A note on harmonic subgraphs in labelled geometric graphs

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Abstract

Let S be a set of n points in general position in the plane, labelled bijectively with the integers $\{0, 1, \ldots, n-1\}$. Each edge (the straight segment that joins two points) is labelled with the sum of the labels of its endpoints. In this note we investigate the maximum size of noncrossing matchings and paths on S, under the requirement that no two edges have the same weight.

1 Introduction

The study of geometric graphs on point sets in the plane has received a considerable amount of attention lately [6, 8]. In particular, matching problems on bicolored point sets have been studied. A folklore result asserts that any point set with $2n$ points, n blue and n red, always admits a *geometric* (that is, no two edges cross) perfect matching, where the edges are straight line segments joining points with different colors. A similar problem attributed to Aharoni and Saks (see [4] and [5]) is the following. Consider a set S of $n = w + b$ points in the plane in general position, where w of them are white and b of them are black. The question is if there is a "near perfect" geometric matching, but in this case we require the edges to be straight line segments joining points with *the same color*. Dumitrescu and Steiger [5] proved that at least 83.33% of the points can always be matched. They also showed that there is a configuration of colored points in which at most 99.36% of the points can be matched.

A classical open problem in Graph Theory is to decide if every tree T is *graceful*, that is, if the n vertices of T can be injectively labeled with the integers $0, 1, \ldots, n-1$, in such a way that the weights of all the edges of T are different, where the weight of an edge is the absolute value of the difference of its endpoints.

In this paper we introduce the study of some geometric problems inspired by graceful trees. Let S be a set of points in convex position, and consider any labeling of the elements of S with the integers

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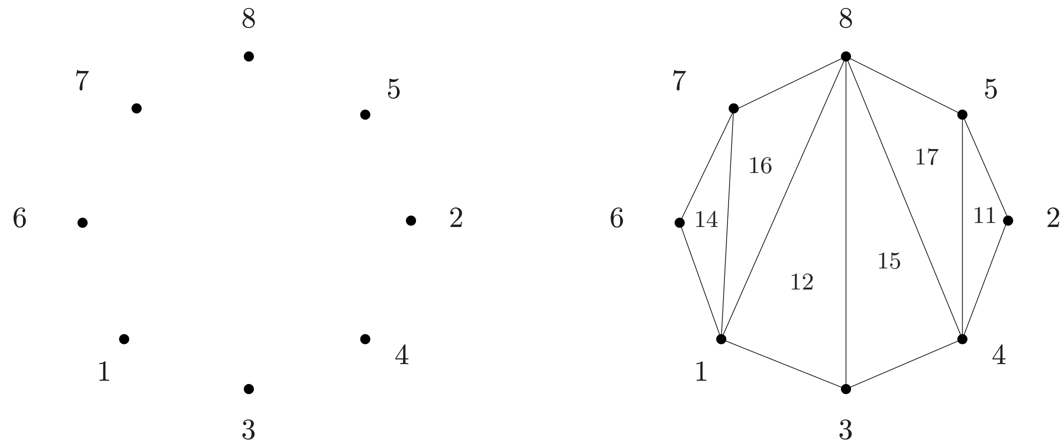

Figure 1: Given the labeled set of 8 points in convex position on the left hand side, on the right hand side we give a *graceful* triangulation: the weight of each triangle (that is, the sum of the labels of its vertices) is different to the weights of all the other triangles.

$0, 1, \ldots, n-1$. Our aim is to find plane geometric graphs whose vertex set is S, e.g. matchings, paths, trees, in such a way that the weights of the edges of our graph are all different, where the weight of each edge is defined in terms of the labels of its endpoints.

Our starting point was the following question raised by Urrutia at the Indonesia–Japan Joint Conference on Combinatorial Geometry and Graph Theory in 2003: does there exist a triangulation of S such that the weights of all the triangles of S are different, where the weight of a triangle is the sum of the labels of its vertices? (see Fig. 1).

In this paper, we let the *weight* of an edge be the sum of the labels of its endpoints. A graph with all the edges having different weights is *harmonic* (and it is $\pmod m$–*harmonic*) if all the edges have different weights modulo m.

We consider the following problem.

Question 1 *What is the maximum size of a noncrossing harmonic matching (or path) which is always guaranteed in a configuration of n labelled points in general position in the plane?*

We start by noting that the interest on sparse harmonic substructures must naturally focus on paths or matchings, as opposed to trees, since any spanning star is harmonic.

As it is often the case in combinatorial geometry, in this work we shall focus in the case in which the point set is in convex position.

A remarkable result of Balogh, Pittel and Salazar [3] almost answers the question for the case in which the points are randomly labelled.

