An extended lower bound on the number of $(\leq k)$–edges to generalized configurations of points and the pseudolinear crossing number of K_n

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Abstract

Recently, Aichholzer, García, Orden, and Ramos derived a remarkably improved lower bound for the number of $(\leq k)$–edges in an n–point set, and as an immediate corollary an improved lower bound on the rectilinear crossing number of K_n. Our aim in this self–contained note is to show that following the approach of circular sequences results in a more general treatment that extends their main result. We prove that their lower bound holds in the more general setting of $(\leq k)$–pseudoedges in generalized configurations of points. In particular, it follows that the pseudolinear (and consequently the rectilinear) crossing number of K_n is at least $0.37968\binom{n}{4} + O(n^3)$.

1 Introduction

Two important, closely related open problems in discrete geometry are the determination of the minimum number of $(\leq k)$–edges in a set S of points in general position in the plane, and the determination of the rectilinear crossing number of K_n. The close relationship between these problems was independently unveiled by Ábrego and Fernńandez Merchant [1], and by Lovász, Vesztergombi, Wagner, and Welzl [11]. For a concise, up–to–date overview on these and related problems see the recent monograph [9].

A j–edge, for $0 \leq j \leq \lfloor \frac{n-2}{2} \rfloor$, is a segment spanned by two points $p, q \in S$ such that (exactly) j points of S lie in the open halfspace defined by the line that goes through p and q. A $(\leq k)$–edge is a j–edge for $0 \leq j \leq k$.

The rectilinear crossing number $\overline{\text{cr}}(G)$ of a graph G is the minimum number of crossings in a *rectilinear* drawing of G, that is, a drawing in which each edge is a straight segment. In a more general vein, a drawing of G is *pseudolinear* if each edge can be extended to a pseudoline in such a way that the resulting set is an arrangement of pseudolines. The *pseudolinear* crossing number $\widetilde{\text{cr}}(G)$ of G is the minimum number of edge crossings in a pseudolinear drawing of G.

Trivially, for every graph G, $\widetilde{\text{cr}}(G) \leq \overline{\text{cr}}(G)$. Thus, any lower bound for $\widetilde{\text{cr}}(G)$ is also a lower bound for $\overline{\text{cr}}(G)$.

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A recent major breakthrough from Aichholzer et al. [5] is a remarkable improvement on the minimum number of $(\le k)$–edges in an n–point set. As an immediate corollary, they obtain an improved lower bound for $\overline{\text{cr}}(K_n)$.

Our aim in this self–contained note is to show that their lower bound on the number of $(\le k)$–edges also holds in the more general setting of $(\le k)$–pseudoedges in generalized configurations of points.

2 Circular sequences, $(\le k)$–pseudoedges, and $\widetilde{\text{cr}}(K_n)$

Recall that a *circular sequence* $\mathbf{\Pi}$ is a doubly–infinite sequence $(\ldots, \pi_{-1}, \pi_0, \pi_1, \ldots)$ of permutations on n elements, such that any two consecutive permutations differ by a transposition of neighboring elements, and such that for every i, π_i is the reverse permutation of $\pi_{i+\binom{n}{2}}$. Thus $\mathbf{\Pi}$ has period $2\binom{n}{2}$, and the whole information of $\mathbf{\Pi}$ is contained in any of its halfperiods (which we call *n–halfperiods*).

Let $\Pi = \{\pi_0, \pi_1, \ldots, \pi_{\binom{n}{2}}\}$ be an n–halfperiod of a circular sequence $\mathbf{\Pi}$. Thus, for each $i \ge 1$, π_{i-1} differs from π_i by a transposition of adjacent elements, whose *initial* and *final* permutations are π_{i-1} and π_i, respectively. A transposition of Π (or of any subsequence of a halfperiod of $\mathbf{\Pi}$) is a *k–transposition* if it swaps elements in positions k and $k+1$, and a *$(\le k)$–transposition* if it is an i–transposition for either some $i \le k$ or some $i \ge n-k$. Let $\mathcal{N}_k(\Pi)$ (respectively $\mathcal{N}_{\le k}(\Pi)$) denote the set of all k–transpositions (respectively, $(\le k)$–transpositions) in Π. Let $N_k(\Pi) := |\mathcal{N}_k(\Pi)|$, and $N_{\le k}(\Pi) := |\mathcal{N}_{\le k}(\Pi)|$.

In a generalized configuration S of n points, a *j–pseudoedge*, for $0 \le j \le \lfloor \frac{n-2}{2} \rfloor$, is a pseudosegment spanned by two points p, q, such that the pseudoline that goes through p and q divides $S \setminus \{p, q\}$ into two sets, one of size j and one (obviously) of size $n-j-2$. As with $(\le k)$–edges, a *$(\le k)$–pseudoedge* is a k–pseudoedge with $j \le k$. We let $\chi_{\le k}(S)$ denote the number of $(\le k)$–pseudoedges in S.

