Infinite families of crossing–critical graphs with given average degree

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Abstract. In their paper on minimal graphs with crossing number at least k (or, equivalently, k–crossing–critical graphs), Richter and Thomassen described a construction of an infinite family of 4–regular 3–crossing–critical graphs. They also showed that no infinite family of 6–regular k–crossing–critical graphs exists. In this paper we generalize the Richter and Thomassen construction, and prove the following result: for each rational number $q \in [4,6)$ there is an infinite family of graphs that are k–crossing–critical for the same integer k, each of which has average degree q. Moreover, we show that for each rational number $q \in [4,6)$ this statement holds for infinitely many values of k.

1 Introduction

In [2], Richter and Thomassen established several interesting results on the set $\mathcal{M}_k$ of graphs which are k–crossing–critical, or, equivalently, which are minimal with respect to having crossing number at least k. A graph G is *k–crossing–critical* if the crossing number $\mathrm{cr}(G)$ of G is at least k, and $\mathrm{cr}(G-e) < k$ for every edge e of G. If G is k–crossing–critical for some integer k, then G is *crossing–critical.*

One of the main results in [2] is that if G is k–crossing–critical, then its crossing number is at most $2.5k+16$. An interesting corollary of this is that if G is any graph with crossing number k, then there is an edge e of G such that $\mathrm{cr}(G-e) \geq (2k-37)/5$.

Richter and Thomassen also investigated infinite families of graphs that belong to $\mathcal{M}_k$ for the same value of k. They proved that if $\mathcal{M}_k$ has infinitely many r–regular simple graphs, then $r = 4$ or 5, and exhibited an infinite class of 4–regular simple graphs in $\mathcal{M}_3$.

Moreover, it is easy to check that the argument used to prove Corollary 2 in [2] yields the following general result.

Proposition 1 *Let $\epsilon > 0$, and let k be an integer. Then $\mathcal{M}_k$ has finitely many simple graphs with average degree at least $6+\epsilon$.*

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This raises the following question. Let $\epsilon > 0$ be given. Is there an integer k such that $\mathcal{M}_k$ has an infinite family of simple graphs with average degree at least $6 - \epsilon$? In other words, is there an integer k such that there exists an infinite family of k–crossing–critical simple graphs, each of which has average degree at least $6 - \epsilon$?

In this paper we show that the answer to this question is yes. Actually, we show that the infinite family can be chosen so that all its graphs have the same average degree. Moreover, we show that this average degree can be any number in $[4, 6)$ fixed in advance, with the obvious constraint that it be rational.

Theorem 2 *For each rational number $q \in [4, 6)$, there is an integer $k(q)$ and an infinite family $\mathcal{G}_q$ of simple 4–connected graphs such that (i) each graph in $\mathcal{G}_q$ has average degree q, and (ii) each graph in $\mathcal{G}_q$ is $k(q)$–crossing–critical. Moreover, all the graphs in $\mathcal{G}_q$ have the same crossing number.*

Moreover, we show that for each rational number $q \in [4, 6)$, there are infinitely many values of $k(q)$ for which this statement holds.

Theorem 3 *For each rational number $q \in [4, 6)$, there is an increasing sequence $\{k_1(q), k_2(q), \ldots\}$ of integers with the following property. For each $k_i(q)$ there exists an infinite family $\mathcal{G}_q^i$ of simple 4–connected graphs such that (i) each graph in $\mathcal{G}_q^i$ has average degree q, and (ii) each graph in $\mathcal{G}_q^i$ is $k_i(q)$–crossing–critical. Moreover, for each fixed i, all the graphs in $\mathcal{G}_q^i$ have the same crossing number.*

The first construction of crossing–critical graphs appeared in [3]. Širáň's construction yields graphs with multiple edges. An infinite family of crossing–critical simple 3–connected graphs was then exhibited by Kochol in [1].

This paper is organized as follows. In Section 2 we describe the three–parameter family of graphs $T(m, n, r)$. These graphs play the major role in the proofs of Theorems 2 and 3. The average degree of $T(m, n, r)$ depends only on m and r, and, by adjusting these two parameters, it can be made equal to any rational number in $[4, 6)$. As we show in Section 2, there is an upper bound for the crossing number of $T(m, n, r)$ which does not depend on n. On the other hand, we show in Section 3 that the graphs $T(m, n, r)$ are crossing–critical if n is sufficiently large compared to m and r. All this information is then gathered in Section 4, where Theorems 2 and 3 are proved. Section 5 contains some final remarks.