Theorem 2 ([3]) *Let S be an n–point set in convex position, where each point is independently labelled at random with an integer in $\{0, 1, \ldots, n-1\}$. Then with high probability there is a noncrossing $\pmod n$–harmonic matching covering at least $n - (n \log n)^{1/3}$ points. Furthermore, with probability at least $(n \log n)^{-1/3}$ there is a perfect noncrossing $\pmod n$–harmonic matching (or, if n is odd, a matching covering $n-1$ points).*

It was also conjectured in [3] that with high probability there is always a perfect (mod n)-matching (or, if n is odd, a matching covering $n-1$ points). This conjecture was supported by numerous computer simulations.

Theorem 2 and some other heuristic arguments led us to the following conjecture.

Conjecture 3 *In any set of n points in convex position, labelled bijectively with the integers in $\{0, 1, \ldots, n-1\}$, there is a noncrossing path (and consequently a matching) of size $\Theta(n)$.*

A straightforward application of the Erdős–Szekeres theorem yields that there always exists a noncrossing harmonic path (and consequently a matching) of size $\Omega(n^{1/2})$. It is quite surprising that (up to a constant factor) this is the best known lower bound.

Our results handle a variant of this problem: the separable matching case. A matching M (of a point set in convex position) is *separable* if its vertex set can be split into two sets U, U' of equal size, such that (i) the vertices of U (and consequently of U') appear consecutively in the convex hull; and (ii) every edge of M has an endpoint in U and the other endpoint in U'. Separable matchings are equivalently characterized by the property that there is a line that crosses every edge of the matching. This alternative definition has the advantage that it seamlessly extends to paths: a path is *separable* if there is a line that crosses all its edges. Separable matchings appear naturally when searching for large matchings and paths in point sets in convex position; see for instance [1, 7].

We shall focus our attention on separable matchings and paths. Since we are interested in lower bounds, we may as well assume that the point set is divided (by a line) into two sets $\mathcal{P}, \mathcal{Q}$ of equal size; here, we search for $(\mathcal{P}, \mathcal{Q})$*–matchings*, that is, matchings in which every edge has a vertex in $\mathcal{P}$ and another vertex in $\mathcal{Q}$ ($(\mathcal{P}, \mathcal{Q})$–paths are defined similarly: the vertices of such a path alternate between $\mathcal{P}$ and $\mathcal{Q}$). Our aim was to consider the following question.

Question 4 *Let $\mathcal{P}, \mathcal{Q}$ be n–sets of points, such that $\mathcal{P} \cup \mathcal{Q}$ is in convex position. Suppose that the points of $\mathcal{P}$ are labelled $0, 1, 2, \ldots, n-1$ (in the cyclic order in which they appear by the convex hull), and that the labels of $\mathcal{Q}$ form any permutation of $0, 1, 2, \ldots, n-1$. What is the number of edges in a maximum size noncrossing harmonic $(\mathcal{P}, \mathcal{Q})$–path (or matching)?*

Note that in these questions, the labelling map is two–to–one. As we have observed above, from any given path it is trivial to obtain a matching of the same order, so we shall consider only the question for paths. The main result in this note is the following.

Theorem 5 *If n is sufficiently large, then there is a noncrossing harmonic $(\mathcal{P}, \mathcal{Q})$–path with at least $(1/9)n^{2/3}$ points.*

2 Proof of Theorem 5

Throughout the proof, $\mathcal{P} = \{p_0, p_1, \ldots, p_{n-1}\}$ and $\mathcal{Q} = \{q_0, q_1, \ldots, q_{n-1}\}$ are set points, with $\mathcal{P} \cup \mathcal{Q}$ in convex position, and such that the points in $\mathcal{P} \cup \mathcal{Q}$ appear in the convex hull in the order $p_0, p_1, \ldots, p_{n-1}, q_{n-1}, q_{n-2}, \ldots, q_0$. Also, the *labeling map* $\ell : \mathcal{P} \cup \mathcal{Q} \to \{0, 1, \ldots, n-1\}$ satisfies $\ell(p_i) = i$ for every i, and $(\ell(q_0), \ell(q_1), \ldots, \ell(q_{n-1}))$ is a permutation of $(0, 1, \ldots, n-1)$ (see Figure 1).

