Shellable drawings and the cylindrical crossing number of K_n

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Abstract

The Harary-Hill Conjecture states that the number of crossings in any drawing of the complete graph K_n in the plane is at least $Z(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$. In this paper, we settle the Harary-Hill conjecture for *shellable drawings*. We say that a drawing D of K_n is *s-shellable* if there exist a subset $S = \{v_1, v_2, \ldots, v_s\}$ of the vertices and a region R of D with the following property: For all $1 \leq i < j \leq s$, if D_{ij} is the drawing obtained from D by removing $v_1, v_2, \ldots v_{i-1}, v_{j+1}, \ldots, v_s$, then v_i and v_j are on the boundary of the region of D_{ij} that contains R. For $s \geq n/2$, we prove that the number of crossings of any s-shellable drawing of K_n is at least the long-conjectured value $Z(n)$. Furthermore, we prove that all cylindrical, x-bounded, monotone, and 2-page drawings of K_n are s-shellable for some $s \geq n/2$ and thus they all have at least $Z(n)$ crossings. The techniques developed provide a unified proof of the Harary-Hill conjecture for these classes of drawings.

Abstract

In this paper, we prove the Harary-Hill conjecture for *shellable drawings* of the complete graph. For a drawing D of K_n and $i < j$, let D_{ij} be the drawing obtained from D by removing $v_1, v_2, \ldots v_{i-1}, v_{j+1}, \ldots, v_s$. We say that D is *s-shellable* if there exist a subset $S = \{v_1, v_2, \ldots, v_s\}$ of the vertices and a region R of D with the following property: For all $1 \leq i < j \leq s$, v_i and v_j are on the boundary of the region of D_{ij} that contains R. We prove that if $s \geq n/2$ the number of crossings of any s-shellable drawing of K_n is at least the long-conjectured value of the crossing number of K_n, namely $Z(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$. We also prove that every cylindrical drawing of K_n is s-shellable for some $s \geq n/2$. From these results it follows that the cylindrical crossing number of K_n is $Z(n)$, thus settling the Harary-Hill conjecture for cylindrical drawings of K_n. Moreover, the techniques developed provide an unified proof verifying the Harary-Hill conjecture for 2-page, monotone, and x-bounded drawings.

1 Introduction

In the late 1950s, the British artist Anthony Hill got interested in producing drawings of the complete graph K_n with the least possible number of edge crossings. His general technique, explained in a paper he wrote jointly with Harary [7], is best described by drawing K_n on a cylinder as follows. Draw a cycle with $\lceil n/2 \rceil$ vertices on the rim of the top lid, and a cycle with the remaining $\lfloor n/2 \rfloor$ vertices on the rim of the bottom lid. Then draw the remaining edges joining vertices on the same lid using the straight line joining them across the lid. Finally, for any two vertices on distinct lids, draw the edge joining them along the geodesic that connects them on the side of the cylinder. (See Figure 1, left, for a planar representation of such a drawing.) It is an elementary exercise to show that such a drawing of K_n has exactly $Z(n) := \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$ crossings. The Harary-Hill constructions are a particular instance of *cylindrical* drawings (see formal definition in Section 3).

At about the same time as the Harary-Hill paper was published, Blažek and Koman got independently interested in drawing K_n with as few crossings as possible [5]. In their construction (see Figure 1, right), they start by drawing a cycle as a regular n-gon, and then drawing all diagonals with positive slope (as straight line segments) and all other edges outside the cycle. The Blažek-Koman construction also yields drawings of K_n with exactly $Z(n)$ crossings, and it is a particular instance of *2-page* drawings (see below for the definition).

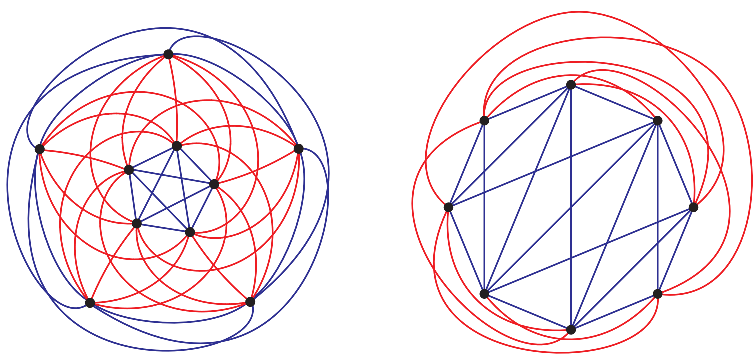

Figure 1: *Left*: Harary-Hill construction for 10 points. (A cylindrical drawing.) *Right*: Blažek-Koman construction for 8 points. (A 2-page drawing.)

