ZARANKIEWICZ'S CONJECTURE IS FINITE FOR EACH FIXED m

ROBIN CHRISTIAN, R. BRUCE RICHTER, AND GELASIO SALAZAR

Abstract. Zarankiewicz's Crossing Number Conjecture states that the crossing number $\mathrm{cr}(K_{m,n})$ of the complete bipartite graph $K_{m,n}$ equals $Z(m,n) := \lfloor m/2 \rfloor \lfloor (m-1)/2 \rfloor \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor$, for all positive integers m, n. This conjecture has only been verified for $\min\{m,n\} \leq 6$, for $K_{7,7}, K_{7,8}$, $K_{7,9}$, and $K_{7,10}$, and for $K_{8,8}$, $K_{8,9}$, and $K_{8,10}$. We determine, for each positive integer m, an integer $N_0 = N_0(m)$ with the following property: if $\mathrm{cr}(K_{m,n}) = Z(m,n)$ for all $n \leq N_0$, then $\mathrm{cr}(K_{m,n}) = Z(m,n)$ for every n. This yields, for each fixed integer m, a finite algorithm that either proves that $\mathrm{cr}(K_{m,n}) = Z(m,n)$ for every n, or else finds a counterexample.

1. Introduction

Perhaps the most tantalizing open crossing number problem is what appears to be the first question in the field, namely the Brick Factory Problem, considered by Turán back in 1944. In current terminology, Turán asked:

Question 1.1 (P. Turán)**.** What is the crossing number $\mathrm{cr}(K_{m,n})$ of the complete bipartite graph $K_{m,n}$?

We recall that the crossing number $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise crossings of edges in a drawing of G in the plane.

To date, the best exact result concerning $\mathrm{cr}(K_{m,n})$ is Kleitman's 1970 work showing that, for $n \geq 5$, $\mathrm{cr}(K_{5,n}) = 4\lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor$ [6]. Woodall showed that $\mathrm{cr}(K_{7,7}) = 81$ and $\mathrm{cr}(K_{7,9}) = 144$ in 1993 [9]. All these confirm, for the indicated values, what has become known as Zarankiewicz' Conjecture:

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor .$$

We set $Z(m,n)$ to be the number on the right hand side of this equation. Zarankiewicz gave a drawing of $K_{m,n}$ that has $Z(m,n)$ crossings [10], so the difficulty is to prove that $\mathrm{cr}(K_{m,n}) \geq Z(m,n)$.

Of relevance to us is the overall approach taken by Kleitman and the details of Woodall's arguments. Kleitman showed that, if there is an n so that $\mathrm{cr}(K_{5,n})$ is smaller than $Z(5,n)$, then there is such an n that is either 5 or 7. (A simple, but useful, observation is that, if m is odd and $\mathrm{cr}(K_{m,n}) = Z(m,n)$, then $\mathrm{cr}(K_{m+1,n}) = Z(m+1,n)$. By symmetry, the same statement holds for the second coordinate n.) He then gave special arguments to deal with the two small cases.

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Woodall introduced cyclic-order graphs to determine (by computer) $\mathrm{cr}(K_{7,7})$ and $\mathrm{cr}(K_{7,9})$. He pointed out that Kleitman's approach does not obviously carry over to now be able to conclude that, for all n, $\mathrm{cr}(K_{7,n}) = Z(7,n)$.

Our main result is the following. Let $N_0(m) := ((2Z(m))^{m!}(m!)!)^4$.

Theorem 1.2. *Let m be a positive integer. If, for every $n \le N_0(m)$, $\mathrm{cr}(K_{m,n}) = Z(m,n)$, then, for every n, $\mathrm{cr}(K_{m,n}) = Z(m,n)$.*

Since determining the crossing number of any particular graph is a finite problem, our main theorem has the satisfying conclusion that it is a finite problem to determine, for a given m, whether or not, for each $n \ge m$, it is true that $\mathrm{cr}(K_{m,n}) = Z(m,n)$.

Unfortunately, our method is not practical, even for $n = 5$.

Very little of substance has been published concerning $\mathrm{cr}(K_{m,n})$. Turán considered the "Brick Factory Problem" when he was working in a labour camp during World War II [7]. He mentioned the problem to Zarankiewicz in 1952 and the latter published a paper claiming a proof of what is now known as his conjecture in 1953 [10]. In 1966 and 1967, Kainen and Ringel noticed an unpluggable gap in Zarankiewicz' argument (see [5]).

The case $K_{3,n}$ is quite easy and an elegant proof based on Turán's Theorem about the number of edges in triangle-free graphs is presented in [2]. DeKlerk et al. used Woodall's cyclic-order graphs to set up a quadratic program with 6! variables whose solution requires state-of-the-art quadratic programming methods. The result shows that $\mathrm{cr}(K_{7,n}) \ge .968Z(7,n) - \Theta(n)$ [3]. DeKlerk, Pasechnik, and Schrijver used an improved version of the same method to prove that $\mathrm{cr}(K_{9,n}) \ge .9667Z(9,n) - \Theta(n)$ [4]. This roughly implies that $\mathrm{cr}(K_{m,n}) \ge .8594Z(m,n)$.