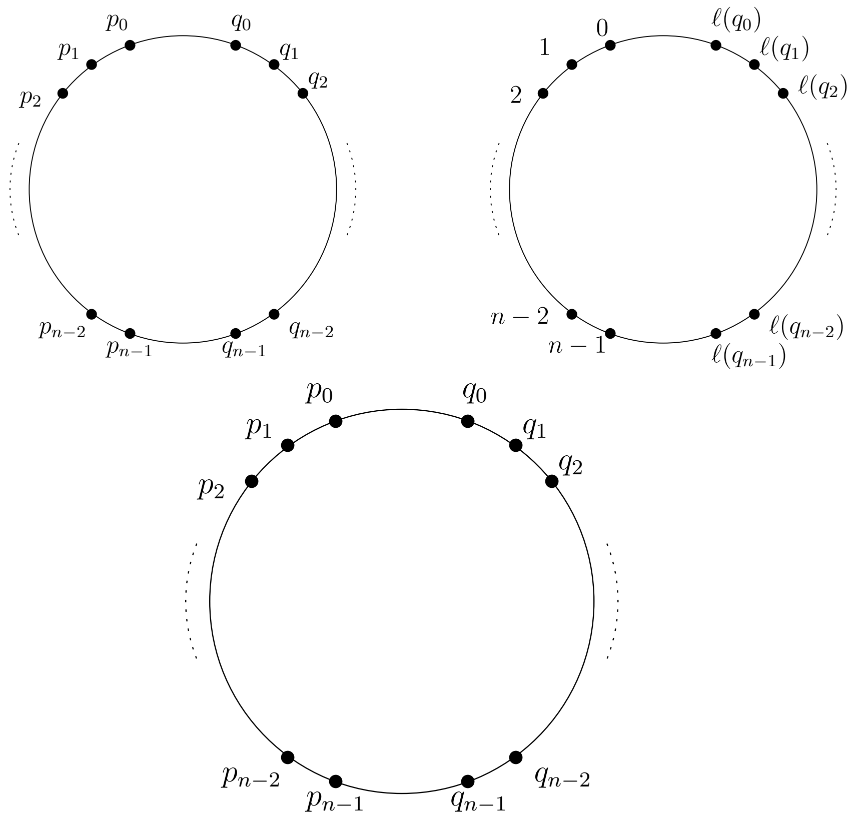

Figure 2: Configuration analyzed in Theorem 5, with $\mathcal{P} = \{p_0, p_1, \ldots, p_{n-1}\}$ and $\mathcal{Q} = \{q_0, q_1, \ldots, q_{n-1}\}$. The assumption on the labels is (see right hand side figure) that each p_i has label $\ell(p_i) = i$, whereas the label $\ell(q_j)$ of each q_j is arbitrary, as long as $\ell(q_0), \ell(q_1), \ldots, \ell(q_{n-1})$ is a permutation of $0, 1, \ldots, n-1$.

In order to avoid a cumbersome analysis involving floors and ceilings, we assume in the proof that $n^{1/3}$ is an integer divisible by 3.

Suppose first that $\ell(q_0), \ell(q_1), \ldots, \ell(q_{n-1})$ has a decreasing subsequence $\ell(q_{j_1}), \ell(q_{j_2}), \ldots, \ell(q_{j_s})$ of size $s = (1/3)n^{2/3}$ (note that the assumption on the divisibility of $n^{1/3}$ by 3 implies that s is indeed an integer). It is readily checked that then the noncrossing path $(q_{j_1}, p_0, q_{j_4}, p_1, q_{j_7}, p_2, q_{j_{10}}, p_3, \ldots, q_{j_{s-2}}, p_{s/3-1})$ is harmonic. Since this path has $2(s/3) = (2/3)n^{2/3}$ vertices, in this case we are clearly done. Thus we may assume that no such decreasing subsequence exists.

The nonexistence of such a decreasing subsequence allows us to establish the existence of a relatively small collection of pairwise disjoint increasing subsequences, whose union is relatively large.

Claim *There exist $t \leq (1/6)n^{1/3}$ pairwise disjoint sequences $S_1, S_2, \ldots, S_t$, each of which is a subsequence of $\ell(q_0), \ell(q_1), \ldots, \ell(q_{n-1})$, such that $|S_1| \geq |S_2| \geq \ldots \geq |S_t|$, and $\sum_{i=1}^{t} |S_i| = (1/9)n^{2/3}$.*

Proof of Claim. First we note that since, by assumption $\ell(q_0), \ell(q_1), \ldots, \ell(q_{n-1})$ does not have a decreasing subsequence of size $(1/3)n^{2/3}$, it follows that $\ell(q_0), \ell(q_1), \ldots, \ell(q_{n-1})$ can be decomposed into $m < (1/3)n^{2/3}$ increasing subsequences $T_1, T_2, \ldots, T_m$, labelled so that $|T_1| \geq |T_2| \geq \ldots \geq |T_m|$.