To this date, no drawing of K_n with fewer than $Z(n)$ crossings is known. Moreover, all general constructions (for arbitrary values of n) known with exactly $Z(n)$ crossings are obtained from insubstantial alterations of either the Harary-Hill or the Blažek-Koman constructions (a few exceptions are known, but only for some small values of n). The tantalizingly open *Harary-Hill conjecture* $\mathrm{cr}(K_n) = Z(n)$ has been confirmed only for $n \leq 12$ [10].

The main contribution of this paper is the introduction of *shellable* drawings, a large class of drawings for which (as we shall show) the Harary-Hill conjecture holds. Shellability captures the essential features of 2-page drawings we previously used [1, 3] to prove that the 2-page crossing number of K_n is $Z(n)$, and allows us to extend the lower bound to a larger family of drawings, including cylindrical, monotone, and x-bounded drawings (see definitions below).

If a drawing D of a graph is regarded as a subset of the plane, then a *region* of D is a connected component of $\mathbb{R}^2 \setminus D$. (If D is an embedding, then the regions of D are the faces). A drawing D of K_n is s-*shellable* if there exists a subset $S = \{v_1, v_2, \ldots, v_s\}$ of the vertices and a region R of D with the following property. For $1 \leq i < j \leq s$, if D_{ij} denotes the drawing obtained from D by removing $v_1, v_2, \ldots v_{i-1}, v_{j+1}, v_{j+2}, \ldots, v_s$, then for all $1 \leq i < j \leq s$, the vertices v_i and v_j are on the boundary of the region of D_{ij} that contains R. The set S is an s-*shelling* of D *witnessed* by R.

The core of this paper is the following statement, whose proof is given in Section 2.

Theorem 1. *Let D be an s-shellable drawing of K_n, for some $s \geq n/2$. Then D has at least $Z(n)$ crossings.*

We use this to settle the Harary-Hill conjecture for several classes of drawings:

- In a *2-page book drawing* (or simply *2-page drawing*), the vertices are placed on a line (the *spine* of the *book*), and each edge (except for its endvertices) lies entirely on an open halfplane spanned by the spine (one of the 2 *pages* of the book). (See Figure 2, right.)
- Following Schaefer [12], in a *cylindrical drawing* of a graph, there are two concentric circles that host all the vertices, and no edge is allowed to intersect these circles, other than at its endvertices. (Schaefer defines cylindrical drawings only for bipartite graphs, but his definition obviously applies to arbitrary graphs). (See Figure 2, left.)

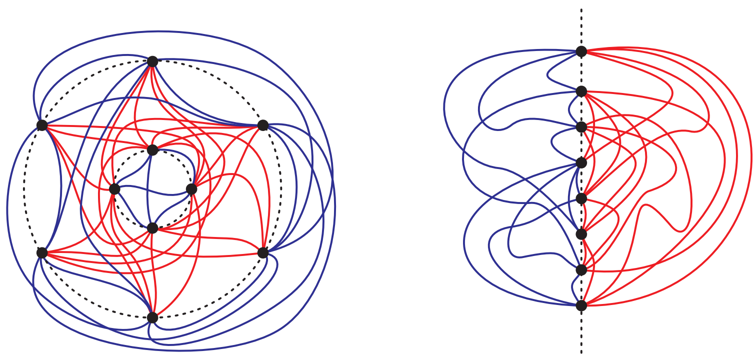

Figure 2: *Left:* A cylindrical drawing of K_{10}. *Right:* A 2-page drawing of K_8.

We remark that Hill's drawings can be naturally regarded as cylindrical drawings. Indeed, even though in Hill's drawings the edges joining consecutive rim vertices are placed on the rims, such drawings are easily adapted to this definition, since those edges can be drawn arbitrarily close to a rim.