In outline, our argument proceeds as follows. Let m be a fixed positive integer. In Section 2, we introduce *templates.* A drawing of $K_{m,k}$ is a *template of rank m and order k* if no two degree m vertices have the same rotation (in that section, we recall the definition of a rotation). We show that, for each positive integer n, there is an optimal drawing of $K_{m,n}$ that can be obtained from some template B of rank m and order k, by duplicating its degree m vertices (to create a total of n vertices) so that the crossing number of the resulting optimal drawing of $K_{m,n}$ is determined by information contained in the template and the distribution of the duplicate vertices. A key observation is that, for each fixed m, up to isomorphism there are only finitely many templates to consider.

In Section 3, we associate to each template B a quadratic program $\mathrm{QP}(B)$, whose minimum $\mathrm{Min}(B)$ sheds light on the possibility that a drawing of $K_{m,n}$ arising from B is a counterexample to Zarankiewicz's Conjecture.

The proof of Theorem 1.2 has two main parts: the quite easy Proposition 4.1 treats the case of a template for which the minimum of $QP(B)$ is smaller than $Z(m)/4$ (Section 4), while the rather more technical Proposition 5.1 treats the case $\min QP(B) \ge Z(m)/4$ (Section 5). These are put together in the very short Section 6 to prove Theorem 1.2.

2. Drawings and rotations

In this section we do some preliminary work to show that the drawings we need to consider have certain useful properties that reduce the computation of the crossing number to that of a template plus some arithmetic.

Let G be a graph and let D be a drawing of G. We use $\mathrm{cr}(D)$ to denote the number of crossings in D. Thus, the crossing number $\mathrm{cr}(G)$ is the minimum of the $\mathrm{cr}(D)$, over all drawings D of G in the plane. The drawing D is *optimal* if $\mathrm{cr}(D) = \mathrm{cr}(G)$.

Let M be the part of the bipartition of $K_{m,n}$ having the n vertices of degree m; these will be called the *degree-m* vertices (even if $n = m$). We let $[m] = \{0, 1, \dots, m-1\}$ denote the other part of the bipartition. For each degree-m vertex u of $K_{m,n}$, we let $\mathrm{cr}_D(u)$ denote the number of crossings that involve an edge incident with u. If v is a degree-m vertex distinct from u, then we let $\mathrm{cr}_D(u,v)$ denote the number of crossings that involve an edge incident with u and an edge incident with v.

In a drawing D of $K_{m,n}$, each degree-m vertex v induces a natural cyclic order of $[m]$ by considering the clockwise cyclic order of the edges incident with v and using their ends in $[m]$ as their labels. This cyclic order is the *rotation $\pi_D(v)$ of v in D*. There are $(m-1)!$ cyclic permutations of $[m]$, and so this is also the number of possible rotations occurring among the degree-m vertices.

The number $Z(m)$ mentioned earlier is defined to be

$$Z(m) := \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor .$$

It is well-known, and important for us, that if two degree-m vertices u and v have the same rotation a drawing D of $K_{m,2}$, then D has at least $Z(m)$ crossings (for a proof, see [9]). Thus, for any drawing D of $K_{m,n}$, if u and v are distinct degree-m vertices with $\pi_D(u) = \pi_D(v)$, then $\mathrm{cr}_D(u,v) \geq Z(m)$.

Let D be a drawing of $K_{m,n}$ and suppose u and v are distinct degree-m vertices with $\pi_D(u) = \pi_D(v)$ and $\mathrm{cr}_D(u) \geq \mathrm{cr}_D(v)$. We obtain the drawing D' of $K_{m,n}$ from D by deleting u from D and adding a new vertex u' placed near v so that u' mimics v: all the edges incident with u' go very near the corresponding edges incident with v. It is a routine exercise to show that this can be done so that $\mathrm{cr}_{D'}(u', v) = Z(m)$ and, for all other vertices w of $K_{m,n}$, $\mathrm{cr}_{D'}(u,w) = \mathrm{cr}_D(v,w)$. A straightforward verification shows that $\mathrm{cr}(D') \leq \mathrm{cr}(D)$.

For a drawing D of $K_{m,n}$, the degree-m vertices u and v are *duplicates in D* if:

(1) $\pi_D(u) = \pi_D(v)$;
(2) $\mathrm{cr}_D(u,v) = Z(m)$; and
(3) for every other degree-m vertex w, $\mathrm{cr}_D(u,w) = \mathrm{cr}_D(v,w)$.

We have proved the following.

