Drawings of $C_m \times C_n$ with one disjoint family II [†]

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Abstract. A long–standing conjecture states that the crossing number of the Cartesian product of cycles $C_m \times C_n$ is $(m-2)n$, for every m, n satisfying $n \geq m \geq 3$. A crossing is *proper* if it occurs between edges in different principal cycles. In this paper drawings of $C_m \times C_n$ with the principal n–cycles pairwise disjoint or the principal m–cycles pairwise disjoint are analyzed, and it is proved that every such drawing has at least $(m-2)n$ proper crossings. As an application of this result, we prove that the crossing number of $C_m \times C_n$ is at least $(m-2)n/2$, for all integers m, n such that $n \geq m \geq 4$. This is the best general lower bound known for the crossing number of $C_m \times C_n$.

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1. Introduction

The *crossing number* $cr(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane. A long–standing conjecture is that the crossing number of the Cartesian product $C_m \times C_n$ is at least $(m-2)n$, if $n \geq m \geq 3$ [5]. This has been proved only for $m \leq 6$ [8, 4, 3, 7, 6, 1, 9], and for the special case $m = n = 7$ [2].

It is not difficult to exhibit drawings of $C_m \times C_n$ with exactly $(m-2)n$ crossings, which is thus an upper bound. General lower bounds are a lot more difficult to compute.

The *Cartesian product* $C_m \times C_n$ is a 4–regular graph with mn vertices $v_{i,j}$, where $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$. The vertices are labeled in such a way that the four vertices adjacent to $v_{i,j}$ are $v_{i-1,j}, v_{i+1,j}, v_{i,j-1}$, and $v_{i,j+1}$, where indices i and j are read modulo m and n, respectively.

The edge set of $C_m \times C_n$ is naturally partitioned into m edge sets of *principal blue* n–cycles and n edge sets of *principal red* m–cycles. We label the blue cycles $(v_{i,j}), 0 \leq j \leq n-1$, by B_i, $0 \leq i \leq m-1$, and the red cycles $(v_{i,j})$, $0 \leq i \leq m-1$, by R_j, $0 \leq j \leq n-1$.

We remark that if $m \neq n$, then the edges in $C_m \times C_n$ and $C_n \times C_m$ are colored in a different way.

A crossing in a drawing of $C_m \times C_n$ is *proper* if no principal cycle contains both edges involved in the crossing. In Section 2 we give a short proof of the following statement.

Theorem 1. *Let m, n be such that $n \geq m \geq 3$. Then every drawing of $C_m \times C_n$ such that either the principal m–cycles are pairwise disjoint or the principal n–cycles are pairwise disjoint has at least $(m-2)n$ proper crossings.*

In [10], the following weaker version of Theorem 1 was proved: Every drawing of $C_m \times C_n$ with either the principal m–cycles pairwise disjoint or the principal n–cycles pairwise disjoint has at least $(m-2)n$ crossings. By Theorem 1, these $(m-2)n$ crossings can be chosen to be proper. This refinement is crucial in the proof of the following result.

Theorem 2. *If m, n are positive integers such that $n \geq 4$, then $cr(C_m \times C_n) \geq (m-2)n/2$.*

This clearly implies that if $n \geq m \geq 4$, then $cr(C_m \times C_n)$ is at least half its conjectured value. Moreover, since $C_m \times C_n$ and $C_n \times C_m$ are isomorphic, this shows that if $n > m \geq 4$, then $cr(C_m \times C_n)$ is strictly greater than half its conjectured value.

2. Proof of Theorem 1

In a drawing of $C_m \times C_n$, a cycle C *separates* the cycles C' and C'' if C' and C'' are contained in different components of $\mathbb{R}^2 \setminus C$.

Lemma 3. *Suppose that* $\min\{p, q\} \geq 3$. *Let $\mathcal{D}$ be a drawing of $C_p \times C_q$, such that the q p–cycles are pairwise disjoint and no p–cycle separates two p–cycles. Then $\mathcal{D}$ has at least $(p-2)q$ proper crossings.*

Proof. This proof is divided in 3 steps.
Step 1. *Definition of force.* The red p–cycles are naturally cyclically ordered $R_0, R_1, \ldots, R_{q-1}$, so that each of the blue q–cycles has an edge that joins a vertex of R_{k-1} with a vertex of R_k, for $k = 0, 1, \ldots, q-1$, where the indices are read modulo q.

Let H_k denote the subgraph of $C_p \times C_q$ induced by the vertices in $R_{k-1} \cup R_k$, for each $k = 0, 1, \ldots, q-1$. Thus, H_k has $2p$ red edges and p blue edges, every red edge is in two of the H_k, and every blue edge is in exactly one of the H_k.

The *force* $f(H_k)$ of H_k is the total number of proper crossings of the following types:

(1) a crossing of a blue edge in H_k with an edge in $R_k \cup R_{k+1}$;
(2) a crossing of a blue edge in H_{k+1} with an edge in H_k; and
(3) a crossing of a blue edge in H_k with a blue edge in H_k.

