

[40] M. Yannakakis, Embedding planar graphs in four pages, *Journal of Computer and System Sciences*, 38(1) (1989), 36–67.

[41] K. Zarankiewicz, On a Problem of P. Turán Concerning Graphs, *Fund. Math.* **41** (1954), 137–145,