Drawings of C_m X C_n with one disjoint family*

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Abstract. We show that every drawing of $C_m \times C_n$ with either the m n–cycles pairwise disjoint or the n m–cycles pairwise disjoint has at least $(m-2)n$ crossings, for every m, n satisfying $n \geq m \geq 3$. This supports the long standing conjecture by Harary et al. that the crossing number of $C_m \times C_n$ is $(m-2)n$.

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1. Introduction

The *Cartesian product* $C_m \times C_n$ of the cycles of sizes m and n is a 4–regular graph with mn vertices $v_{i,j}$, where $0 \le i \le m-1$ and $0 \le j \le n-1$. The vertices are labeled in such a way that the four vertices adjacent to $v_{i,j}$ are $v_{i-1,j}, v_{i+1,j}, v_{i,j-1}$, and $v_{i,j+1}$, where indices are read modulo m and n respectively. Thus, the edge set of $C_m \times C_n$ can be partitioned in m n–cycles plus n m–cycles.

Harary et al. [3] conjectured that the crossing number of $C_m \times C_n$ is $(m-2)n$, for every m, n satisfying $n \ge m \ge 3$. This has been proved for every $n \ge m$ for each $m = 3, 4, 5$, and 6 (cf. [7], [2], [4], [8], respectively). In all these papers one crucial step is to show that every drawing of $C_m \times C_n$ with the n m–cycles pairwise disjoint has at least $(m-2)n$ crossings (for the cases $m = 3, 4, 5$, and 6). The main result of this paper is that this statement holds for arbitrary m. Moreover, we show that the same conclusion is obtained if the m n–cycles are pairwise disjoint.

Theorem 1. *Let m, n be such that $n \ge m \ge 3$. Then every drawing of $C_m \times C_n$ such that either the n m–cycles are pairwise disjoint or the m n–cycles are pairwise disjoint has at least $(m-2)n$ crossings.*

The crossing number $cr(G)$ of a graph G is the minimum number of crossings of edges in a drawing of G in the plane. It is well–known [9] that the crossing number of a graph is attained by a *good* drawing, that is, by a drawing in which (i) no edge crosses itself, (ii) no adjacent edges cross, (iii) no two edges cross each other more than once, and (iv) no three edges cross in the same point.

The crossing numbers of very few families of graphs are known exactly. Of particular interest is the crossing number of $C_m \times C_n$, which has received a good deal of attention since Harary et al. conjectured that $cr(C_m \times C_n) \ge (m-2)n$ for all m, n such that $n \ge m \ge 3$. It is not difficult to give drawings of $C_m \times C_n$ with exactly $(m-2)n$ crossings (see [3]), for every m, n satisfying $n \ge m \ge 3$, but, as mentioned above, the exact value of $cr(C_m \times C_n)$ remains an unsolved problem, except for a few values of m.

For each m, n satisfying $n > m \ge 3$, all the drawings known of $C_m \times C_n$ with exactly $(m-2)n$ crossings have the n m–cycles pairwise disjoint. Similarly, if $m \ge 5$, then, in all the drawings known of $C_m \times C_n$ with exactly $(m-2)n$ crossings, at least $m-4$ of the n–cycles are pairwise disjoint. Thus, Theorem 1 represents an interesting complement to our knowledge of $cr(C_m \times C_n)$, since it establishes that no drawing in which either the n m–cycles are pairwise disjoint or the m n–cycles are pairwise disjoint has fewer than $(m-2)n$ crossings.