- A drawing is *monotone* if each vertical line intersects each edge at most once. (See Figure 3, right.)
- A drawing is *x-bounded* if by labelling the vertices $v_1, v_2, \ldots, v_n$ in increasing order of their x-coordinates, for all $1 \leq i < j \leq n$ the edge $v_i v_j$ is contained in the strip bounded by the vertical line that contains v_i and the vertical line that contains v_j. (See Figure 3, left.)

In Section 3, we find a condition on drawings of K_n that guarantees that they are s-shellable for some $s \geq n/2$. Then we show that if D is a crossing minimal 2-page, cylindrical, monotone, or x-bounded drawing, then D satisfies this condition, thus settling (in view of Theorem 1) the Harary-Hill conjecture for all these families of drawings. Section 4 contains some concluding remarks.

2 k-edges in shellable drawings and proof of Theorem 1

We recall that in a *good* drawing of a graph, no two edges share more than one point and no edge crosses itself. It is easy to show that every crossing minimal drawing of a graph is good.

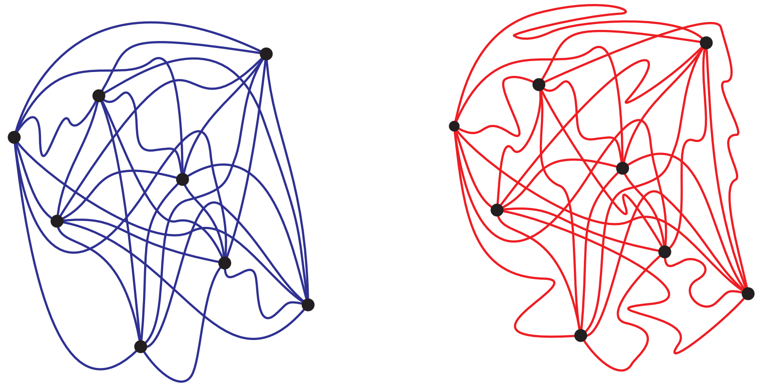

Figure 3: *Left:* A monotone drawing of K_8. *Right:* An x-bounded drawing of K_8.

We generalized the geometrical concept of a k-edge to arbitrary (topological) good drawings of K_n [1, 3], as follows. Let D be a good drawing of K_n, pq a directed edge of D, and r a vertex of D distinct from p and q. Then pqr denotes the oriented closed curve defined by concatenating the edges pq, qr, and rp. An oriented, simple, and closed curve in the plane is oriented *counterclockwise* (respectively, *clockwise*) if the bounded region it encloses is on the left (respectively, right) hand side of the curve. Further, r is *on the left* (respectively, *right*) *side of* pq if pqr is oriented counterclockwise (respectively, clockwise). We say that the edge pq is a k-edge of D if it has exactly k points of D on one side (left or right), and thus $n-2-k$ points on the other side. Hence, as in the geometric setting, a k-edge is also an $(n-2-k)$-edge. The direction of the edge pq is no longer relevant and every edge of D is a k-edge for some unique k such that $0 \leq k \leq \lfloor n/2 \rfloor - 1$.

Following our previous work [1, 3], if D is a good drawing of K_n, then for each $0 \leq k \leq \lfloor n/2 \rfloor - 1$ we define the set of *$\leq k$-edges* of D as all j-edges in D for $j = 0, \ldots, k$. The number of $\leq k$-edges of D is denoted by

$$E_{\leq k}(D) := \sum_{j=0}^{k} E_j(D).$$

Similarly, we denote the number of *$\leq\leq k$-edges* of D by

$$E_{\leq\leq k}(D) := \sum_{j=0}^{k} E_{\leq j}(D) = \sum_{j=0}^{k} \sum_{i=0}^{j} E_i(D) = \sum_{i=0}^{k} (k+1-i) E_i(D). \tag{1}$$

It is known [1, 3] that if D is a good drawing, then D has exactly

$$2 \sum_{k=0}^{\lfloor n/2 \rfloor - 2} E_{\leq\leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n-2}{2} \right\rfloor - \frac{1}{2} (1 + (-1)^n) E_{\leq\leq \lfloor n/2 \rfloor - 2}(D) \tag{2}$$

crossings. Thus we now concentrate on bounding $E_{\leq\leq k}(D)$. We need a few more definitions. If D_y is the drawing of K_{n-1} obtained from D by deleting a vertex y, then an edge non-incident to y is (D, D_y)-*invariant* if for some $0 \leq k \leq \lfloor (n-3)/2 \rfloor$ it is a k-edge in both D and D_y. We let $E_{\leq k}(D, D_y)$ denote the number of (D, D_y)-invariant $\leq k$-edges.