2 The Main Construction

Our aim in this section is to define the three–parameter family of graphs $T(m, n, r)$.

Let m, n, r, be nonnegative integers, such that (i) m is positive and even; (ii) n is odd; (iii) $n \geq m$; and (iv) $r \leq m - 1$. The graph $T(m, n, r)$ has mn vertices, labeled $v(i, j)$, for i, j such that $0 \leq i \leq m - 1$, $0 \leq j \leq n - 1$.

Remark *Operations on the indices of the vertices are performed modulo m and n, respectively.*

The adjacency rules for the vertices in $T(m,n,r)$ are given as follows:

(i) For every $i \in \{0,\ldots,m-2\}$, i even, and $j \in \{0,\ldots,n-1\}$, there is a *blue edge of depth* $i+1$ that joins $v(i,j)$ to $v(i+1,j')$ if and only if $j-j' \in \{0,1\}$.

(ii) For every $i \in \{1,\ldots,m-3\}$, i odd, and $j \in \{0,\ldots,n-1\}$, there is a *blue edge of depth* $i+1$ that joins $v(i,j)$ to $v(i+1,j')$ if and only if $j'-j \in \{0,1\}$.

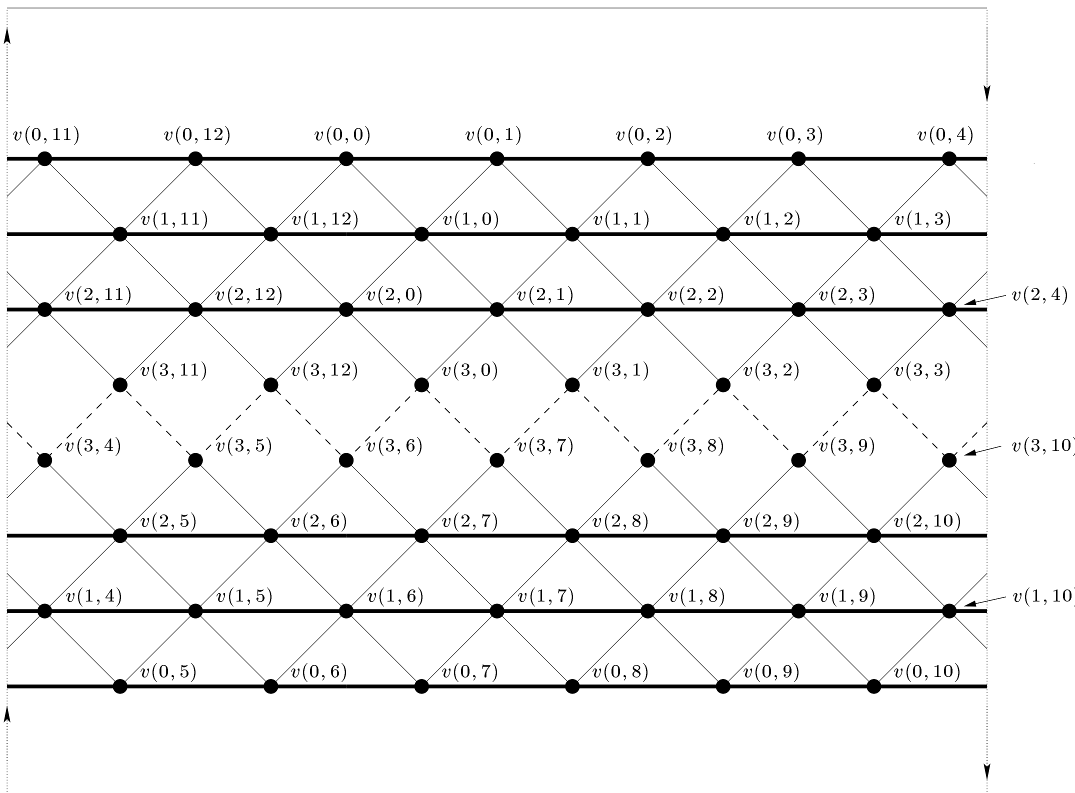

FIGURE 1. A drawing of $T(4,13,2)$ in the Möbius band. Thick segments correspond to red edges, thin segments correspond to blue edges, and dotted segments correspond to green edges. The left and right sides of the rectangle are to be identified as specified by the corresponding arrows. After the identification, the upper and lower sides of the rectangle yield the boundary of the Möbius band.