In a related result, Richter and Thomassen [5] have shown that if $(\mathcal{C}_1, \mathcal{C}_2)$ is an (m, n)–mesh with both $\mathcal{C}_1$ and $\mathcal{C}_2$ pairwise disjoint, then the total number of intersections of the curves in $\mathcal{C}_1 \cup \mathcal{C}_2$ is at least $mn + (m-2)n$ (an (m, n)–*mesh* is a pair $(\mathcal{C}_1, \mathcal{C}_2)$ of families of closed curves such that each curve in $\mathcal{C}_1$ intersects each curve in $\mathcal{C}_2$). A direct consequence of this result is that every drawing of $C_m \times C_n$ with the m n–cycles pairwise disjoint *and* the n m–cycles pairwise disjoint has at least $(m-2)n$ crossings. However, we cannot expect (m, n)–meshes to be as helpful in the study of drawings of $C_m \times C_n$ in which only one family is pairwise disjoint. This follows from the fact that an optimal (m, n)–mesh is not in general the underlying set of curves of a drawing of $C_m \times C_n$. For instance, certain optimal

$(3, n)$–meshes have one family pairwise disjoint, but the curves in one family intersect the curves in the other family in the wrong cyclic order.

Besides its value as an independent result, Theorem 1 is very important in connection with the calculation of the exact crossing number of $C_m \times C_n$ for small values of m. As mentioned above, the crossing number of $C_m \times C_n$ is known exactly for every $n \geq m$, for each m satisfying $3 \leq m \leq 6$. The cases $m = 3, 4, 5$, and 6 were solved in [7], [2], [4], and [8] respectively, using an induction with base cases $cr(C_3 \times C_3) = 3, cr(C_4 \times C_4) = 8$, $cr(C_5 \times C_5) = 15$, and $cr(C_6 \times C_6) = 24$ respectively. In each of [7], [2], [4], and [8], an important part of the inductive step is to deal with the case in which the n m–cycles are pairwise disjoint. Roughly speaking, half of the difficulty in these inductions lies in the case where the n m–cycles are pairwise disjoint, and the other half lies in the case where some pair of m–cycles intersect. Thus, we hope that Theorem 1 will be very helpful in connection with the computation of $cr(C_m \times C_n)$ for more values of m and n.

This paper is organized as follows. In Section 2 we state Lemmas 2 and 3 and Proposition 4. Lemmas 2 and 3 imply Theorem 1, and Proposition 4 is the main ingredient of the proofs of Lemmas 2 and 3. In Section 3 we prove Lemmas 2 and 3, and in Section 4 we prove Proposition 4. Section 5 contains some final remarks.

2. Proof of Theorem 1

To help comprehension, we colour the edges of $C_m \times C_n$ as follows: colour the m n–cycles blue and the n m–cycles red. Thus, if $m \neq n$, the graphs $C_m \times C_n$ and $C_n \times C_m$ are coloured in a different way. That is, even though these graphs are isomorphic, the relative order of m and n does matter for our present discussion. We remark that although most of the time we deal with graphs $C_p \times C_q$ where $q \geq p$, in general we do not assume that this inequality holds.

A drawing $\mathcal{D}$ of $C_m \times C_n$ is a *red–disjoint drawing* if the red cycles are pairwise disjoint in $\mathcal{D}$. Similarly, $\mathcal{D}$ is a *blue–disjoint* drawing if the blue cycles are pairwise disjoint. For brevity, we refer to red–disjoint drawings as rd–drawings and to blue–disjoint drawings as bd–drawings. The minimum number of crossings in an rd–drawing of $C_m \times C_n$ is denoted by $cr_{rd}(C_m \times C_n)$, and the minimum number of crossings in a bd–drawing of $C_m \times C_n$ is denoted by $cr_{bd}(C_m \times C_n)$. Using this notation, Theorem 1 can be written in the following way.

Theorem 1. *Let m, n be such that $n \geq m \geq 3$. Then $cr_{rd}(C_m \times C_n) \geq (m-2)n$ and $cr_{bd}(C_m \times C_n) \geq (m-2)n$.*

We prove the two inequalities of Theorem 1 separately, as Lemmas 2 and 3.

