Bounding the Crossing Number of a Graph in terms of the Crossing Number of a Minor with Small Maximum Degree

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Abstract. We show that if G has a minor M with maximum degree at most 4, then the crossing number of G in a surface Σ is at least one fourth the crossing number of M in Σ. We use this result to show that every graph embedded on the torus with representativity $r \geq 6$ has Klein bottle crossing number at least $\lfloor 2r/3 \rfloor^2/64$.

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1. Introduction

In his survey of open problems in topological graph theory [2], Dan Archdeacon quotes Paul Seymour as saying: "Isn't it a shame that crossing numbers don't work well with minors?". This is an allusion to the fact that the crossing number $cr(G/e)$ of the graph G/e obtained by contracting an edge e from a graph G can be strictly larger than $cr(G)$. Moreover, for every fixed constant c it is not difficult to exhibit a graph G and a minor M of G such that $cr(M)$ is at least $c \cdot cr(G)$.

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A *drawing* of a graph G in a surface Σ is a representation of G in Σ in which (different) vertices are represented by (different) points, edges are represented by arcs (so that the end points of a representation of an edge e are precisely the points that represent the vertices incident with e), and such that, except for its endpoints, no representation of an edge contains a representation of a vertex.

The *crossing number* $cr_\Sigma(G)$ of a graph G in a surface Σ is the minimum number of pairwise crossings of edges in a drawing of G in Σ.

One of the main goals in this paper is to prove that the crossing number $cr_\Sigma(G)$ of a graph G in a surface Σ cannot be arbitrarily smaller than the crossing number $cr_\Sigma(M)$ of a minor M of G, provided that the maximum degree $\Delta(M)$ of M is at most four.

Theorem 1. *Let G be a graph, and let M be a minor of G such that $\Delta(M) \leq 4$. Then, for every surface Σ, $cr_\Sigma(G) \geq (1/4)cr_\Sigma(M)$.*

As an example of the usefulness of this result, we apply it to bound the Klein bottle crossing number of a toroidal graph G in terms of how densely G embeds on the torus. The *representativity* (or *face–width*) $r(G)$ of a graph G embedded on a surface Σ is the smallest number k such that Σ contains a noncontractible closed curve that intersects G in k points. This concept was introduced by Robertson and Seymour in their work on graph minors [5], and is widely accepted as a measure of how densely a graph is embedded on a surface.

Theorem 2. *Every graph embedded on the torus with representativity $r \geq 6$ has Klein bottle crossing number at least $\lfloor 2r/3 \rfloor^2/64$.*

In this paper we investigate the crossing numbers of graphs in the plane $\mathbb{R}^2$, the projective plane $\mathbb{P}$, and the Klein bottle $\mathbb{K}$.

It is trivial to check that, for every graph G, $cr_{\mathbb{R}^2}(G) \geq cr_{\mathbb{P}}(G) \geq cr_{\mathbb{K}}(G)$ (moreover, both inequalities are strict if $cr_{\mathbb{P}}(G) > 0$). Therefore Theorem 2 also holds if we substitute the Klein bottle crossing number either by the projective plane crossing number or by the plane crossing number.

Every toroidal graph also embeds on every orientable surface of genus at least two, and on every nonorientable surface of genus at least three. Thus it seems natural to investigate drawings of toroidal graphs in the sphere, the projective plane, and the Klein bottle, which are the only compact surfaces in which it may not be possible to embed a toroidal graph.

It is easy to see that the crossing number of a graph in the sphere is the same as its crossing number in $\mathbb{R}^2$. Throughout this paper we follow the trend in the literature, which is to speak about crossing numbers of graphs in the plane rather than in the sphere.

This paper is organized as follows. In Section 2 we prove Theorem 1. The idea of the proof is to find minors G'' and M'' of G and M, respectively, such that $cr_\Sigma(G) \geq cr_\Sigma(G'')$ and $cr_\Sigma(M'') = cr_\Sigma(M)$, and then to show that every drawing $\mathcal{D}_{G''}$ of G'' can be modified to yield a drawing of M'' with at most four times the number of crossings in $\mathcal{D}_{G''}$.

Theorem 2 is proved in Section 3. A crucial ingredient in this proof is the fact that the Cartesian product $C_{\lfloor 2r/3 \rfloor} \times C_{\lfloor 2r/3 \rfloor}$ is a minor of every graph that embeds on the torus with representativity $r \geq 5$ [7]. In view of this result and Theorem 1, in order to prove that the Klein bottle crossing number of a toroidal graph is quadratic in the representativity it suffices to show that $cr_{\mathbb{K}}(C_m \times C_m)$ is quadratic on m.

It is a nice coincidence that the plane, projective plane, and Klein bottle crossing numbers of the graphs $C_m \times C_m$ have been thoroughly investigated [4, 6, 8]. In particular, $cr_{\mathbb{K}}(C_m \times C_m) \geq m^2/16$ for every $m \geq 4$. Theorem 2 follows from this inequality, Theorem 1, and the result in [7] cited in the previous paragraph. The details are carried out in Section 3.