Note that if $|T_i| \geq (1/9)n^{2/3}$ for some i, then there clearly exists a noncrossing harmonic path of size at least $(2/9)n^{2/3}$ in $\mathcal{P} \cup \mathcal{Q}$. Thus we assume that $|T_i| < (1/9)n^{2/3}$ for every i.

Let t be the smallest integer such that $\sum_{i=1}^{t} |T_i| \geq (1/9)n^{2/3}$. We claim that $|T_t| \geq 2n^{1/3}$. For suppose that $|T_t| < 2n^{1/3}$. Then $n = \sum_{i=1}^{m} |T_i| = \sum_{i=1}^{t} |T_i| + \sum_{i=t+1}^{m} |T_i| < (m-t)(2n^{1/3}) + O(n^{2/3}) < (1/3)n^{2/3}(2n^{1/3}) + O(n^{2/3}) = (2/3)n + O(n^{2/3})$, a contradiction. Thus $|T_t| \geq 2n^{1/3}$, as claimed.

Moreover, we claim $t \leq (1/6)n^{1/3}$. Indeed, if $t > (1/6)n^{1/3}$, then $\sum_{i=1}^{t} |T_i| > (1/6)n^{1/3}(2n^{1/3}) = (1/3)n^{2/3}$. By the choice of t this would imply that $|T_t| > (1/3 - 1/9)n^{2/3} = (2/9)n^{2/3}$, a contradiction.

Finally, let $S_i := T_i$ for $i < t$, and let S_t be any subsequence of T_t such that $\sum_{i=1}^{t} |T_i| = (1/9)n^{2/3}$. Clearly, the subsequences $S_1, S_2, \ldots, S_t$ satisfy the required properties. ∎

Let $\ell(q_{i_0}), \ell(q_{i_1}), \ldots, \ell(q_{i_{(1/9)n^{2/3}-1}})$ be the subsequence of $\ell(q_0), \ell(q_1), \ldots, \ell(q_{n-1})$ defined by the union of the subsequences S_i given by the previous Claim.

We shall define a noncrossing harmonic path P, whose points alternate between $\mathcal{P}$ and $\mathcal{Q}$. Moreover, the points of P belong alternately to $\mathcal{P}$ and to $\{q_{i_0}, q_{i_1}, \ldots, q_{i_{(1/9)n^{2/3}-1}}\}$.

Actually, we recursively construct a sequence of noncrossing harmonic paths $P_0, P_1, P_2, \ldots, P_{(1/9)n^{2/3}-1}$, such that P_i has $2i+2$ vertices, and such that P_{i+1} is an extension of P_i. At the end of the process we simply let $P := P_{(1/9)n^{2/3}-1}$ be the required path.

These paths are defined as follows. Let $j_0 := 0$ and $P_0 := (p_{j_0}, q_{i_0})$. Now let $k > 0$, and suppose that $P_{k-1} = (p_{j_0}, q_{i_0}, p_{j_1}, q_{i_1}, \ldots, q_{i_{k-2}}, p_{j_{k-1}} q_{i_{k-1}})$ has been constructed (we remark that $i_0, i_1, \ldots,$ have been all defined already, in the paragraph after the proof of the Claim; our task in the construction of the paths P_i is to define $j_0, j_1, j_2, \ldots$). Then let j_k be the smallest integer in the closed interval $I_k := [j_{k-1} + n^{1/3} + 1, j_{k-1} + 2n^{1/3}]$ such that the (obviously noncrossing) path $P_k := (p_{j_0}, q_{i_0}, p_{j_1}, q_{i_1}, \ldots, p_{j_{k-1}}, q_{i_{k-1}}, p_{j_k}, q_{i_k})$ is harmonic.

We need to show that each j_k can be chosen as required. For $k = 0$ there is nothing to prove, so assume that $k > 0$, and that every j_m with $m < k$ has been chosen with the required properties. Since by inductive assumption P_{k-1} is harmonic, it suffices to show that there is an integer j_k in I_k such that neither $\ell(q_{i_{k-1}}) + j_k$ nor $j_k + \ell(q_{i_k})$ is a label of an edge in P_{k-1}. The first (essential) observation is that $j_{k-1} + 2n^{1/3}$ is less than n (that is, we never run out of points while searching for j_k). This follows since $j_0 = 0$ and $j_r \leq j_{r-1} + 2n^{1/3}$ for every $r \geq 1$, and so $j_k \leq k(2n^{1/3}) \leq (1/9)n^{2/3}(2n^{1/3}) < n$.