Lemma 2.1. *If D is a drawing of $K_{m,n}$, then there is another drawing D' of $K_{m,n}$ so that:*

(1) $\mathrm{cr}(D') \leq \mathrm{cr}(D)$*; and,*
(2) *if u and v are distinct degree-m vertices such that $\pi_{D'}(u) = \pi_{D'}(v)$, then u and v are duplicates in D'.* □

We continue our "tidying up" of drawings of $K_{m,n}$. For any distinct rotations π and π' of $[m]$, there is a drawing D of $K_{m,2}$ so that one degree-m vertex has the rotation π, while the other has the rotation π', and $\mathrm{cr}(D) < Z(m)$. We wish to show that this property can be enforced on the degree-m vertices of $K_{m,n}$. From Lemma 2.1, we may assume that any two vertices with the same rotation are duplicates.

Suppose D is a drawing of $K_{m,n}$ for which there are degree-m vertices u and v so that $\pi_D(u) \neq \pi_D(v)$, and yet $\mathrm{cr}_D(u,v) \geq Z(m)$. Call such a pair $\{\pi_D(u), \pi_D(v)\}$

of rotations a *bad pair*. We exhibit a drawing D' with fewer bad pairs than D and with $\mathrm{cr}(D') \le \mathrm{cr}(D)$.

Choose the labelling so that $\mathrm{cr}_D(u) \le \mathrm{cr}_D(v)$. Obtain D' from D by deleting v and all its duplicates and making them all duplicates of u. Again, a routine check shows that $\mathrm{cr}(D') \le \mathrm{cr}(D)$ and D' has fewer bad pairs than D. Thus, we have proved the following (the first conclusion being Lemma 2.1).

Lemma 2.2. *If D is a drawing of $K_{m,n}$, then there is a drawing D' of $K_{m,n}$ so that $\mathrm{cr}(D') \le \mathrm{cr}(D)$ and, for any two degree-m vertices u and v:*

(1) *if $\pi_{D'}(u) = \pi_{D'}(v)$, then u and v are duplicates in D'; and*
(2) *if $\pi_{D'}(u) \ne \pi_{D'}(v)$, then $\mathrm{cr}_{D'}(u, v) < Z(m)$.* □

Definition 2.3 (Clean drawings)**.** A drawing D' of $K_{m,n}$ is *clean* if (1) and (2) of Lemma 2.2 hold for D'.

Since we are interested in optimal drawings, in view of Lemma 2.2 we may restrict our attention to clean drawings. The great advantage of this is that, in a clean drawing, all the relevant topological information is contained in a subdrawing of bounded size. To formalize this idea, in the next section we introduce the concept of a template.

3. Templates

In this section we introduce templates and, for each template B, a quadratic program $QP(B)$ that contains significant information about drawings of $K_{m,n}$ based on the template B.

Definition 3.1 (Templates)**.** Let m and k be positive integers.

(1) A *template B of rank m and order k* is a drawing of $K_{m,k}$ in which no two degree-m vertices have the same rotation.
(2) Let $\{v_1, v_2, \dots, v_k\}$ be the set of degree-m vertices of B. For $i, j \in \{1, 2, \dots, k\}$, $i \ne j$, let $q^B_{i,j} := \mathrm{cr}_B(v_i, v_j)$, and let $q^B_{i,i} := Z(m)$. Then the integers $q^B_{i,j}$ define a $k \times k$-matrix $\mathbf{Q}^B = (q^B_{i,j})$, the *crossing matrix* of B. (For brevity, we often omit the reference to B, and simply write $q_{i,j}$ and $\mathbf{Q}$).

Remark 3.2. If B is a template of rank m and order k, then $k \le (m-1)!$. This follows since there are $(m-1)!$ distinct cyclic permutations of $[m]$.

Definition 3.3. Let D be a clean drawing of $K_{m,n}$, and let $v_1, v_2, \dots, v_k$ be a maximal set of degree-m vertices whose rotations in D are pairwise distinct. The drawing of $K_{m,k}$ induced by $v_1, v_2, \dots, v_k$ is (evidently) a template B. We say that B is a *base* for D or, equivalently, that D is an *n-extension* of B.

The following proposition shows that, for an n-extension D of a template B, $\mathrm{cr}(D)$ can be calculated knowing only B and, for each rotation, how many vertices have that rotation in D. The first statement is a simple consequence of Lemma 2.2.

Proposition 3.4. (1) Let D be a drawing of $K_{m,n}$. Then there is an n-extension D' of a base template so that $\mathrm{cr}(D') \le \mathrm{cr}(D)$.
(2) Let B be a clean template of rank m and order k, with degree-m vertices $v_1, v_2, \dots, v_k$, and let $\mathbf{Q} = (q_{i,j})$ be the crossing matrix of B. Let D be an

n-extension of B, with: n_i degree-m vertices in D having the same rotation as v_i; $x_i := n_i/n$; $\mathbf{n} = (n_1, n_2, \ldots, n_k)^T$; and $\mathbf{x} = (x_1, x_2, \ldots, x_k)^T$. Then

$$\operatorname{cr}(D) = \sum_{i=1}^{k} q_{i,i}\binom{n_i}{2} + \sum_{1\le i<j\le k} q_{i,j} n_i n_j \tag{3.1}$$

$$= \frac{1}{2}\mathbf{n}^T\mathbf{Q}\mathbf{n} - \frac{nZ(m)}{2} \tag{3.2}$$

$$= \frac{n^2}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} - \frac{nZ(m)}{2}. \tag{3.3}$$