2.1 Ordering the vertices with respect to a boundary point

The *unbounded region* of a drawing D is its unique region with noncompact closure. We refer to the topological boundary of the unbounded region of D simply as the *boundary of* D.

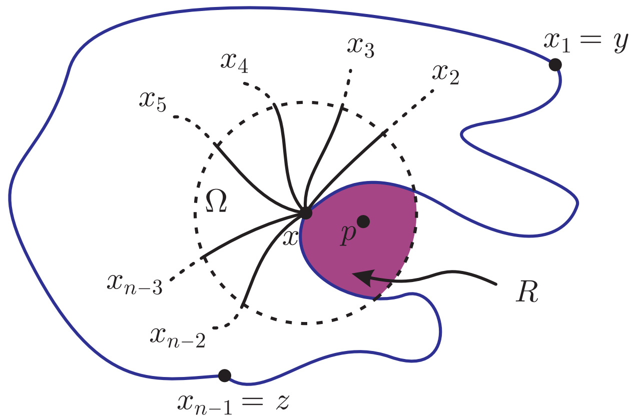

Figure 4: The order induced by x.

Let D be a good drawing of K_n and assume that x is a vertex on the boundary of D. Then there is a natural order of the vertices of D_x induced by the order in which the edges of D *leave* x. Namely, there is a disk Ω with center x and radius $\epsilon > 0$ that intersects D only at the edges incident to x. Moreover, for ϵ small enough, Ω intersects each edge incident to x in a simple connected Jordan curve. (See Figure 4.) Exactly two of these curves, say $xy \cap \Omega$ and $xz \cap \Omega$ for some vertices y and z, are on the boundary of D. Suppose without loss of generality that the triangle xyz is oriented counter-clockwise. Then we can label the vertices of D_x by $x_1, x_2, \ldots, x_{n-1}$ so that $x_1 = y$, $x_{n-1} = z$, and the Jordan curves $xx_1 \cap \Omega, xx_2 \cap \Omega, \ldots, xx_{n-1} \cap \Omega$ appear in counter-clockwise order around x. We refer to this as *the order induced by* x *in* D.

Proposition 2. *Let $n \geq 1$ and consider a good drawing D of the complete graph K_n. Let x be a vertex on the boundary of D, and let $x_1, x_2, \ldots, x_{n-1}$ be the order induced by x in D. Then xx_i and xx_{n-i} are $i-1$-edges of D for $1 \leq i \leq \lfloor (n-2)/2 \rfloor$.*

Proof. Consider a disk Ω as above. Then any point p in Ω and outside the triangle xyz is in the unbounded region of D. (See Figure 4.) This means that p cannot be in the interior of any triangle of D. In particular, if $j < i$, then the triangle xx_jx_i is oriented counter-clockwise as otherwise its interior would contain p. This means that x_j is to the right of xx_i if $j < i$, and to the left if $j > i$. Thus there are exactly $i-1$ vertices to the right of xx_i and $n-1-i$ to the left. This means that xx_i is a $\min(i-1, n-1-i)$-edge of D, implying the result. $\square$

Proposition 3. *Let $0 \leq i-1 \leq k \leq \lfloor (n-3)/2 \rfloor$, D a good drawing of the complete graph K_n, and x and y vertices of D. Let U be a subset of $i-1$ vertices of D not including x and y. Assume that x is on the boundary of the drawing $D(U)$ obtained from D by removing U. Then there exist at least $k-i+2$ edges incident to x and non-incident to vertices in U that are (D, D_y)-invariant $\leq k$-edges.*

Proof. Consider the order $x_1, x_2, \ldots x_{n-i}$ induced by x in $D(U)$. As before, x_ℓ is to the right of xx_j if $\ell < j$, and to the left if $\ell > j$. Thus there are exactly $j-1$ vertices in $D(U)$ to the right of xx_j and $n-i-j$ to the left. Including U, this means that there are at most $i-1+j-1 = i+j-2$ vertices to the right of xx_j in D and at most $i-1+n-i-j = n-j-1$ to the left.