Now the key facts to prove the existence of an integer j_k in I_k such that neither $\ell(q_{i_{k-1}}) + j_k$ nor $j_k + \ell(q_{i_k})$ is a label of an edge in P_{k-1} are the following:

(i) if $q_{i_x}, q_{i_y}, q_{i_z}$ are different vertices in P_{k-1}, and each of them is incident with an edge in P_{k-1} whose label is in $\{\ell(q_{i_{k-1}}) + w \mid w \in I_k\}$ then $\ell(q_{i_x}), \ell(q_{i_y}), \ell(q_{i_z})$ cannot all belong to the same S_i (recall the definition of the subsequences S_i from the Claim above);

(ii) if $q_{i_x}, q_{i_y}, q_{i_z}$ are different vertices in P_{k-1}, and each of them is incident with an edge in P_{k-1} whose label is in $\{w + \ell(q_{i_k}) \mid w \in I_k\}$, then $\ell(q_{i_x}), \ell(q_{i_y}), \ell(q_{i_z})$ cannot all belong to the same S_i.

We prove (i); the proof of (ii) is quite similar. Seeking a contradiction, suppose that $\ell(q_{i_x}), \ell(q_{i_y})$, $\ell(q_{i_z})$ are as in (i), and that they all belong to the same subsequence S_i. Without any loss of generality, $\ell(q_{i_x}) < \ell(q_{i_y}) < \ell(q_{i_z})$, so that $q_{i_x}, q_{i_y}, q_{i_z}$ occur in this order in P_{k-1}. Since q_{i_x} and q_{i_z} are not consecutive in S_i, it follows from the construction of $P_0, P_1, \ldots, P_{k-1}$ that if q_{i_x} and q_{i_z} are adjacent to vertices p_a, p_b in P_{k-1}, respectively, then $b - a > n^{1/3}$. This in turn implies (since $\ell(q_{i_x}) < \ell(q_{i_z})$) that the labels of each edge in P_{k-1} incident with q_{i_x} and of each edge in P_{k-1} incident with q_{i_z} differ by more than $n^{1/3}$. This contradicts the assumption that two such edges have labels in the set $\{\ell(q_{i_{k-1}}) + w \mid w \in I_k\}$, since this last set consists of $n^{1/3}$ consecutive integers.

Since there are at most $(1/6)n^{1/3}$ sequences S_i, (i) implies that at most $2 \cdot (1/6)n^{1/3} = (1/3)n^{1/3}$ of the edge labels in P_{k-1} are contained in $\{\ell(q_{i_{k-1}}) + w \mid w \in I_k\}$. Analogously, (ii) shows that there are at most $(1/3)n^{1/3}$ edge labels in P_{k-1} contained in $\{w + \ell(q_{i_k}) \mid w \in I_k\}$. Since $|I_k| = n^{1/3}$, it follows that there is a $j_k \in I_k$ such that neither $\ell(q_{i_{k-1}}) + j_k$ nor $j_k + \ell(q_{i_k})$ is a label of an edge in P_{k-1}, as required.

Thus the paths $P_0, P_1, \ldots, P_{(1/9)n^{2/3}-1}$ are all well–defined. As we mentioned above, this completes the proof, since $P := P_{(1/9)n^{2/3}-1}$ satisfies the required properties. ∎

3 Concluding remarks and open questions

Most interesting problems remain open. First of all, there is a huge gap between the best lower bound known for the size of a harmonic path (or matching) when the labeling map is a bijection (namely $\Omega(n^{1/2})$), and the value $\Theta(n)$ put forward in Conjecture 3. We strongly believe that the correct bound is $\Theta(n)$, but are still intrigued at the level of sophistication required in the proof of a weaker and restricted version of this statement, namely the one we worked out in Theorem 5.

Using an exhaustive computer search, we have shown that for collections with up to 14 points, if the labeling map is a bijection then a perfect matching always exists. However, for $n = 16, 18, ..., 28$, we were able to find labelling in which no such perfect matching exists. For instance, if $n = 16$, and the point labels are, in cyclic order, $(1, 15, 2, 14, 3, 12, 4, 13, 8, 9, 7, 10, 6, 11, 5,\ 16)$, then there is no harmonic perfect matching. It would be interesting to have a general construction to show that, for every $n \geq 16$, there is a labeling for which no perfect matching exists.

We close with one more conjecture that, not surprisingly, hints at a more than superficial relationship between the existence of large harmonic substructures and a widely studied area, namely sum sets of integers. For any set A of n integers, let $M(A)$ denote the minimum size of a harmonic matching that can be guaranteed if we bijectively label n points in convex position with the elements of A (an analogous function $P(A)$ may be defined for paths, etc.).

Conjecture *For each integer n, the function M is minimized (only) at those sets A that form an arithmetic progression.*

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