Optimal meshes of curves in the Klein bottle*

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Abstract. It is proved that if n is sufficiently large compared to m, and $m \geq 3$, then the minimum number of intersections in an (m,n)–mesh of curves in the Klein bottle equals $mn + \binom{\lfloor m/2 \rfloor}{2} + \binom{\lceil m/2 \rceil}{2}$. The proof is based on Riskin's result that the Cartesian product $C_3 \times C_5$ cannot be embedded in the Klein bottle.

In their work on curve systems [1], Richter and Thomassen introduced the concept of an (m,n)–mesh, and showed its relevance in the study of certain crossing number problems.

An (m,n)–*mesh* is a pair $(\mathcal{B},\mathcal{R})$ of families of (*blue* and *red*, respectively, to help comprehension) closed curves, of sizes m and n, respectively, such that each blue curve intersects each red curve. It is assumed that no point belongs to more than two curves in the mesh.

Here we restrict our attention to (m,n)–meshes in the Klein bottle $\widetilde{\Sigma_2}$. Following Shahrokhi et al. [3], let $i_2^*(\mathcal{B},\mathcal{R})$ denote the total number of intersections in the mesh $(\mathcal{B},\mathcal{R})$, and let $i_2^*(m,n)$ denote the least $i_2^*(\mathcal{B},\mathcal{R})$ over all (m,n)–meshes $(\mathcal{B},\mathcal{R})$ in $\widetilde{\Sigma_2}$.

In [3], Shahrokhi et al. derived the lower bound $i_2^*(m,n) \geq mn + m^2/(544321)^2$, for $n \geq m \geq 3 \cdot 9!/2 + 1$. Recently, Riskin [2] calculated the exact Klein bottle crossing number of the Cartesian products $C_m \times C_n$ for $m = 3,4,5$, and 6. As Riskin observed, for m,n such that $n \geq m \geq 3$, $C_m \times C_n$ can be drawn in $\widetilde{\Sigma_2}$ with $\binom{\lfloor m/2 \rfloor}{2} + \binom{\lceil m/2 \rceil}{2}$ crossings. A drawing of $C_m \times C_n$ with k crossings naturally yields an (m,n)–mesh with $mn + k$ intersections (regard the m n–cycles and the n m–cycles as the curves in a mesh, and the mn vertices and k crossings as intersection points between the curves), and so $i_2^*(m,n) \leq mn + \binom{\lfloor m/2 \rfloor}{2} + \binom{\lceil m/2 \rceil}{2}$.

As Riskin also observed (in the context of crossing numbers of Cartesian products, but his remark also applies to (m,n)–meshes), this implies that, for each $m \geq 3$, if n is sufficiently large, then $i_2^*(m,n) - mn$ depends only on m. Our aim in this note is to show that the upper bound above is the exact value of $i_2^*(m,n)$, if n is sufficiently large compared to m.

Theorem 1 *For each $m \geq 3$, there is an $n_0(m)$ with the following property. For every $n \geq n_0(m)$, $i_2^*(m,n) = mn + \binom{\lfloor m/2 \rfloor}{2} + \binom{\lceil m/2 \rceil}{2}$.*

Proof. First we show that there is an integer Ω such that $i_2^*(3,n) > 3n$, for all $n \geq \Omega$. To see this, we note that if n is sufficiently large, then every $(3,n)$–mesh $(\mathcal{P},\mathcal{Q})$ contains a set Q_1, Q_2,

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Q_3, Q_4, Q_5 of red curves such that each curve in $\mathcal{P}$ intersects these red curves in the given cyclic order. If $i_2^*(\mathcal{P}, \mathcal{Q}) = 3n$, then in particular $i_2^*(\mathcal{P}, \{Q_1, Q_2, \ldots, Q_5\}) = 15$. If in the $(3,5)$–mesh $(\mathcal{P}, \{Q_1, Q_2, \ldots, Q_5\})$ we then replace each intersection by a vertex we obtain an embedding of $C_3 \times C_5$ in $\widetilde{\Sigma}_2$, contradicting Corollary 3.11 in [2].