As expected, in the particular case in which S is a set of n points in the plane in general position, its j–edges agree with the j–pseudoedges one obtains if S is regarded as a generalized configuration (with each pseudoline being a straight line).

Goodman and Pollack [10] established a one–to–one correspondence between circular sequences and generalized configurations of points. Under this setting, each $(\le k)$–pseudoedge corresponds to a $(\le k+1)$–transposition.

Ábrego and Fernández–Merchant [1] (although not explicitly stated there, this also follows from [11]) used this correspondence to derive an expression for the number of crossings in a pseudolinear drawing in terms of its corresponding circular sequence.

In [1], it is proved that if S is a generalized configuration of n points, and $\mathcal{D}$ is the (pseudolinear) drawing of K_n induced by S, then the number $\widetilde{\text{cr}}(\mathcal{D})$ of crossings in $\mathcal{D}$ satisfies $\widetilde{\text{cr}}(\mathcal{D}) = \sum_{1 \le k \le (n-2)/2} (n-2k-3)\chi_{\le k-1}(S) + O(n^3)$. If Π is (any) halfperiod of the circular sequence $\mathbf{\Pi}$ defined by S, then the correspondence between $(\le k)$–pseudoedges and $(\le k+1)$–transpositions implies that $\widetilde{\text{cr}}(\mathcal{D}) = \sum_{1 \le k \le (n-2)/2} (n-2k-3) N_{\le k}(\Pi) + O(n^3)$. As $\widetilde{\text{cr}}(K_n)$ is the minimum of $\widetilde{\text{cr}}(\mathcal{D})$ over all pseudolinear drawings $\mathcal{D}$ of K_n, it follows that

$$\widetilde{\text{cr}}(K_n) \ge \sum_{1 \le k \le (n-2)/2} (n-2k-3) \cdot \left(\min_{\Pi} \{N_{\le k}(\Pi)\} \right) + O(n^3), \qquad (1)$$

where the minimum is taken over all n–halfperiods Π.

3 The main result

Our main result is that the bound obtained in [5] for $(\leq k)$–edges still holds in the more general context of generalized configurations of points.

Theorem 1 *For any generalized configuration S of n points, the number $\chi_{\leq k}(S)$ of $(\leq k)$–pseudoedges of S satisfies*

$$\chi_{\leq k}(S) \geq 3\binom{k+2}{2} + 3\binom{k - \lfloor n/3 \rfloor + 2}{2} - (k - \lfloor n/3 \rfloor + 1)(n \bmod 3).$$

We remark that a routine manipulation shows that the lower bound in Theorem 1 equals the lower bound $3\binom{k+2}{2} + \sum_{j=\lfloor n/3 \rfloor}^{k}(3j - n + 3)$ in [5].

The correspondence between generalized configurations and circular sequences (see Section 2) implies that Theorem 1 is equivalent to the following.

Theorem 2 (Equivalent to Theorem 1) *For any n–halfperiod Π, the number $N_{\leq k}(\Pi)$ of $(\leq k)$–transpositions of Π satisfies*

$$N_{\leq k}(\Pi) \geq 3\binom{k+1}{2} + 3\binom{k - \lfloor n/3 \rfloor + 1}{2} - (k - \lfloor n/3 \rfloor)(n \bmod 3).$$

Using (1) and Theorem 2, a routine calculation gives $\widetilde{\mathrm{cr}}(K_n) \geq (41/108)\binom{n}{4} > 0.37962\binom{n}{4} + O(n^3)$. If for $k > 0.4864n$ we use the lower bound for $N_{\leq k}(\Pi)$ derived in [8], the coefficient 0.37962 gets slightly improved as follows.

Corollary 3 $\widetilde{\mathrm{cr}}(K_n) > 0.379688\binom{n}{4} + O(n^3)$.

Using different techniques, a similar bound was proved in [3] for pseudolinear drawings of K_n that satisfy certain additional conditions.

4 Proof of Theorem 2

If $\pi_i = (\pi_i(1), \ldots, \pi_i(s), \ldots, \pi_i(t), \ldots, \pi_i(n))$, and $1 \leq a < b \leq n$, then we let $\pi_i[a,b]$ denote the subpermutation $(\pi_i(a), \ldots, \pi_i(b))$, and π_i^{-1} is the permutation $(\pi_i(n), \pi_i(n-1) \ldots, \pi_i(1))$.