(iii) For every $i \in \{0,\ldots,r\}$ and $j \in \{0,\ldots,n-1\}$, there is a *red edge of depth* i that joins $v(i,j)$ to $v(i,j')$ if and only if $j-j' \in \{1,n-1\}$.

(iv) For every $j \in \{0,\ldots,n-1\}$, there is a *green edge* that joins $v(m-1,j)$ to $v(m-1,j')$ if and only if $j-j' \in \{(n-1)/2,(n+1)/2\}$.

For convenience, we say that every green edge has depth m.

It is straightforward to check that, for each $i_0 \in \{1, \ldots, m-1\}$, the blue edges of depth i_0 induce a $2n$–cycle, the *blue cycle* B_{i_0} *of depth* i_0. Similarly, for each $i_0 \in \{0, \ldots, r\}$, the red edges of depth i_0 induce an n–cycle, the *red cycle* R_{i_0} *of depth* i_0. Finally, the green edges induce an n–cycle, the *green cycle* G. Thus, for instance, for the graph $T(4, 13, 2)$ in Figure 1, the blue cycle B_2 of depth 2 is the 26–cycle $(v(1,0), v(2,1), v(1,1), v(2,2), v(1,2),$ $v(2,3), \ldots, v(2,12),$ $v(1,12), v(2,0), v(1,0))$.

Thus, each edge in $T(m,n,r)$ belongs either to B_i for some i, $1 \le i \le m-1$, or to R_i for some i, $0 \le i \le r$, or to the green cycle G.

We now establish some basic properties of the graphs $T(m,n,r)$. The proof of the following statement is straightforward.

Proposition 4 *The average degree of* $T(m,n,r)$ *is* $4 + 2r/m$. ∎

It is a routine exercise to exhibit drawings of $T(m,n,r)$ with exactly $\binom{2(m+r)+1}{2}$ crossings. This implies the following.

Proposition 5 *The crossing number of* $T(m,n,r)$ *is at most* $\binom{2(m+r)+1}{2}$. ∎

As we shall show in Section 3, if n is sufficiently large then $T(m,n,r)$ is crossing–critical. In order to prove this property we need to exploit the symmetries of $T(m,n,r)$.

Proposition 6 *Suppose that* e, e' *are edges of the same color and of the same depth in* $T(m,n,r)$. *Then there is an automorphism of* $T(m,n,r)$ *that takes* e *to* e'.

Proof. Let V denote the vertex set of $T(m,n,r)$. Let $\tau : V \to V$ (the *translation*) and $\rho : V \to V$ (the *rotation*) be defined by the following rules: $\tau(v(i,j)) = v(i, j+1)$; $\rho(v(i,j)) = v(i, n-1-j)$ if i is even, and $\rho(v(i,j)) = v(i, n-2-j)$ if i is odd.

It is readily checked that both τ and ρ define automorphisms of $T(m,n,r)$. Moreover, it is easy to check that if e, e' are edges of the same color and of the same depth, then there is an integer j such that either τ^j or $\tau^j \circ \rho$ takes e to e'. ∎

3 $T(m,n,r)$ is crossing–critical for sufficiently large n

Our aim in this section is to show that if n is sufficiently large compared to m and r, then $T(m,n,r)$ is crossing–critical.

To prove the criticality of $T(m,n,r)$, we need to show that for every edge e of $T(m,n,r)$, $\mathrm{cr}(T(m,n,r) - e) < \mathrm{cr}(T(m,n,r))$. We analyze two cases separately. In Lemma 8 we deal with the case where e is either blue or red, and in Lemma 10 we analyze the case where e is green. These lemmas are established as consequences of Propositions 7 and 9, respectively.

Proposition 7 *Let* i_0 *be an integer, and let* $\mathcal{D}$ *be a drawing of* $T(m,n,r)$ *in which either (i) no blue edge of depth* i_0 *is crossed, or (ii) no red edge of depth* i_0 *is crossed. Then* $\mathcal{D}$ *has at least* $\binom{\lfloor((n-1)/2)/2\rfloor}{2} + \binom{\lceil((n-1)/2)/2\rceil}{2}$ *crossings.*

Proof. Suppose that there is a cycle $C_{i_0} \in \{B_{i_0}, R_{i_0}\}$ that is not crossed in $\mathcal{D}$. For each k such that $0 \leq k \leq (n-3)/2$, let P_k be the path $(v(i_0,k), v(i_0+1,k), \ldots, v(m-1,k), v(m-1,k+(n-1)/2), v(m-2,k+(n-1)/2), \ldots, v(i_0+1,k+(n-1)/2), v(i_0,k+(n-1)/2))$.