Now consider the point y, which is equal to x_w for some $1 \leq w \leq n-i$. If $w > k+2-i$, then for $1 \leq j \leq k+2-i$ the edge xx_j has at most $i+j-2 \leq i+(k+2-i)-2 = k$ points to its right and y on its left (because $w > k+2-i \geq j$). If $w \leq k+2-i$, then for $n-k-1 \leq j \leq n-i$

the edge xx_j has at most $n-j-1 \leq n-(n-k-1)-1 = k$ points to its left and y on its right (because $k \leq (n-3)/2 < (n-3+i)/2$ and thus $w \leq k+2-i < n-k-1 \leq j$). In either case, the $k+2-i$ edges xx_j are (D, D_y)-invariant $\leq k$-edges. $\square$

2.2 Bounding the number of $\leq\leq k$-edges in shellable drawings of K_n

We now bound the number of $\leq\leq k$-edges of s-shellable drawings of K_n for a certain interval of k determined by s.

Proposition 4. *Let D be an s-shellable good drawing of the complete graph K_n, in which the region R that witnesses the s-shellability of D is its unbounded region. Then $E_{\leq\leq k}(D) \geq 3\binom{k+3}{3}$ for all $0 \leq k \leq \min(s-2, \lfloor (n-3)/2 \rfloor)$.*

Proof. Let V be the set of vertices of D and $S = \{v_1, v_2, \ldots, v_s\}$ an s-shelling of D witnessed by the unbounded region R. Fix k with $0 \leq k \leq \min(s-2, \lfloor (n-3)/2 \rfloor)$. We prove that

$$E_{\leq\leq i}(D_{1,s-k+i}) \geq 3\binom{i+3}{3} \tag{3}$$

for $0 \leq i \leq k$ by induction on i. For $i = 0$, because S is an s-shelling of D, and the unbounded region witnesses this s-shellability, it follows that v_1 and v_{s-k} are on the boundary of $D_{1,s-k}$. By Proposition 2 each of these two vertices (they are different because $k \leq s-2$) is incident to two 0-edges and they can share at most one 0-edge. That is, $E_{\leq\leq 0}(D_{1,s-k}) \geq 3$. We now compare the following two identities obtained from (1). For $1 \leq r \leq s$ and $0 \leq k' \leq \lfloor (n-s+r)/2 \rfloor$,

$$E_{\leq\leq k'}(D_{1,r}) = \sum_{j=0}^{k'} (k'+1-j) E_j(D_{1,r}) \tag{4}$$

and

$$E_{\leq\leq k'-1}(D_{1,r-1}) = \sum_{j=0}^{k'-1} (k'-j) E_j(D_{1,r-1}). \tag{5}$$

As shown in our previous work [2], for a $j \leq k'$ a j-edge incident to v_r contributes $k'-j$ to (4) and nothing to (5), a $(D_{1,r}, D_{1,r-1})$-invariant edge contributes 1 more to (4) than to (5), and all other edges contribute the same to (4) and (5). Therefore,

$$E_{\leq\leq k'}(D_{1,r}) = E_{\leq\leq k'-1}(D_{1,r-1}) + \sum_{\ell=0}^{k'} (k'+1-\ell) e_\ell(v_r) + E_{\leq k}(D_{1,r}, D_{1,r-1}), \tag{6}$$

where $e_\ell(r)$ is the number of ℓ-edges incident to v_r in $D_{1,r}$.