Lemma 2. *Let m, n be such that $n \geq m \geq 3$. Then $cr_{bd}(C_m \times C_n) \geq (m-2)n$.*

Lemma 3. *Let m, n be such that $n \geq m \geq 3$. Then $cr_{rd}(C_m \times C_n) \geq (m-2)n$.*

Let $\mathcal{D}$ be a drawing of $C_m \times C_n$. We denote by $cr(\mathcal{D})$ the number of crossings of edges in $\mathcal{D}$. If C, C', and C'' are cycles such that C' and C'' are contained in different components of $\mathbb{R}^2 \setminus C$, then C *separates* C' and C'' in $\mathcal{D}$.

The following result is the heart of the proofs of Lemmas 2 and 3.

Proposition 4. *Suppose that* $\min\{p,q\} \geq 3$*. Let* $\mathcal{D}$ *be a drawing of* $C_p \times C_q$*, such that the* q p*–cycles are pairwise disjoint and no* p*–cycle separates two* p*–cycles. Then* $cr(\mathcal{D}) \geq (p-2)q$*.*

It is important to note that in Proposition 4 we do not assume that $p \geq q$.

The proof of Proposition 4 is presented in Section 4. In the next section we use this result to prove Lemmas 2 and 3.

3. Proofs of Lemmas 2 and 3

Proof of Lemma 2. We prove this lemma by induction on $m+n$, the base case being $m=n=3$. To see that $cr_{bd}(C_3 \times C_3) \geq 3$, we note that $cr(C_3 \times C_3) = 3$ (see [3] or [6]), and that $cr_{bd}(C_p \times C_q) \geq cr(C_p \times C_q)$ for all integers p,q.

Let m,n be such that $n \geq m \geq 3$, where m and n are not both equal to three, and suppose that $cr_{bd}(C_{m'} \times C_{n'}) \geq (m'-2)n'$ for all m',n' satisfying $n' \geq m' \geq 3$ and $m'+n' < m+n$. Let $\mathcal{D}$ be a bd–drawing of $C_m \times C_n$. To complete the proof, we show that $cr(\mathcal{D}) \geq (m-2)n$.

First let us suppose that no blue cycle separates two blue cycles in $\mathcal{D}$. In this case we can apply Proposition 4 with $p=n$ and $q=m$ to obtain that $cr(\mathcal{D}) \geq (n-2)m$. Now since $n \geq m$, $(n-2)m \geq (m-2)n$, and so $cr(\mathcal{D}) \geq (m-2)n$.

Now we consider the case in which a blue cycle B separates two blue cycles in $\mathcal{D}$. In this case B crosses every red cycle, and so B has at least n crossings. Thus, if $m=3$, then $cr(\mathcal{D}) \geq n$, as required. Now suppose that $m > 3$. Let $\mathcal{D}'$ be the drawing of $C_{m-1} \times C_n$ obtained by removing B from $\mathcal{D}$. Since B has at least n crossings, $cr(\mathcal{D}) \geq cr(\mathcal{D}') + n$. By the induction hypothesis, $cr(\mathcal{D}') \geq (m-3)n$, and so $cr(\mathcal{D}) \geq (m-3)n + n = (m-2)n$. ∎

Proof of Lemma 3. We note that if $n=m$, then the blue cycles and the red cycles are interchangeable, and so it follows from Lemma 2 that $cr_{rd}(C_m \times C_m) \geq (m-2)m$ for every $m \geq 3$. For each fixed m, we prove that $cr_{rd}(C_m \times C_n) \geq (m-2)n$ by induction on n, the base case being $cr_{rd}(C_m \times C_m) \geq (m-2)m$. Thus, we let $n > m$, and assume that $cr_{rd}(C_m \times C_{n-1}) \geq (m-2)(n-1)$. Let $\mathcal{D}$ be an rd–drawing of $C_m \times C_n$. We finish the proof by showing that $cr(\mathcal{D}) \geq (m-2)n$.