Remark. If $q = 3$, then a crossing of a blue edge in H_k with an edge of R_{k+1} is counted in $f(H_k)$ as type (1) and in $f(H_{k-1})$ as type (2). In this case we note that, since every blue edge of H_k can cross the cycle R_{k+1} only an even number of times, we can let one such crossing contribute to $f(H_k)$ and another such crossing contribute to $f(H_{k-1})$.

It is not difficult to check that no crossing counted in $f(H_k)$ is counted in $f(H_j)$ if $j \neq k$. Therefore the number $cr_p(\mathcal{D})$ of proper crossings in $\mathcal{D}$ is at least $\sum_{0 \leq k \leq q-1} f(H_k)$.

Therefore, in order to complete the proof of Lemma 3, it suffices to show that, for each $k = 0, 1, \ldots, q-1$, $f(H_k) \geq p-2$. By the symmetry it suffices to show that $f(H_0) \geq p-2$.

Step 2. *Counting the contributions to $f(H_0)$ of crossings of types (1) and (3).* Let $\mathcal{B}$ denote the collection of blue edges in H_0. Let $\mathcal{B}_{R_0 \cup R_1}$ consist of the edges in $\mathcal{B}$ that intersect $R_0 \cup R_1$. Let $\mathcal{B}_{\mathcal{M}}$ be a maximal collection of pairwise disjoint edges in $\mathcal{B} \setminus \mathcal{B}_{R_0 \cup R_1}$.

Clearly, the crossings of edges in $\mathcal{B}_{R_0 \cup R_1}$ contribute to $f(H_0)$ in at least $|\mathcal{B}_{R_0 \cup R_1}|$. Since each edge in $\mathcal{B} \setminus (\mathcal{B}_{R_0 \cup R_1} \cup \mathcal{B}_{\mathcal{M}})$ crosses at least one edge in $\mathcal{B}_{\mathcal{M}}$, it follows that the edges in $\mathcal{B} \setminus (\mathcal{B}_{R_0 \cup R_1} \cup \mathcal{B}_{\mathcal{M}})$ contribute to $f(H_0)$ in at least $p - (|\mathcal{B}_{R_0 \cup R_1}| + |\mathcal{B}_{\mathcal{M}}|)$. Therefore, there are at least $|\mathcal{B}_{R_0 \cup R_1}| + p - (|\mathcal{B}_{R_0 \cup R_1}| + |\mathcal{B}_{\mathcal{M}}|) = p - |\mathcal{B}_{\mathcal{M}}|$ crossings of types (1) and (3). Thus, to finish the proof it suffices to show that there are at least $|\mathcal{B}_{\mathcal{M}}| - 2$ crossings of type (2).

Step 3. *There are at least $|\mathcal{B}_{\mathcal{M}}| - 2$ crossings of type (2).* Let H_0' denote the subgraph of H_0 that consists of R_{q-1}, R_0, and the edges in $\mathcal{B}_{\mathcal{M}}$. Since R_1 is disjoint from H_0' and is not separated by R_{q-1} or R_0 from the other, there is a region F of H_0' that contains R_1. The blue edges in H_0' are pairwise disjoint, and so at least $|\mathcal{B}_{\mathcal{M}}| - 2$ vertices on R_0 are not incident with F; therefore each of the at least $|\mathcal{B}_{\mathcal{M}}| - 2$ edges that join these vertices to R_1 crosses the boundary of F. We note that each of these $|\mathcal{B}_{\mathcal{M}}| - 2$ crossings is of type (2), as required. ∎

Proof of Theorem 1. A drawing $\mathcal{D}$ of $C_m \times C_n$ is *red–disjoint* (respectively *blue–disjoint*) if the red (respectively blue) cycles are pairwise disjoint in $\mathcal{D}$. The minimum number of proper crossings in a red–disjoint (respectively blue–disjoint) drawing of $C_m \times C_n$ is denoted by $cr_{p,rd}(C_m \times C_n)$ (respectively $cr_{p,bd}(C_m \times C_n)$). Using this notation, Theorem 1 states that if $n \geq m \geq 3$, then $cr_{p,rd}(C_m \times C_n) \geq (m-2)n$ and $cr_{p,bd}(C_m \times C_n) \geq (m-2)n$.

We show that $cr_{p,bd}(C_m \times C_n) \geq (m-2)n$ by induction on $m+n$. The base case $m+n = 6$ (that is, $m = n = 3$) follows since every drawing of $C_3 \times C_3$ has at least 3 proper crossings (see [8]). Let m, n be such that $n \geq m \geq 3$, where m and n are not both equal to three, and suppose that $cr_{p,bd}(C_{m'} \times C_{n'}) \geq (m'-2)n'$ for all m', n' satisfying $n' \geq m' \geq 3$ and $m'+n' < m+n$. Let $\mathcal{D}$ be a bd–drawing of $C_m \times C_n$. We now show that the number $cr_p(\mathcal{D})$ of proper crossings in $\mathcal{D}$ is at least $(m-2)n$.