Section 4 contains some concluding remarks.

2. Proof of Theorem 1

It is easy to check that it suffices to prove Theorem 1 for the case when both G and M are connected. We also note that we can assume that $cr_\Sigma(M) > 0$, as otherwise Theorem 1 trivially follows.

Along the proof we define auxiliary graphs G', G'', M', M''. The relationship between these graphs and the graphs G and M is illustrated in Diagram 1.

We divide the proof in three main steps.

Step 1. *There is a graph M'' with the following properties: (i) M'' is a minor of G, (ii) $cr_\Sigma(M'') = cr_\Sigma(M)$, and (iii) all the vertices of M'' have degree three or four.*

Let us say that a path in a graph is *superfluous* if all its internal vertices have degree two. A superfluous path is *maximal* if it is not a proper subpath of a superfluous path.

Let M' be the graph obtained from M by repeatedly removing a vertex of degree one until no such vertices remain.

Since $cr_\Sigma(M) > 0$, it follows that M is not a tree. Therefore M' does not consist of a single vertex. Moreover, all the vertices of M' have degree at least two.

Let M'' denote the minor of M' obtained by substituting each maximal superfluous path in M' by an edge that joins both ends of the path. It is straightforward to check that M'' satisfies the required conditions.

Step 2. *There is a graph G'' with the following properties: (i) M'' is a minor of G'', (ii) $cr_\Sigma(G) \geq cr_\Sigma(G'')$, and (iii) all the vertices of G'' have degree three or four.*

Since M'' is a minor of G, it follows that M'' is obtained by performing a series of edge contractions, edge deletions, and vertex deletions on G. We note that it is possible to perform all edge and vertex deletions first, to obtain a graph G', and then to perform a series of edge contractions on G' to obtain M''. Clearly, $cr_\Sigma(G) \geq cr_\Sigma(G')$.

Since M'' has no vertices of degree one, it follows that G' has no vertices of degree one. On the other hand, since M'' is obtained from G' by performing a series of edge contractions, it follows that $\Delta(G') \leq 4$.

Let G'' be the minor of G' obtained by substituting each maximal superfluous path in G' by an edge that joins both ends of the path. It is straightforward to check that M'' is also a minor of G'', and that $cr_\Sigma(G'') = cr_\Sigma(G')$. Therefore $cr_\Sigma(G) \geq cr_\Sigma(G'')$.

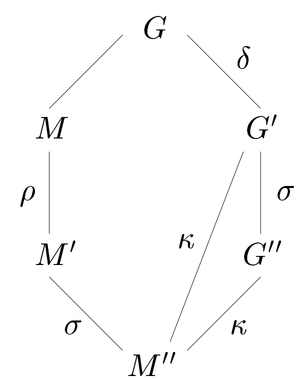

Diagram 1. Hasse diagram of the graphs G, G', G'', M, M', M'' in the proof of Theorem 1. The label of each edge (joining the upper level graph H to the lower level graph K) indicates the operation performed on H to obtain its minor K. The operation "repeatedly remove a vertex of degree one until no such vertices remain" is denoted ρ. The operation "substitute each maximal superfluous path by an edge that joins the ends of the path" is denoted σ. The operation "delete a (certain) set of vertices an edges" is denoted δ. The operation "perform a (certain) series of edge contractions" is denoted κ.

By construction, G'' has no vertices of degree two. Since G' has no vertices of degree one, it follows that G'' has no vertices of degree one. On the other hand, it is readily checked that $\Delta(G') \leq 4$ implies that $\Delta(G'') \leq 4$. Thus all the vertices of G'' have degree three or four.

Step 3. *$cr_\Sigma(G'') \geq (1/4)cr_\Sigma(M'')$, and so $cr_\Sigma(G) \geq (1/4)cr_\Sigma(M)$.* To help comprehension, we color the edges in G''. An edge that gets contracted in order to get M'' is colored blue, and all other edges in G'' are colored red.

Since $\Delta(M'') \leq 4$, it follows that no edge incident with a vertex of degree four in G'' is blue. Therefore each blue edge of G'' is incident with two vertices of degree three, and is adjacent to no other blue edge.

Let $\mathcal{D}_{G''}$ be an optimal drawing of G'' in Σ, that is, a drawing whose number of crossings $cr_\Sigma(\mathcal{D}_{G''})$ equals $cr_\Sigma(G'')$. Our goal is to modify $\mathcal{D}_{G''}$ to get a drawing $\mathcal{D}_{M''}$ of M'' with at most $4cr_\Sigma(G'')$ crossings.

Let e be a fixed blue edge, and let u, v be its incident vertices. Let e', e'' be the two red edges incident with u. Let N_u be a very small neighborhood of u. Delete the subarcs of e' and e'' drawn in the interior of N_u. Then, following e very closely, join the loose ends of e' and e'' to v. Finally, delete e and u.