If some red cycle R separates two red cycles, then every blue cycle crosses R, and so Lemma 3 follows easily from the induction hypothesis. Therefore we may assume that no red cycle separates two red cycles. Then Proposition 4 applies with $p=m$ and $q=n$, and so we obtain $cr(\mathcal{D}) \geq (m-2)n$. ∎

4. Proof of Proposition 4

The proof of Proposition 4 is largely based on the proof of Lemma 3 in [4], which establishes Proposition 4 for the special case $q \geq p$, $p=5$.

Proof of Proposition 4. The red p–cycles are naturally cyclically ordered as $R_1, R_2, \ldots, R_q$, so that each of the blue q–cycles has an edge that joins a vertex of R_{i-1} with a vertex of R_i, for $i = 1, 2, \ldots, q$. We remark that indices are read modulo q throughout the whole proof.

Let H_i denote the subgraph of $C_p \times C_q$ induced by the vertices in $R_{i-1} \cup R_i$, for each $i = 1, 2, \ldots, q$. Thus, H_i has $2p$ red edges and p blue edges, every red edge is in two of the H_i, and every blue edge is in exactly one of the H_i.

As in [4], we define the *force* $f(H_i)$ of H_i in $\mathcal{D}$ as the total number of crossings in $\mathcal{D}$ of the following types:

(1) a crossing of a blue edge in H_i with an edge in $R_i \cup R_{i+1}$;
(2) a crossing of a blue edge in H_{i+1} with an edge in H_i;
(3) a crossing of a blue edge in H_i with a blue edge in H_i; and
(4) a self–intersection of R_i.

It is not difficult to check that no crossing counted in $f(H_i)$ is counted in $f(H_j)$ if $i \neq j$. Thus it follows that the number $cr(\mathcal{D})$ of crossings in $\mathcal{D}$ is at least $\sum_{1 \leq i \leq q} f(H_i)$. Therefore, in order to complete the proof of Proposition 4, it suffices to show that, for each $i = 1, 2, \ldots, q$, $f(H_i) \geq p - 2$. By the symmetry it suffices to show that $f(H_1) \geq p - 2$.

Now let $I = I_{H_1,R_2}$ be the collection of blue edges of H_1 that intersect R_2. Let $f_I(H_1)$ be the total number of crossings that contribute to $f(H_1)$ that involve a blue edge of H_1 and a red edge of R_2, and let $f_{\overline{I}}(H_1)$ be the total number of crossings that contribute to $f(H_1)$ that do not contribute to $f_I(H_1)$.

Thus, $f(H_1) = f_I(H_1) + f_{\overline{I}}(H_1)$, and so it suffices to show that $f_I(H_1) + f_{\overline{I}}(H_1) \geq p - 2$. We claim that the statements in Claims 5 and 6 below combine to prove this last inequality. For if $|I_{H_1,R_2}| \geq p - 2$, then, by Claim 5, $f_I(H_1) \geq 2(p-2) > p - 2$, since $p \geq 3$. On the other hand, if $|I_{H_1,R_2}| < p - 2$, then by Claims 5 and 6 we obtain $f_I(H_1) + f_{\overline{I}}(H_1) \geq p - 2 + |I_{H_1,R_2}| \geq p - 2$. Thus Claims 5 and 6 complete the proof of Proposition 4. ∎

Claim 5. $f_I(H_1) \geq 2|(I_{H_1,R_2})|$.

Proof. Since no two of R_q, R_1, and R_2 intersect and no one separates the other two, each edge in I_{H_1,R_2} crosses R_2 at least twice. ∎

Claim 6. *Suppose that* $|I_{H_1,R_2}| < p - 2$. *Then* $f_{\overline{I}}(H_1) \geq (p-2) - |I_{H_1,R_2}|$.

Proof. Let H_1' denote the subgraph $H_1 - I_{H_1,R_2}$. Note that the cycles R_q and R_1 are still contained in H_1', but these cycles may contain some vertices of degree two (subdivisions) in H_1'. Since any such subdivisions are irrelevant to the discussion, we eliminate (i.e., unsubdivide) them from H_1', and call H_1'' the obtained graph. The cycles R_q and R_1 induce naturally two cycles R_q'' and R_1'' in H_1''. We note that each of R_q'' and R_1'' has exactly $p - |I_{H_1,R_2}| \geq 3$ vertices. As in H_1, we colour the edges in R_q'' and R_1'' red, and we colour all the other edges in H_1'' blue.