Suppose that no blue cycle separates two blue cycles in $\mathcal{D}$. Then we can apply Lemma 3 with $p = n$ and $q = m$ to obtain that $cr_p(\mathcal{D}) \geq (n-2)m \geq (m-2)n$, as required. Thus we assume that a blue cycle B separates two blue cycles in $\mathcal{D}$. In this case B crosses every red cycle, and so B has at least n proper crossings. If $m = 3$, then $cr_p(\mathcal{D}) \geq n = (m-2)n$, as required, so suppose that $m \geq 4$. Let $\mathcal{D}'$ be the drawing of $C_{m-1} \times C_n$ that results by deleting B from $\mathcal{D}$. By the induction hypothesis, $cr_p(\mathcal{D}') \geq ((m-1)-2)n$, and since $cr_p(\mathcal{D}) \geq cr_p(\mathcal{D}') + n$, it follows that $cr_p(\mathcal{D}) \geq (m-2)n$, as required.

The inequality $cr_{p,rd}(C_m \times C_n) \geq (m-2)n$ is proved analogously. ∎

3. Proof of Theorem 2

A drawing is *good* if every intersection of edges is a crossing rather than tangential, no edge crosses itself, no two adjacent edges cross, and no two edges cross more than once. It is a routine exercise to show that if $\mathcal{D}$ is a drawing of G with the minimum number of crossings, then $\mathcal{D}$ is good.

Proof of Theorem 2. Let $\mathcal{D}$ be a good drawing of $C_m \times C_n$. Denote by $P_{i,j}$ the 2–path joining $v_{i,j-1}$ to $v_{i,j+1}$. A 2–path $P_{i,j}$ is *bad* if the rotation scheme around its middle vertex is red–blue–red–blue. Let $\mathcal{W}_j = \{P_{i,j}\}_{i=0}^{m-1}$, $0 \leq j \leq n-1$, and let $\mathcal{P}_j$ be a maximal subset of pairwise disjoint 2–paths in $\mathcal{W}_j$. Let $\mathcal{B}_j$ be the set of those 2–paths in $\mathcal{P}_j$ which are bad, and let $\mathcal{R}_j$ be the set of those 2–paths in $\mathcal{P}_j \setminus \mathcal{B}_j$ which are crossed by an edge in $R_{j-1} \cup R_j \cup R_{j+1}$.

Let $\mathcal{C}_j = \mathcal{P}_j \setminus (\mathcal{B}_j \cup \mathcal{R}_j)$. We claim that there are at least $|\mathcal{C}_j| - 2$ proper red–red crossings that involve only edges in $R_{j-1} \cup R_j \cup R_{j+1}$. Since the 2–paths in $\mathcal{C}_j$ are pairwise disjoint, and no 2–path in $\mathcal{C}_j$ is bad or crosses an edge in $R_{j-1} \cup R_j \cup R_{j+1}$, the ends of each 2–path $P_{i,j}$ in $\mathcal{C}_j$ can be joined by a green edge, drawn very close to $P_{i,j}$, so that the resulting green–blue 3–cycles are pairwise disjoint, and such that no edge in $R_{j-1} \cup R_j \cup R_{j+1}$ crosses an edge in a 3–cycle. Thus, the 3–cycles form, together with R_{j-1}, R_j, and R_{j+1}, a drawing of $C_3 \times C_{|\mathcal{C}_j|}$ where no edge in the 3–cycles is crossed. Since by Theorem 1 such a drawing must have at least $|\mathcal{C}_j| - 2$ proper crossings, the claim follows.

A proper crossing is *associated* to $\mathcal{W}_j$ if either (i) each edge involved in the crossing belongs to $P_{i,j}$ for some i or to R_{j-1}, R_j, or R_{j+1}; or (ii) it occurs between a blue edge not in $\mathcal{W}_j$ and an edge in R_j. Each 2–path in $\mathcal{R}_j$ contains an edge involved in a red–blue crossing of type (i) associated to $\mathcal{W}_j$, and, by Jordan's Curve Theorem, for each 2–path in $\mathcal{B}_j$ there is a crossing of types (i) or (ii) associated to $\mathcal{W}_j$. These observations show that there are at least $|\mathcal{R}_j| + |\mathcal{B}_j|$ red–blue crossings associated to $\mathcal{W}_j$. Since there are at least $m - |\mathcal{P}_j|$ blue–blue and at least $|\mathcal{C}_j| - 2 = (|\mathcal{P}_j| - (|\mathcal{B}_j| + |\mathcal{R}_j|)) - 2$ red–red proper crossings associated to $\mathcal{W}_j$, it follows that there are at least $m-2$ crossings associated to $\mathcal{W}_j$. A moment's thought shows that, since $n \geq 4$, no crossing is associated to $\mathcal{W}_j$ for more than two values of j. Thus, the total number of crossings in $\mathcal{D}$ is at least $(1/2)\sum_{j=0}^{n-1}(m-2) = (m-2)n/2$. ∎

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