Thus, each P_k has both end vertices in C_{i_0}, and the $(n-1)/2$ paths P_k are pairwise disjoint (as subgraphs of $T(m,n,r)$). Moreover, if $j \neq k$, then the end vertices (say u_j, v_j) of P_j and the end vertices (say u_k, v_k) of P_k occur in C_{i_0} in the cyclic order u_j, u_k, v_j, v_k.

Since by assumption no edge in C_{i_0} is crossed in $\mathcal{D}$, it follows that each path P_k is contained in one of the two components of $\mathbb{R}^2 \setminus C_{i_0}$. On the other hand, if P_j and P_k are in the same component of $\mathbb{R}^2 \setminus C_{i_0}$, then some edge in P_j crosses some edge in P_k (this follows from the statement at the end of the previous paragraph).

Therefore, if there are s P_k's in the bounded region of $\mathbb{R}^2 \setminus C_{i_0}$ and t P_k's in the unbounded region of $\mathbb{R}^2 \setminus C_{i_0}$, then the paths P_k have at least $\binom{s}{2} + \binom{t}{2}$ crossings in total. Since this sum is minimized when s and t are as close as possible to each other, the result follows. ∎

Lemma 8 *Let e be a blue or red edge in $T(m,n,r)$. If n is sufficiently large compared to m and r, then $cr(T(m,n,r)-e) < cr(T(m,n,r))$.*

Proof. In view of Proposition 6, it suffices to show that, if n is sufficiently large, then in every optimal drawing of $T(m,n,r)$ (i) for each i, $1 \leq i \leq m-1$, some blue edge of depth i is crossed, and (ii) for each i, $0 \leq i \leq r$, some red edge of depth i is crossed.

Let $\mathcal{D}$ be a drawing of $T(m,n,r)$ in which there is an i_0, $1 \leq i_0 \leq m-1$, such that no blue edge of depth i_0 is crossed. Then, by Proposition 7, $\mathcal{D}$ has at least $\binom{\lfloor((n-1)/2)/2\rfloor}{2} + \binom{\lceil((n-1)/2)/2\rceil}{2}$ crossings. For every n sufficiently large, this number is greater than $\binom{2(m+r)+1}{2}$, which by Proposition 5 is an upper bound for the crossing number of $T(m,n,r)$. Thus, $\mathcal{D}$ cannot be optimal if there is an i_0, $1 \leq i_0 \leq m-1$, such that no blue edge of depth i_0 is crossed. This proves (i). Statement (ii) is proved by a virtually identical argument. ∎

Proposition 9 *Let $\mathcal{D}$ be a drawing of $T(m,n,r)$ in which no green edge is crossed. Then $\mathcal{D}$ has at least n crossings.*

Proof. Let $\mathcal{D}$ be a drawing of $T(m,n,r)$ in which no green edge is crossed. It is easy to see that all vertices in $T(m,n,r)$ not in G must be contained in the same component of $\mathbb{R}^2 \setminus G$.

Consider the n pairwise internally disjoint 2–paths $Q_k = (v(m-1,k), v(m-2,k+1), v(m-1,k+1))$, $k = 0,\ldots,n-1$. For each $k \in \{0,\ldots,n-1\}$, the ends (namely $v(m-1,k)$ and $v(m-1,k+1)$) of Q_k and the ends (namely $v(m-1,k+(n+1)/2)$ and $v(m-1,k+(n+1)/2+1)$) of $Q_{k+(n+1)/2}$ occur in G in the cyclic order $v(m-1,k), v(m-1,k+(n+1)/2), v(m-1,k+1), v(m-1,k+(n+1)/2+1)$. Thus, since Q_k and $Q_{k+(n+1)/2}$ are drawn in the same component of $\mathbb{R}^2 \setminus G$, it follows that some edge in Q_k crosses in $\mathcal{D}$ some edge in $Q_{k+(n+1)/2}$, for each $k \in \{0,\ldots,n-1\}$. ∎

Lemma 10 *Let e be a green edge in $T(m,n,r)$. If n is sufficiently large compared to m and r, then $cr(T(m,n,r)-e) < cr(T(m,n,r))$.*

Proof. In view of Proposition 6, it suffices to show that if n is sufficiently large compared to m and r, then in every optimal drawing of $T(m,n,r)$ some green edge is crossed. Equivalently, it suffices to show that if $\mathcal{D}$ is a drawing of $T(m,n,r)$ in which no green edge is crossed, and n is sufficiently large compared to m and r, then $\mathcal{D}$ is not optimal.