Let $m \geq 3$ be fixed. It follows from the upper bound for $i_2^*(m,n)$ that there is an integer n_0 (which can be assumed to satisfy $n_0 > \Omega$) such that $i_2^*(m,n) - mn$ is the same for all $n \geq n_0$. We complete the proof by showing that $i_2^*(m, n_0+1) \geq m(n_0+1) + \binom{\lfloor m/2 \rfloor}{2} + \binom{\lceil m/2 \rceil}{2}$.

Let $(\mathcal{B}, \mathcal{R})$ be an (m, n_0+1)–mesh that satisfies $i_2^*(\mathcal{B}, \mathcal{R}) = i_2^*(m, n_0+1)$. Then each curve R in $\mathcal{R}$ has exactly (the forced red–blue) m intersections, as otherwise $i_2^*(\mathcal{B}, \mathcal{R} \setminus \{R\}) - mn_0$ would be smaller than $i_2^*(\mathcal{B}, \mathcal{R}) - m(n_0+1)$, which is impossible, since the choice of n_0 guarantees that $i_2^*(\mathcal{P}, \mathcal{Q}) - mn_0 \geq i_2^*(m, n_0+1) - m(n_0+1) = i_2^*(\mathcal{B}, \mathcal{R}) - m(n_0+1)$ for each (m, n_0)–mesh $(\mathcal{P}, \mathcal{Q})$.

We claim that no three blue curves in $\mathcal{B}$ are pairwise disjoint. For suppose that $\mathcal{B}_0$ is a pairwise disjoint 3–collection of blue curves. Since (by the previous paragraph) each red curve intersects each blue curve exactly once, and no two red curves intersect, it follows that $(\mathcal{B}_0, \mathcal{R})$ is a $(3, n_0+1)$–mesh with $i_2^*(\mathcal{B}_0, \mathcal{R}) = 3(n_0+1)$, which is impossible, since $n_0 > \Omega$.

Let G be the graph with vertex set $\mathcal{B}$, where two vertices (curves) are adjacent iff they intersect each other. By the result in the previous paragraph, G contains no independent set of size three. Hence, by Turán's theorem, G has at least $\binom{\lfloor m/2 \rfloor}{2} + \binom{\lceil m/2 \rceil}{2}$ edges, and so $(\mathcal{B}, \mathcal{R})$ has at least $m(n_0+1) + \binom{\lfloor m/2 \rfloor}{2} + \binom{\lceil m/2 \rceil}{2}$ intersections. Therefore $i_2^*(m, n_0+1) = i_2^*(\mathcal{B}, \mathcal{R}) \geq m(n_0+1) + \binom{\lfloor m/2 \rfloor}{2} + \binom{\lceil m/2 \rceil}{2}$. ∎

Since for all m, n such that $n \geq m \geq 3$, there exist drawings of $C_m \times C_n$ in the Klein bottle with exactly $\binom{\lfloor m/2 \rfloor}{2} + \binom{\lceil m/2 \rceil}{2}$ crossings, Theorem 1 yields the following.

Corollary 2 *For each $m \geq 3$, there is an $n_0(m)$ with the following property. For every $n \geq n_0(m)$, the Klein bottle crossing number of $C_m \times C_n$ equals $\binom{\lfloor m/2 \rfloor}{2} + \binom{\lceil m/2 \rceil}{2}$.* ∎

Riskin [2] proved that Corollary 2 holds with $n_0 = m$, for $m = 3, 4, 5$, and 6, and conjectured that the Klein bottle crossing number of $C_m \times C_n$ depends only on m (assuming $n \geq m \geq 3$). It follows from our results that if Riskin's conjecture is true, then the Klein bottle crossing number of $C_m \times C_n$ is $\binom{\lfloor m/2 \rfloor}{2} + \binom{\lceil m/2 \rceil}{2}$, for m, n satisfying $n \geq m \geq 3$. We certainly endorse this stronger conjecture.

References

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