Proof. We only need to prove Item (2). For $i = 1, 2, \ldots, k$, let S_i denote the set of degree-m vertices in D that have the same rotation as v_i. Let u, w be distinct degree-m vertices in D, and let i, j be the integers such that $u \in S_i$ and $w \in S_j$. Since D is clean, it follows that $\operatorname{cr}_D(u, w) = q_{i,j}$. Now we note that $\operatorname{cr}(D) = (1/2)\sum_{(u,w)} \operatorname{cr}_D(u, w)$, where the summation is over all ordered pairs (u, w) of distinct degree-m vertices in D. Since $|S_i| = n_i$ for $i = 1, 2, \ldots, k$, Equation (3.1) easily follows. Finally, Equations (3.2) and (3.3) follow from (3.1) by straightforward manipulations, since each $q_{i,i}$ is $Z(m)$. □

An elementary counting argument using $\operatorname{cr}(K_{3,n}) = Z(3, n)$ yields that $\operatorname{cr}(K_{m,n})$ is of order m^2n^2, whereas the term $Z(m) \cdot n/2$ in (3.2) is of order m^2n. This is a key hint of the importance of investigating the quadratic expression $\mathbf{x}^T\mathbf{Q}\mathbf{x}$ (as in (3.3)). Motivated by this, we introduce the quadratic program QP(B) associated to a template B.

Definition 3.5. Let B be a template of rank m and order k, and let $\mathbf{Q} = (q_{i,j})$ be the crossing matrix of B. The *quadratic program* QP*(B) associated to B* is:

$$\text{Minimize } f_B(\mathbf{x}) := \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} = \frac{1}{2}\sum_{i,j=1}^{k} q_{i,j}x_ix_j \text{ over all } \mathbf{x} = (x_1, x_2, \ldots, x_k) \text{ s.t.}$$

$$(a)\ \sum_{i=1}^{k} x_i = 1; \text{ and } (b)\ x_i \ge 0 \text{ for } i = 1, 2, \ldots, k.$$

We use the notation Min(B) for the minimum of QP(B). Theorem 1.2 follows easily from two statements (namely Proposition 4.1 for the case Min$(B) < Z(m)/4$ and Proposition 5.1 for the case Min$(B) \ge Z(m)/4$) about whether template B has an extension that is a counterexample to Zarankiewicz' Conjecture.

We finish this section by proving the existence of an optimal solution for QP(B), each of whose coordinates can be expressed as a quotient of integers that are bounded above by an explicit function of m. Let $N_2(m) := (2Z(m))^{(m-1)!}(m!)!$.

Proposition 3.6. Let B be a clean template of rank m and order k. Then there exist nonnegative integers $p_1, p_2, \ldots, p_k$, with $p := \sum_{i=1}^{k} p_i \le N_2(m)$, such that $(p_1/p,\ p_2/p \ldots, p_k/p)$ achieves the minimum for QP(B).

Proof. Among all optimal solutions for QP(B), let $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_k^*)$ be one in which the number of nonzero coordinates is least possible. By rearranging the columns and rows of $\mathbf{Q}$ if necessary, we may assume that there is an s, $1 \le s \le k$, such that $x_i^* > 0$ for $i \le s$, and $x_i^* = 0$ for $i > s$. If $s = 1$ then we are clearly done (setting $p_1 = p = 1$, and $p_2 = p_3 = \cdots = p_k = 0$ yields the required optimal solution), and so we assume that $s \geq 2$.

Since $\mathbf{x}^*$ is an optimal solution for QP(B), it immediately follows that $(x_1^*, x_2^*, \ldots, x_s^*)$ is an optimal solution for the quadratic program QP* on the variables $x_1, x_2, \ldots, x_s$ obtained from QP(B) by forcing all of $x_{s+1}, \ldots, x_k$ to be 0.

Using the constraint $\sum_{i=1}^{s} x_i = 1$, we can eliminate the variable x_s in QP*. The constraint $x_s > 0$ becomes $\sum_{i=1}^{s-1} x_i < 1$. Thus it follows that $(x_1^*, x_2^*, \ldots, x_{s-1}^*)$ is an optimal solution for a quadratic program QP** of the form:

$$\min\{g(x_1, \ldots, x_{s-1}) \mid \sum_{i=1}^{s-1} x_i < 1, \text{ and, for } i = 1, 2, \ldots, s-1, \; x_i > 0\}.$$

Because the feasible domain is open, $\nabla g(x_1^*, \ldots, x_{s-1}^*) = \mathbf{0}$. In fact, we now show $(x_1^*, x_2^*, \ldots, x_{s-1}^*)$ is the unique solution of $\nabla g = \mathbf{0}$.