Clearly, the drawing $\mathcal{D}_{M''}$ that results from performing this operation on every blue edge is a drawing of M''. We remark that $cr_\Sigma(\mathcal{D}_{M''}) \geq cr_\Sigma(M'')$, since there may be drawings of M'' with fewer than $cr_\Sigma(\mathcal{D}_{M''})$ crossings.

It is easily checked that every blue–blue crossing in $\mathcal{D}_{G''}$ yields four crossings in $\mathcal{D}_{M''}$, every blue–red crossing in $\mathcal{D}_{G''}$ yields two crossings in $\mathcal{D}_{M''}$, and every red–red crossing in $\mathcal{D}_{G''}$ yields one crossing in $\mathcal{D}_{M''}$. Therefore each crossing in $\mathcal{D}_{G''}$ yields at most four crossings in $\mathcal{D}_{M''}$, and so $cr_\Sigma(\mathcal{D}_{M''}) \leq 4cr_\Sigma(\mathcal{D}_{G''})$. Therefore $cr_\Sigma(\mathcal{D}_{G''}) \geq (1/4)cr_\Sigma(\mathcal{D}_{M''})$.

Since $cr_\Sigma(G) \geq cr_\Sigma(G'') = cr_\Sigma(\mathcal{D}_{G''})$ and $cr_\Sigma(\mathcal{D}_{M''}) \geq cr_\Sigma(M'') = cr_\Sigma(M)$, it follows that $cr_\Sigma(G) \geq (1/4)cr_\Sigma(M)$. ∎

3. Proof of Theorem 2

Let G be a graph embedded on the torus with representativity $r \geq 6$. By Theorem 2 in [7], G contains the 4–regular graph $C_{\lfloor 2r/3 \rfloor} \times C_{\lfloor 2r/3 \rfloor}$ as a minor. Thus, by Theorem 1, $cr_{\mathbb{K}}(G) \geq (1/4)cr_{\mathbb{K}}(C_{\lfloor 2r/3 \rfloor} \times C_{\lfloor 2r/3 \rfloor})$.

A standard counting argument based on the inequality $cr_{\mathbb{K}}(C_4 \times C_4) \geq 1$ (established in [4]) shows that $cr_{\mathbb{K}}(C_m \times C_m) \geq m^2/16$ for every $m \geq 4$. Since $\lfloor 2r/3 \rfloor \geq 4$, it follows that $cr_{\mathbb{K}}(C_{\lfloor 2r/3 \rfloor} \times C_{\lfloor 2r/3 \rfloor}) \geq \lfloor 2r/3 \rfloor^2/16$. Therefore $cr_{\mathbb{K}}(G) \geq \lfloor 2r/3 \rfloor^2/64$. ∎

4. Concluding Remarks

It follows immediately from Theorem 2 that if a graph G embeds on the torus with representativity at least six, then G does not embed on the Klein bottle. In a related result, Maharry [3] has shown that representativity four is enough to guarantee the nonembeddability of a toroidal graph in the Klein bottle.

As mentioned in Section 1, Theorem 2 also applies if we substitute $cr_{\mathbb{K}}$ by either $cr_{\mathbb{R}^2}$ or $cr_{\mathbb{P}}$. Moreover, using the bounds $cr_{\mathbb{P}}(C_m \times C_m) \geq (1/8)m^2$ for $m \geq 4$ [8], and $cr_{\mathbb{R}^2}(C_m \times C_m) \geq (5/7)m^2$ for $m \geq 7$ (derived from a standard counting argument from the result $cr_{\mathbb{R}^2}(C_7 \times C_7) = 35$ in [1]), the proof of Theorem 2 can be slightly modified to yield the following.

Theorem 3. *Every graph embedded on the torus with representativity $r \geq 6$ has projective plane crossing number at least $\lfloor 2r/3 \rfloor^2/32$.*

Theorem 4. *Every graph embedded on the torus with representativity $r \geq 11$ has plane crossing number at least $(5/28)\lfloor 2r/3 \rfloor^2$.*

An important step in the proof of Theorem 1 is the observation that M'' can be obtained by edge contraction operations from G'', in such a way that each vertex in G'' is incident with at most one edge that gets contracted in order to obtain M''. It is easily seen that the condition $\Delta(M) \leq 4$ is essential to guarantee the existence of graphs M'', G'' with these properties. However, as Bruce Richter has observed in a personal communication to the authors, the proof of Theorem 1 can be slightly modified to yield a proof of the following statement.

Lemma 5. *Let G be a graph, and let M be a minor of G. Let G' be the subgraph of G such that M is obtained by contracting a set E of edges in G'. Suppose that the subgraph of G' induced by E is a matching. Then, for every surface Σ, $cr_\Sigma(G) \geq (1/(\Delta(M)-2)^2) cr_\Sigma(M)$.*

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