Now, choose i such that $1 \leq i \leq k$ and assume that

$$E_{\leq\leq i-1}(D_{1,s-k+i-1}) \geq 3\binom{i+2}{3}. \tag{7}$$

By (6) for $k' = i$ and $r = s-k+i$, we have that

$$E_{\leq\leq i}(D_{1,s-k+i}) = E_{\leq\leq i-1}(D_{1,s-k+i-1}) + \sum_{\ell=0}^{i} (i+1-\ell) e_\ell(v_{s-k+i}) + E_{\leq i}(D_{1,s-k+i}, D_{1,s-k+i-1}), \tag{8}$$

We separately bound each term of the right-hand side of (8). The first term is bounded in (7). For the second term, Proposition 2 (for $x = v_{s-k+i}$ is on the boundary of $D_{1,s-k+i}$) implies that $e_\ell(v_{s-k+i}) = 2$ and thus

$$\sum_{\ell=0}^{i}(i+1-\ell)e_\ell(v_{s-k+i}) = \sum_{\ell=0}^{i}(i+1-\ell)2 = 2\binom{i+2}{2}. \tag{9}$$

Finally, we show that

$$E_{\leq i}(D_{1,s-k+i}, D_{1,s-k+i-1}) \geq \sum_{\ell=1}^{i+1}(i-\ell+2) = \binom{i+2}{2}. \tag{10}$$

We use Proposition 3 for the drawing $D_{\ell,s-k+i}$, $x = v_\ell, y = v_{s-k+i}$, and $U = \{v_1, v_2 \ldots, v_{\ell-1}\}$. Note that $k \leq s-2$ implies $1 \leq \ell \leq i+1 < s-k+i$ and thus v_ℓ and v_{s-k+i} are different and do not belong to $\{v_1, v_2, \ldots v_{\ell-1}\}$. Moreover, v_ℓ and v_{s-k+i} are on the boundary of $D_{1,s-k+i}$ because S is an s-shelling of D. Also, $D_{\ell,s-k+i}$ has $n-s+(s-k+i) = n-k+i$ vertices and thus we must check that $0 \leq \ell-1 \leq i \leq (n-k+i-3)/2$. The first two inequalities hold because $1 \leq \ell \leq i+1$. The last inequality follows from $k \leq \min(s-2, \lfloor (n-3)/2 \rfloor) \leq \lfloor (n-3)/2 \rfloor$, which implies $k+i \leq 2k \leq n-3$. Therefore, Proposition 3 implies that for $1 \leq \ell \leq i+1$ there are at least $i-\ell+2$ edges incident to v_ℓ and non-incident to $v_1, v_2, \ldots, v_{\ell-1}$ (so all these edges are different) that are $(D_{s-k+i-1}, D_{s-k+i})$-invariant $\leq i$-edges. $\square$

2.3 Proof of Theorem 1

Let D be an s-shellable drawing of K_n, for some $s \geq n/2$. By using a suitable inversion, if needed, we transform D into a drawing D', with the same number of crossings as D, such that the region that witnesses the s-shellability of D' is the unbounded region. Since $\min(s-2, \lfloor (n-3)/2 \rfloor) = \lfloor (n-3)/2 \rfloor$, it follows from Proposition 4 that $E_{\leq\leq k}(D') \geq 3\binom{k+3}{3}$ for all $0 \leq k \leq \lfloor (n-3)/2 \rfloor$.

Since D' is a good drawing, then by (2) D' has exactly

$$2\sum_{k=0}^{\lfloor n/2 \rfloor -2} E_{\leq\leq k}(D') - \frac{1}{2}\binom{n}{2}\left\lfloor \frac{n-2}{2} \right\rfloor - \frac{1}{2}\left(1+(-1)^n\right) E_{\leq\leq \lfloor n/2 \rfloor -2}(D')$$

crossings. Using this fact, a straightforward calculation [1, 3] shows that if D' is a drawing of K_n that satisfies $E_{\leq\leq k}(D') \geq 3\binom{k+3}{3}$ for all $0 \leq k \leq \lfloor (n-3)/2 \rfloor$, then D' has at least $Z(n)$ crossings. $\square$

3 Verifying the Harary-Hill conjecture for 2-page, cylindrical, monotone, and x-bounded drawings

The workhorse of this section is a property of a drawing that guarantees its shellability:

Lemma 5. *Let D be a drawing of K_n. Suppose that $C = v_1v_2 \ldots v_s$ is a cycle that satisfies the following: (i) the edge v_sv_1 has no crossings; and (ii) for $k = 1, \ldots, s-1$ all crossings in the edge v_kv_{k+1} involve edges v_iv_j with $i < k$ and $j > k+1$. Then D is s-shellable.*