As g is a quadratic function, $\nabla g(x_1, x_2, \ldots, x_{s-1}) = \mathbf{0}$ is a system of linear equations, so its solution set is a subspace W of $\mathbb{R}^{s-1}$. The mean value theorem implies that, for every $(u_1, u_2, \ldots, u_{s-1}) \in W$, $g(u_1, u_2, \ldots, u_{s-1}) = g(x_1^*, x_2^*, \ldots, x_{s-1}^*)$.

The closure of the feasible region for QP** is a compact set. If W has positive dimension, then W must have a point on the boundary of the feasible region; any such point yields an optimal solution of QP(B) having fewer non-zero components than $(x_1^*, \ldots, x_k^*)$, a contradiction.

We now know that $(x_1^*, x_2^*, \ldots, x_{s-1}^*)$ is the unique solution of the system of $s-1$ linear equations in $s-1$ unknowns $\nabla g = \mathbf{0}$, or:

$$\sum_{i=1}^{s-1} \Big(q_{i,\ell} + q_{s,s} - q_{i,s} - q_{\ell,s} \Big) x_i = q_{s,s} - q_{\ell,s}, \qquad \text{for} \quad \ell = 1, 2, \ldots, s-1.$$

Since $0 \leq q_{i,j} \leq Z(m)$ for all $i, j \in \{1, 2, \ldots, k\}$, in absolute value each of these coefficients is at most $2Z(m)$. Cramer's Rule and the permutation expansion for the determinant imply the (equal) denominators are all at most $(2Z(m))^{s-1}(s-1)!$. Since $s \leq k \leq (m-1)!$, this is easily seen to be at most N_2. □

4. The case Min(B) $< Z(m)/4$

In this section, we treat the easier of the two parts of the proof of Theorem 1.2 by showing that if B is a template with Min(B) $< Z(m)/4$, then there is a (not very large) n so that $\mathrm{cr}(K_{m,n}) < Z(m,n)$. Recall that $N_0(m) = ((2Z(m))^{m!}(m!)!)^4$.

Proposition 4.1. Let m be any positive integer, and let B be a clean template of rank m. If Min(B) $< Z(m)/4$, then, for each $n \geq N_0(m)$, there is an n-extension D of B such that $\mathrm{cr}(D) < Z(m,n)$. In particular, for each $n \geq N_0(m)$, $\mathrm{cr}(K_{m,n}) < Z(m,n)$.

Proof. First we note that $m \geq 3$, as otherwise $Z(m) = 0$, and we obtain the contradiction that Min(B) < 0.

Let $v_1, v_2, \ldots, v_k$ be the degree-m vertices and let $\mathbf{Q} = (q_{i,j})$ be the crossing matrix of B. By Proposition 3.6, there exist integers $p_1, p_2, \ldots, p_k$, with $p = \sum_{i=1}^{k} p_i \leq N_2$, such that $\mathbf{x} = (p_1/p, p_2/p, \ldots, p_k/p)$ is an optimal solution of QP(B).

We observe that $p \geq 2$, as otherwise $(1/2)\mathbf{x}^T\mathbf{Q}\mathbf{x} = Z(m)/2$, a contradiction.

By hypothesis, $f_B(\mathbf{x}) = Z(m)/4 - \epsilon$, where $\epsilon > 0$. Since $f_B(\mathbf{x}) = (1/2)\mathbf{x}^T\mathbf{Q}\mathbf{x}$, and $\mathbf{Q}$ is an integral matrix, it follows that $f_B(\mathbf{x}) = L/2p^2$ for some integer L, and therefore $\epsilon = \ell/4p^2$ for some integer ℓ. In particular, $\epsilon \geq 1/4p^2$.

Let n be any positive integer, and let r, s be the unique integers such that $n = rp + s$, with $0 \leq s < p$. Let $n_i = rp_i + \delta_i$ for $i = 1, 2, 3, \ldots, k$, with s of the δ_i equal to 1 and the rest 0. Note that $n = \sum_{i=1}^{k} n_i$ and, for each i, $n_i \leq (r+1)p_i$. Consider an n-extension D of B, in which, for $i = 1, 2, \ldots, k$, there are n_i degree-m vertices with rotation $\pi_B(v_i)$. From Proposition 3.4(2), where $\mathbf{n} = (n_1, n_2, \ldots, n_k)^T$,

$$\operatorname{cr}(D) = \frac{1}{2}\mathbf{n}^T\mathbf{Q}\mathbf{n} - \frac{nZ(m)}{2}.$$

Since each $n_i \leq (r+1)p_i = ((r+1)p)(p_i/p) \leq (n+p)(p_i/p)$, and $(1/2)\mathbf{x}^T\mathbf{Q}\mathbf{x} \leq (Z(m)/4) - (1/4p^2)$, it follows that

$$\operatorname{cr}(D) \leq (n+p)^2\left(\frac{Z(m)}{4} - \frac{1}{4p^2}\right) - \frac{nZ(m)}{2}.$$

Now observe that $Z(m)(n-1)^2/4 \leq Z(m,n) + (Z(m)/4)$, that is, $\big((n-1)^2 - 1\big)(Z(m)/4) \leq Z(m,n)$. Therefore, using $(n+p)^2 = \big((n-1)^2 - 1\big) + 2n + 2np + p^2$, we readily get

$$\operatorname{cr}(D) \leq Z(m,n) + \big(2np + p^2\big)\frac{Z(m)}{4} - \frac{(n+p)^2}{4p^2}. \tag{4.1}$$

Claim 4.2. For $n \geq N_0$, $(n+p)^2/(4p^2) > (2np+p^2)(Z(m)/4)$.