Let $\mathcal{D}$ be a drawing of $T(m,n,r)$ in which no green edge is crossed. By Proposition 9, $\mathcal{D}$ has at least n crossings. On the other hand, by Proposition 5, an optimal drawing of $T(m,n,r)$ has at most $\binom{2(m+r)+1}{2}$ crossings. Therefore, if n is sufficiently large compared to m and r, then $\mathcal{D}$ cannot be optimal. ∎

Since every edge in $T(m,n,r)$ is either blue, red, or green, Lemmas 8 and 10 imply the main result in this section.

Theorem 11 *If n is sufficiently large compared to m and r, then $T(m,n,r)$ is crossing–critical.* ∎

4 Proofs of Theorems 2 and 3

We finally proceed to the proofs of Theorems 2 and 3.

Theorem 2 *For each rational number $q \in [4,6)$, there is an integer $k(q)$ and an infinite family $\mathcal{G}_q$ of simple 4–connected graphs such that (i) each graph in $\mathcal{G}_q$ has average degree q, and (ii) each graph in $\mathcal{G}_q$ is $k(q)$–crossing–critical. Moreover, all the graphs in $\mathcal{G}_q$ have the same crossing number.*

Proof. Let q be a rational number in $[4,6)$. Clearly, q can be written as $4+r/p$, where r,p are nonnegative integers, and $r<2p$. Let $m=2p$. Thus, q can be written as $q=4+2r/m$, where m is a positive, even integer, and r satisfies $0 \leq r \leq m-1$.

Consider the family $\mathcal{F} = \{T(m,n,r) \mid T(m,n,r)$ is crossing–critical$\}$. By Theorem 11, $\mathcal{F}$ is infinite. By Proposition 4, each graph in $\mathcal{F}$ has average degree $q = 4+2r/m$. Since each graph in $\mathcal{F}$ has crossing number at most $\binom{2(m+r)+1}{2}$ (by Proposition 5), it follows that $\mathcal{F}$ contains a subcollection of graphs with the same crossing number $k(q)$. We finish the proof by noting that all the graphs in $\mathcal{F}$ are 4–connected. ∎

Theorem 3 *For each rational number $q \in [4,6)$, there is an increasing sequence $\{k_1(q), k_2(q), \ldots\}$ of integers with the following property. For each $k_i(q)$ there exists an infinite family $\mathcal{G}_q^i$ of simple 4–connected graphs such that (i) each graph in $\mathcal{G}_q^i$ has average degree q, and (ii) each graph in $\mathcal{G}_q$ is $k_i(q)$–crossing–critical. Moreover, for each fixed i, all the graphs in $\mathcal{G}_q^i$ have the same crossing number.*

Proof. The construction of $\mathcal{F}$ in the previous proof is based on a particular representation of q as $q=4+r/p$. Taking from this expression r and $m=2p$, we constructed an infinite family of graphs of the form $T(m,n,r)$, and proved that it satisfied the properties in the statement of Theorem 2.

If we start with a different representation of q as $q = 4 + r'/p'$, with $r' \neq r$, we obtain a different infinite family of graphs that satisfies the statement of Theorem 2. Since there are infinitely many such representations of q, the result follows. ∎

5 Concluding Remarks

The construction described above completely settles the question of the existence of infinite families of crossing–critical graphs with average degree in the interval $[4, 6)$. However, two important questions remain.

Question 1 *Let q be a rational number in $(3, 4)$. Does there exist an integer k and an infinite family of graphs, each of which has average degree q and is k–crossing–critical?*

Question 2 *Does there exist an infinite family of 5–regular graphs that are k–crossing–critical for the same integer k?*

Recently, Benny Pinontoan and Bruce Richter (private communication) answered Question 2 by the affirmative. Question 1 remains open.

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