Now let $e = uv$ be any blue edge of H_1'', with $u \in V(R_q'')$ and $v \in V(R_1'')$. As we go along e from v to u, let a be the first point in e that is in R_q'' (u can be such a point), and let b be the last point in e before a that is in R_1'' (v can be such a point). Let τe denote the segment of e from a to b. If $b \neq v$, then e crosses R_1'' at b. Let $\tau H_1''$ denote the subset of the plane consisting of R_q'', R_1'', and all the τe, for e a blue edge of H_1''.

There are exactly $p - |I_{H_1,R_2}|$ blue edges going from vertices of R_1'' to R_2. Since R_2 is disjoint from $\tau H_1''$ and is not separated by either R_q'' or R_1'' from the other, there is a region F of $\tau H_1''$ that contains R_2 and whose boundary contains some of the blue part of $\tau H_1''$.

Let us suppose that there are exactly k vertices on R_1'' incident with F. Then there must be at least $(p - |I_{H_1,R_2}|) - k$ crossings of blue edges of H_2 with the boundary of this region. Clearly, these crossings contribute to $f_{\overline{I}}(H_1)$. Thus, $f_{\overline{I}}(H_1) \geq (p - |I_{H_1,R_2}|) - k$. Consequently, if $k \leq 2$, then $f_{\overline{I}}(H_1) \geq (p-2) - |I_{H_1,R_2}|$, as required.

We now consider the possibility that $k \geq 3$. Since we have already found $(p - |I_{H_1,R_2}|) - k)$ crossings that contribute to $f_{\overline{I}}(H_1)$, it follows that we must find an additional $((p - 2) - |I_{H_1,R_2}|) - ((p - |I_{H_1,R_2}|) - k) = k - 2$ crossings that contribute to $f_{\overline{I}}(H_1)$.

The boundary of F is a closed curve whose intersection with R_1'' is a connected arc. Let s_1 be the point where we start as we go along the portion P of the boundary of F in R_1''. As we go along P, we traverse edges and possibly self–intersections of R_1'', and we go through vertices $w_1, w_2, \ldots, w_l$ of R_1'', in this order, and finally finish at a point s_2. We remark that s_1 and s_2 need not be vertices of R_1'', but in any case $l \geq k-2$. The vertices $w_1, \ldots, w_l$ are in the interior of the segment of R_1'' that is in the boundary of F.

For each $i = 1, \ldots, l$, let e_i be the blue edge from R_q'' to w_i. If $b_i = w_i$, then the segment τe_i goes from w_i to R_q'', and so τe_i must go through the region F. But this is impossible, since in such a case we would have incorrectly identified F. Therefore $b_i \neq w_i$, and so e_i crosses R_1'' at b_i.

Therefore, there are an additional $l \geq k-2$ crossings that contribute to $f_{\overline{I}}(H_1)$, and so $f_{\overline{I}}(H_1) \geq ((p - |I_{H_1,R_2}|) - k) + (k-2) = (p-2) - |I_{H_1,R_2}|$, as required. ∎

5. Comments

As mentioned in Section 1, dealing with the case where the n m–cycles are pairwise disjoint is a substantial part of the argument to show inductively that $C_m \times C_n = (m-2)n$, for $m = 3, 4, 5$, and 6. Recently [1] it has been shown that $cr(C_7 \times C_7) = 35$ (as conjectured), and so it seems plausible that, in a proof of the conjecture $cr(C_7 \times C_n) = 5n$, it will be a major relief that the case where the n 7–cycles are pairwise disjoint has been dealt with in Theorem 1. A similar observation holds for a potential proof of the general conjecture $cr(C_m \times C_n) = (m-2)n$.

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