Proof. First we note that $m \geq 3$ implies

$$(2Z(m))^{m!} \geq 2Z(m)(2Z(m))^{(m-1)!},$$

which trivially implies

$$\left((2Z(m))^{m!}\right)^4 \geq 2Z(m)\left((2Z(m))^{(m-1)!}\right)^4.$$

Multiplying both sides by $((m!)!)^4$ shows $N_0 \geq 2Z(m)N_2^4$.

Note that $n \geq N_0$ and $N_2 \geq p$, so $n \geq 2Z(m)p^4$. Obviously, $2n > p$. Since $p \geq 2$, $p(p-1) \geq p$, and so

$$2np(p-1) > p^2.$$

Thus, $2np^2 > 2np + p^2$.

Consequently,

$$\frac{(n+p)^2}{4p^2} > \frac{n^2}{4p^2} \geq \frac{n(2p^4Z(m))}{4p^2} > \frac{(2np+p^2)Z(m)}{4}. \qquad \square$$

Inequality (4.1) and Claim 4.2 show that if $n \geq N_0$, then $\operatorname{cr}(D) < Z(m,n)$, completing the proof of the proposition. $\square$

5. The case $\mathrm{Min}(B) \geq Z(m)/4$

In this section, we aim for the other half of the proof of Theorem 1.2, now considering a template B for which $\mathrm{Min}(B) \geq Z(m)/4$. Let

$$N_1(m) := (Z(m)+2)^{2m!}(2m!)!\,.$$

Proposition 5.1. Let B be a template with $\mathrm{Min}(B) \geq Z(m)/4$. If there is an n-extension D of B so that $\mathrm{cr}(D) < Z(m,n)$, then there is such an n-extension for which $n \leq N_1$.

The rest of this section is devoted to proving Proposition 5.1. Throughout, B, D, and n are as in the hypotheses of the proposition. For $i = 1, 2, \ldots, k$, let n_i be the number of vertices having rotation π_i in D and let $x_i = n_i/n$. Equation (3.2) shows that, for $\mathbf{x} = (x_1, \ldots, x_k)^T$,

$$\mathrm{cr}(D) = \frac{n^2}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} - \frac{Z(m)n}{2}\,.$$

Since $\mathbf{x}$ is a feasible point for $\mathrm{QP}(B)$, $\frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x}$ is at least $\mathrm{Min}(B)$ and, therefore, at least $Z(m)/4$. Thus,

$$\mathrm{cr}(D) \geq \frac{Z(m)n^2}{4} - \frac{Z(m)n}{2}\,. \tag{5.1}$$

As $Z(m,n) = Z(m)Z(n)$ and, when n is even, $Z(n) = n(n-2)/4 = (n^2/4) - (n/2)$, we see that, if n is even, $\mathrm{cr}(D) \geq Z(m,n)$. In fact, we have proved the following.

Observation 5.2. For every even integer s, if D' is an s-extension of B, then $\mathrm{cr}(D') \geq Z(m,s)$. In particular, n is odd. □

For $i = 1, 2, \ldots, k$, if v is a degree-m vertex with rotation π_i in D, then set $t_i = 2\mathrm{cr}_D(v) - Z(m)n$. (We note that t_i is independent of the choice of v.)
Then

$$4\mathrm{cr}(D) = \sum_v 2\mathrm{cr}_D(v) = \sum_{i=1}^{k}(t_i + Z(m)n)n_i\,, \text{ or} \tag{5.2}$$

$$\sum_{i=1}^{k} t_i n_i = 4\mathrm{cr}(D) - Z(m)n^2\,. \tag{5.3}$$

For any degree-m vertex v with rotation π_i, $\mathrm{cr}_D(v)$ may be computed as follows:

$$\mathrm{cr}_D(v) = Z(m)(n_i - 1) + \sum_{j\neq i} q_{i,j}n_j\,.$$

Replacing $\mathrm{cr}_D(v)$ with $(Z(m)n + t_i)/2$ and rearranging, we get

$$t_i = 2Z(m)n_i - 2Z(m) - Z(m)n + \sum_{j\neq i} 2q_{i,j}n_j\,.$$

Since $n = \sum_{i=1}^{k} n_i$, this becomes

$$t_i = Z(m)n_i - 2Z(m) + \sum_{j\neq i}(2q_{i,j} - Z(m))n_j\,. \tag{5.4}$$

We define ε by $\mathrm{cr}(D) = Z(m,n) - \varepsilon$, so $\varepsilon > 0$. Substituting this into (5.3), we get