We extend the definition of $\mathcal{N}_k(\Pi)$ to any subsequence Π' of Π: $\mathcal{N}_k(\Pi')$ is the set of k–transpositions of Π *whose final permutation is in* Π'. Clearly there is no conflict with this definition if we regard Π as a subsequence of itself. Moreover, if Π is partitioned into $\Pi_0, \Pi_1, \ldots, \Pi_r$, so that Π is the concatenation $\Pi_0\Pi_1 \ldots \Pi_r$, then $\mathcal{N}_k(\Pi)$ equals the disjoint union $\bigcup_i \mathcal{N}_k(\Pi_i)$.

An *extreme point* of Π is one that occupies positions 1 or n in some π_i.

If Π and $\overline{\Pi}$ are n–halfperiods such that $N_{\leq k}(\overline{\Pi}) \leq N_{\leq k}(\Pi)$ for every $k = 1, \ldots, \lfloor n/2 \rfloor$, then we write $\overline{\Pi} \preceq \Pi$.

In order to give a self–contained proof of Theorem 2, we include a proof of the following result, which was proved in [7].

Lemma 4 *Any halfperiod minimal with respect to $\preceq$ has exactly 3 extreme points.*

Proof. Let $\Pi = (\pi_0, \pi_1, \ldots, \pi_{\binom{n}{2}})$ be an n–halfperiod of a circular sequence $\mathbf{\Pi}$, minimal with respect to $\preceq$, where $\pi_i = (\pi_i(1), \pi_i(2), \ldots, \pi_i(n))$. We note it suffices to show that $\pi_0(1)$ and $\pi_0(n)$ swap either when $\pi_0(1)$ is in position 1 or when $\pi_0(n)$ is in position n.

We may assume without any loss of generality (otherwise work instead with the n–halfperiod $(\pi_0^{-1}, \pi_1^{-1}, \ldots, \pi_{\binom{n}{2}}^{-1}))$ that in Π the element $\pi_0(1)$ reaches position $\lceil n/2 \rceil$ (say in permutation π_ℓ) before $\pi_0(n)$ reaches position $\lfloor n/2 \rfloor + 1$. We claim that $\pi_0(1)$ swaps with the elements $\pi_\ell(1), \pi_\ell(2), \ldots, \pi_\ell(\lceil n/2 \rceil - 1)$ in the given order. For suppose this is not the case. Let x, y be the first pair that swaps after $\pi_0(1)$ has swapped (in this order) with both x and y. Note that Π may be modified, if necessary, without losing its $\preceq$–minimality, so that the swap between $\pi_0(1)$ and x is put on hold until y is a neighbor of x. So we may assume that, in Π, just before $\pi_0(1)$ swapped with either x or y, x and y were neighbors. If we had swapped x and y back then, and kept Π otherwise unchanged, the result would be an n–halfperiod strictly $\preceq$–smaller than Π, a contradiction.

By the same argument, $\pi_0(1)$ must swap with the elements in $\pi_\ell(\lceil n/2 \rceil + 1), \pi_\ell(\lceil n/2 \rceil + 2), \ldots, \pi_\ell(n)$ in the given order. Thus, if $\pi_0(n) = \pi_\ell(n)$, we are done. Otherwise, $\pi_0(n) = \pi_\ell(i)$ for some i, $\lceil n/2 \rceil + 1 < i < n$. The argument in the previous paragraph shows that $\pi_0(n)$ must swap with the elements in $\pi_\ell(n), \pi_\ell(n-1), \ldots, \pi_\ell(i-1)$ in the given order. If instead of allowing $\pi_0(n)$ to move, we leave it in position n, so that it swaps there with $\pi_0(1)$, and then let it swap with all the elements in $\pi_\ell(n), \pi_\ell(n-1), \ldots, \pi_\ell(i-1)$, the result is an n–halfperiod strictly $\preceq$–smaller than Π, a contradiction. ∎

If $\Pi = (\pi_0, \ldots, \pi_{\binom{n}{2}})$ is an n–halfperiod, and s, t are nonnegative integers such that $s \leq t \leq \binom{n}{2}$, then we let $\Pi[s, t]$ denote the subsequence $(\pi_s, \ldots, \pi_t)$.

We now prove our version of the main ingredient for the improved bound in [5].