Proof. Let R be a region of D containing the edge v_sv_1 on its boundary. Let $1 \leq i < j \leq s$ and define D_{ij} as before. Let R' be the region of D_{ij} that contains R. Since the vertices $v_1, v_2, \ldots, v_{i-1}, v_{j+1}, v_{j+2}, \ldots, v_s$, and consequently any edge incident to one of these vertices, are removed to obtain D_{ij}, then v_1 and v_s are in the interior of R'. Moreover, it follows from the crossing properties of the edges of C that the edges $v_1v_2, v_2v_3, \ldots, v_{i-1}v_i, v_jv_{j+1}, v_{j+1}v_{j+2}, \ldots, v_{s-1}v_s$ are not intersected by any edge of D_{ij}. Hence the paths $v_i, v_{i-1}, \ldots, v_1$ and $v_j, v_{j+1}, \ldots, v_s$ are completely contained in R' and thus v_i and v_j are on the boundary of R. Therefore, $\{v_1, v_2, \ldots, v_s\}$ is an s-shelling of D witnessed by R. □

We need the full strength of Lemma 5 to show that monotone and x-bounded drawings satisfy the Harary-Hill conjecture. However, it seems worth stating the following weaker form, which is all we need to show that the Harary-Hill conjecture holds for 2-page and cylindrical drawings:

Corollary 6. *If a drawing D of K_n has a crossing-free cycle C of size s then D is s-shellable.* □

We are finally ready to verify the Harary-Hill conjecture for several classes of drawings.

Theorem 7. *Every cylindrical drawing of K_n has at least $Z(n)$ crossings.*

Proof. Let D be a crossing-minimal cylindrical drawing of K_n. Out of the two concentric cycles that contain all the vertices, let ρ be one that contains at least $n/2$ vertices. Let $v_1, v_2, \ldots, v_s$ be the vertices on ρ, in counterclockwise order. Since no two edges cross each other more than once (this follows since D is crossing-minimal) and no edge crosses ρ, it follows that the cycle $v_1v_2 \ldots v_sv_1$ is uncrossed in D. Since $s \geq n/2$, the result follows by Theorem 1 and Corollary 6. □

A 2-page drawing is a particular kind of a cylindrical drawing, namely, a degenerate one with all vertices on one of the concentric circles. Thus Theorem 7 immediately implies our previous result [1, 3] for 2-page drawings:

Corollary 8. *Every 2-page drawing of K_n has at least $Z(n)$ crossings.* □

It is straightforward to check that any x-bounded drawing D of K_n satisfies the conditions of Lemma 5. Thus the Harary-Hill conjecture holds for x-bounded drawings:

Theorem 9. *Every x-bounded drawing of K_n has at least $Z(n)$ crossings.* □

Since every monotone drawing is obviously x-bounded, this implies the Harary-Hill conjecture for monotone drawings (previously proved by the authors [2] and by Balko et al. [4]):

Corollary 10. *Every monotone drawing of K_n has at least $Z(n)$ crossings.* □

4 Concluding remarks

Cylindrical drawings of K_n were previously investigated by Richter and Thomassen [11]. In that paper, they determined the number of crossings in a cylindrical drawing of $K_{m,m}$ with one chromatic class on the inner circle and the other chromatic class on the outer circle. From their result it follows that a cylindrical drawing of K_{2m} in which the edges joining vertices on the same circle are not drawn on the annulus (bounded by the two circles) has at least $Z(2m)$ crossings.

As we observed in Section 1, the 2-page and the cylindrical constructions (possibly with some insubstantial alterations) are the only known drawings of K_n with $Z(n)$ crossings for arbitrary values of n. In his interesting entry at `mathoverflow.net`, Kynčl [8] asks about the existence of alternative constructions, and observes that there is a plethora of drawings with $Z(n) + O(n^3)$ crossings (noting that Moon showed that a random spherical drawing of K_n has expected crossing number $(1/64)n(n-1)(n-2)(n-3) = Z(n) + O(n^3)$).

Balko et al. [4] noted that there are cylindrical drawings D that do not satisfy the bound $E_{\leq\leq k}(D) \geq 3\binom{k+3}{3}$. However, as shown in this paper, for every such drawing there exists a second drawing D' obtained from D by an appropriate inversion (and thus with the same number of crossings) that satisfies $E_{\leq\leq k}(D') \geq 3\binom{k+3}{3}$.

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