$$\sum_{i=1}^{k} t_i n_i = 4Z(m)(n-1)^2/4 - 4\varepsilon - Z(m)n^2\,, \text{ or} \tag{5.5}$$

$$\sum_{i=1}^{k} t_i n_i = Z(m)(-2n+1) - 4\varepsilon\,. \tag{5.6}$$

In this framework, we see that, if $\mathrm{cr}(D) < Z(m,n)$, then, for some $\varepsilon > 0$ and some integers $t_1, \dots, t_k$, there are non-negative integers $n_1, \dots, n_k$ so that $n = \sum_{i=1}^{k} n_i$ is odd, and both (5.4) and (5.6) both hold.

Conversely, suppose there exist $\varepsilon > 0$, integers $t_1, t_2, \dots, t_k$, and non-negative integers $n_1, \dots, n_k$ so that $n = \sum_i n_i$ is odd and both (5.4) and (5.6) hold. Let D' be an n-extension of B so that there are n_i vertices with rotation π_i. From the above remarks, (5.4) implies that $t_i = 2\mathrm{cr}_{D'}(v) - Z(m)n$, for each vertex v having rotation π_i. Therefore, (5.3) implies $\sum_i t_i n_i = 4\mathrm{cr}(D') - Z(m)n^2$. This together with (5.6) implies $\mathrm{cr}(D') = Z(m,n) - \varepsilon < Z(m,n)$.

We summarize the above remarks as follows.

Proposition 5.3. Let B be a template with rotations $\pi_1, \dots, \pi_k$ and suppose $\mathrm{Min}(B) \ge Z(m)/4$. Then there is an n-extension D of B so that $\mathrm{cr}(D) < Z(m,n)$ if and only if, for some integers $t_1, \dots, t_k$, there is a solution in non-negative integers $n_1, n_2, \dots, n_k, r$ to:

(1) for $i = 1, \dots, k$,

$$t_i = Z(m)n_i - 2Z(m) + \sum_{j \ne i} (2q_{i,j} - Z(m))n_j\,; \tag{5.7}$$

(2) the inequality $\sum_{i=1}^{k} n_i t_i \le Z(m)(1-2n) - 4$, where $n = \sum_i n_i$; and
(3) $n = 2r+1$. □

By itself, Proposition 5.3 is not enough to prove Proposition 5.1. We need to show:

(1) that only finitely many possible k-tuples $(t_1, t_2, \dots, t_k)$ need be considered; and
(2) that, for each one, if it yields a solution $n_1, n_2, \dots, n_k$, then $\sum_{i=1}^{k} n_i \le N_1$.

The remainder of this section is devoted to these two points.

Recall that D is an n-extension of B so that $\mathrm{cr}(D) < Z(m,n)$ and n is odd. There are n_i vertices with rotation π_i, $i = 1, 2, \dots, k$. If v is a degree-m vertex with rotation π_i, then we can obtain an $(n+1)$-extension D^+ of B by adding a duplicate of v and an $(n-1)$-extension D^- of B by deleting v. It is easy to compute $\mathrm{cr}(D^-) = \mathrm{cr}(D) - \mathrm{cr}_D(v)$ and $\mathrm{cr}(D^+) = \mathrm{cr}(D) + \mathrm{cr}_D(v) + Z(m)$. (For the latter, the duplicate has the same crossings as v plus its crossings with edges incident with v.)

Since $n-1$ and $n+1$ are both even, Observation 5.2 implies $\mathrm{cr}(D^-) \geq Z(m, n-1)$ and $\mathrm{cr}(D^+) \geq Z(m, n+1)$. It follows that

$$\begin{aligned} Z(m, n+1) &\leq \mathrm{cr}(D) + \mathrm{cr}_D(v) + Z(m) \\ &< Z(m,n) + \mathrm{cr}_D(v) + Z(m), \text{ or,} \\ Z(m)\left(\frac{n^2-1}{4}\right) &< Z(m)\left(\frac{(n-1)^2}{4}\right) + \mathrm{cr}_D(v) + Z(m), \end{aligned}$$

so that

$$t_i = 2\mathrm{cr}_D(v) - Z(m)n \geq -3Z(m) - 2\,.$$

Likewise, $Z(m, n-1) \leq \mathrm{cr}(D^-)$ implies that $t_i \leq -Z(m) - 2$. Accordingly, for each $i = 1, 2, \dots, k$, $-3Z(m)-2 \leq t_i \leq -Z(m)-2$, so there are indeed only finitely many choices for $(t_1, t_2, \dots, t_k)$.