Proposition 5 *Let Π be an n–halfperiod. Let s, t be integers, $0 \leq s \leq t \leq \binom{n}{2}$, and $k < n/2$. Then $N_k(\Pi[0, s]]) + N_{n-k}(\Pi[s+1, t]) + N_k(\Pi[t+1, \binom{n}{2}]]) \geq 3k - n$.*

Proof. Let $U := \pi_0[1, k] \cap \pi_s[k+1, n]$, and $V := \pi_0[n-k+1, n] \cap \pi_s[n-k+1, n]$. It is straightforward to see that $N_k(\Pi[0, s]) \geq |U|$, $N_{n-k}(\Pi[s+1, t]) + N_k(\Pi[t+1, \binom{n}{2}]) \geq |V|$, and $|V| \geq k - ((n-2k) + |U|)$. The claimed inequality follows. ∎

Proof of Theorem 2. We proceed by induction on n. The base case $n = 3$ is trivial. We may assume $\Pi = (\pi_0, \ldots, \pi_{\binom{n}{2}})$ is $\preceq$–minimal. By Lemma 4, Π has exactly 3 extreme points, say p, q, and r. Let $m := \binom{n}{2}$. By considering, if necessary, another n–halfperiod of the doubly–infinite sequence generated by Π, w.l.o.g q moves from position k to position $k-1$ from π_{m-1} to π_m, while r, p are at positions 1 and n, respectively. That is, $\pi_{m-1}(1) = \pi_m(1) = r, \pi_{m-1}(n) = \pi_m(n) = p$, and $\pi_{m-1}(k) = \pi_m(k-1) = q$. Since $\pi_m = (\pi_0)^{-1}$, then $\pi_0(1) = p, \pi_0(n) = r$, and $\pi_0(n-k+2) = q$. Thus the swaps between p, q, and r occur as follows: first q and r (at positions $n-1$ and n), then p and r (at positions 1 and 2), and finally p and q (at positions $n-1$ and n).

Let π_s be the permutation in which r first enters position $k-1$ (that is, $\pi_{s-1}(k) = \pi_s(k-1) = r$), and let π_t be the permutation in which p first enters position $n-k+2$ (that is, $\pi_{t-1}(n-k+1) = \pi_t(n-k+2) = p$). Clearly, $s < t$. Note that $\pi_s(1) = p, \pi_s(n) = q$, $\pi_t(1) = r$, and $\pi_t(n) = q$.

A transposition in Π that involves p, q, or r is a (p, q, r)–*transposition.*

Let $\Lambda := \lambda_0, \lambda_1, \ldots, \lambda_{\binom{n-3}{2}}$ be the $(n-3)$–halfperiod that results from removing from Π the $3n-6$ permutations that result from a (p, q, r)–transposition, and then removing

p, q, and r from each of the resulting $\binom{n-3}{2}$ permutations. Let I denote the natural injection from Λ to Π (thus, for instance, $I(\lambda_0) = \pi_0$), and define $\iota : \{0, \ldots, \binom{n}{2}\} \to \{0, \ldots, \binom{n-3}{2}\}$ by the condition $\iota(i) = j$ iff $I(\lambda_i) = \pi_j$. Let s' be the largest i such that $\iota(i) < s$, and let t' be the largest j such that $\iota(j) < t$.

It is straightforward to check that if λ_i is a final permutation of a transposition in $\mathcal{N}_{\leq k-2}(\Lambda) \cup \mathcal{N}_{k-1}(\Lambda[0, s']) \cup \mathcal{N}_{n-k+2}\ (\Lambda[s'+1, t']) \cup \mathcal{N}_{k-1}(\Lambda[t'+1, \binom{n-3}{2}])$, then $I(\lambda_i)$ is a non–(p, q, r)–transposition in $\mathcal{N}_{\leq k}(\Pi)$. Clearly there are exactly $6k - 3$ (p, q, r)–transpositions in $\mathcal{N}_{\leq k}(\Pi)$, and so $N_{\leq k}(\Pi) \geq N_{\leq k-2}(\Lambda) + N_{k-1}(\Lambda[0, s']) + N_{n-k+2}(\Lambda[s'+1, t']) + N_{k-1}(\Lambda[t'+1, \binom{n-3}{2}]) + (6k - 3)$.

By the induction hypothesis it follows that $N_{\leq k-2}(\Lambda) \geq 3\binom{k-1}{2} + 3\binom{k-\lfloor n/3 \rfloor -1}{2} - (k - \lfloor n/3 \rfloor - 2)(n \bmod 3)$, and by Proposition 5, $N_{k-1}(\Lambda[0, s']) + N_{n-k+2}(\Lambda[s'+1, t']) + N_{k-1}(\Lambda[t'+1, \binom{n-3}{2}]) \geq 3(k-1) - (n-3) = 3k - n$. Thus $N_{\leq k}(\Pi) \geq 3\binom{k-1}{2} + 3\binom{k-\lfloor n/3 \rfloor -1}{2} - (k - \lfloor n/3 \rfloor - 2)(n \bmod 3) + \max\{3k-n, 0\} + 6k - 3 = 3\binom{k+1}{2} + 3\binom{k-\lfloor n/3 \rfloor +1}{2} - (k - \lfloor n/3 \rfloor)(n \bmod 3)$. ∎

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