Finally, suppose that $t_1, t_2, \dots, t_k$ are such that, for each $i = 1, 2, \dots, k$, $-3Z(m)-2 \leq t_i \leq -Z(m) - 2$ and that there are non-negative integers $n_1, n_2, \dots, n_k, r$ satisfying the constraints (1)–(3) in Proposition 5.3. We show that there is such a solution $n_1, n_2, \dots, n_k, r$ so that $\sum_{i=1}^k n_i \leq N_1$. The following result from [8] does this for us.

Theorem 5.4 (Von Zur Gathen-Sieveking, 1978)**.** *Let* $\mathbf{A}, \mathbf{b}, \mathbf{C}$ *and* $\mathbf{d}$ *be* $p \times r$*-,* $p \times 1$*-,* $q \times r$*-, and* $q \times 1$*-matrices respectively with integer entries. Let* s *be the rank of* $\mathbf{A}$*, and let* T *be the rank of the matrix* $\binom{\mathbf{A}}{\mathbf{C}}$*. Let* M *be an upper bound on the absolute values of those* $(T-1)\times(T-1)$*- or* $T \times T$*-subdeterminants of the matrix*

$$\begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{C} & \mathbf{d} \end{pmatrix}$$

which are formed with at least s *rows from* $(\mathbf{A}, \mathbf{b})$*. If* $\mathbf{Ax} = \mathbf{b}$ *and* $\mathbf{Cx} \geq \mathbf{d}$ *have a common integral solution, then they have one in which each coordinate is bounded by* $(T+1)M$*.*

In our context, we have the k equalities (5.7), plus $n = 2r+1$, yielding $k+1$ equations in $k+1$ unknowns. We also have $k+1$ inequalities (non-negativity of the n_i and (5.6)) in the same $k+1$ unknowns. Therefore, the rank of $\binom{\mathbf{A}}{\mathbf{C}}$ is obviously at most $(k+1)$. For M, we note that the absolute value of each entry in the matrix $\left(\begin{smallmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{C} & \mathbf{d} \end{smallmatrix}\right)$ is bounded by $3Z(m)+2$. Thus, any subdeterminant (having at most $k+1$ rows) of this matrix is bounded in absolute value by $(3Z(m)+2)^{k+1}(k+1)!$. We may take this value for M.

Theorem 5.4 implies that, if the constraints in Proposition 5.3 admit an integral solution, then it admits an integral solution in which each n_i is at most $(2k+3)(Z(m)+2)^{2k+2}(2k+2)!$. For such a solution, $\sum_{i=1}^k n_i \leq k(2k+3)(Z(m)+2)^{2k+2}(2k+2)!$. Now since $k \leq (m-1)!$, it follows that $\sum_{i=1}^k n_i \leq (m-1)!(2(m-1)!+3)(Z(m)+2)^{2(m-1)!+2}(2(m-1)!+2)!$. It is easy to check that this last expression is at most N_1. This completes the proof of Proposition 5.1.

6. Proof of Theorem 1.2

In this short section, we use Propositions 4.1 and 5.1 to prove Theorem 1.2.

Proof of Theorem 1.2. Let m, n be positive integers. By hypothesis, if $n \leq N_0$, then $\mathrm{cr}(K_{m,n}) = Z(m,n)$. Thus we let $n > N_0$, and finish the proof by showing that $\mathrm{cr}(K_{m,n}) = Z(m,n)$. As we have observed, Zarankiewicz' construction shows that $\mathrm{cr}(K_{m,n}) \leq Z(m,n)$, so we need to prove the reverse inequality $\mathrm{cr}(K_{m,n}) \geq Z(m,n)$.

Let D be an optimal drawing of $K_{m,n}$. Our aim is to show that $\mathrm{cr}(D) \geq Z(m,n)$.

Lemma 2.2 implies that we may assume that D an n-extension of a template B.

If $\mathrm{Min}(B) < Z(m)/4$, then it follows from Proposition 4.1 that $\mathrm{cr}(K_{m,N_0}) < Z(m,N_0)$, contradicting the hypothesis of the theorem.

Therefore $\mathrm{Min}(B) \geq Z(m)/4$. By hypothesis, $\mathrm{cr}(K_{m,n'}) = Z(m,n')$ for all $n' \leq N_0$. It is easy to check that, for every $m \geq 3$, $N_1 \leq N_0$. In particular it follows that, for every $n' \leq N_1$, every n'-extension of B has at least $Z(m,n')$ crossings. Therefore Proposition 5.1 applies, and so $\mathrm{cr}(D) \geq Z(m,n)$, as required. □

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Department of Combinatorics & Optimization, University of Waterloo, Waterloo, ON, N2L 3G1

E-mail address: r3christ@math.uwaterloo.ca

Department of Combinatorics & Optimization, University of Waterloo, Waterloo, ON, N2L 3G1

E-mail address: brichter@uwaterloo.ca

Instituto de Fisica, Universidad Autonoma de San Luis Potosi, San Luis Potosi, SLP, Mexico 78000

E-mail address: gsalazar@ifisica.uaslp.mx