On the crossing number of $C_m \times C_n$

GELASIO SALAZAR

*Department of Mathematics and Statistics, Carleton University,
Ottawa, Ontario K1S 5B6, Canada*

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Abstract. We show that the M–crossing number $cr_M(C_m \times C_n)$ of $C_m \times C_n$ behaves asymptotically according to $\lim_{n\to\infty}\{cr_M(C_m \times C_n)/((m-2)n)\} = 1$, for each $m \geq 3$. This result reinforces the conjecture $cr(C_m \times C_n) = (m-2)n$ if $3 \leq m \leq n$, which has been proved only for $m \leq 6$.

1. INTRODUCTION

The *crossing number* $cr(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane. Given two simple curves C and D that intersect at a point p, we say that C and D are *tangent* at p if the rotation scheme around p consists of two subarcs of C followed by two subarcs of D; otherwise we say that they *cross* at p. Since every tangential intersection in a drawing of a graph G can be easily removed to yield a drawing of G with fewer intersections, in order to calculate the crossing number of G it suffices to consider drawings of G where every intersection of edges is a crossing.

Calculating the crossing number of a given graph is in general an elusive problem. As Garey and Johnson have proved, the problem of determining the crossing number of an arbitrary graph is NP–complete [6]. Thus, as Garey and Johnson point out, every serious effort to find the crossing number of a graph with more than just a few vertices must exploit special features of the structure of the particular graph.

The crossing numbers of very few families of graphs are known exactly. Long standing conjectures involve the crossing numbers of the complete graphs K_n and the complete bipartite graphs $K_{m,n}$ (see [1]). These conjectures remain open except for a few cases: $cr(K_n)$ is known only for $n \leq 12$ [13], and $cr(K_{m,n})$ has been determined only for those m, n satisfying $\min\{m, n\} \leq 6$, and for $K_{7,7}, K_{7,8}, K_{7,9}$, and $K_{7,10}$ [8], [13].

Another family of graphs whose crossing numbers have received a good deal of attention is the collection of the Cartesian products of cycles. For each integer l, let C_l denote the cycle of size l. The *Cartesian product* $C_m \times C_n$ of the cycles C_m and C_n is the graph with vertices $v_{i,j}$, for $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$, such that each $v_{i,j}$ is adjacent only to the four vertices $v_{i-1,j}, v_{i+1,j}, v_{i,j-1}$, and $v_{i,j+1}$ (indices are read modulo m and n respectively). Thus, the set of edges of $C_m \times C_n$ can be partitioned into m collections of

n edges (each collection corresponding to the edges of an n–cycle) plus n collections of m edges (each collection corresponding to the edges of an m–cycle).

Harary et al. [7] conjectured that the crossing number of $C_m \times C_n$ is $(m-2)n$, for all m, n satisfying $3 \leq m \leq n$. This has been proved only for m, n satisfying $n \geq m$, $m \leq 6$ [11], [5], [4], [10], [9], [2], [12], and for the special case $m = n = 7$ [3]. In this paper we present a result that supports the conjecture, based on the asymptotic behaviour of the minimum crossing numbers of wide classes of drawings of $C_m \times C_n$.

In this paper we restrict our attention to drawings of $C_m \times C_n$ such that the induced drawings of any two of the m n–cycles have no more than M intersections, where M is any fixed integer. A drawing with this property is an *M–drawing*. The *M–crossing number* $cr_M(C_m \times C_n)$ of $C_m \times C_n$ is the minimum number of crossings in an M–drawing of $C_m \times C_n$. Our main goal in this paper is to show the following.

Main Theorem. *Let M be fixed. Then, for any $m \geq 3$,*

$$\lim_{n \to \infty} \frac{cr_M(C_m \times C_n)}{(m-2)n} = 1.$$

This result is especially appealing because the calculation of the crossing number of $C_m \times C_n$ has proved to be a very hard problem. The techniques developed by Richter and Thomassen [10] have made it possible to calculate the crossing numbers of $C_5 \times C_5$ [10], $C_6 \times C_6$ [2], and $C_7 \times C_7$ [3], and even though there is some hope that these techniques will help to compute $cr(C_m \times C_n)$ for more values of m and n, at the moment this seems to be an increasingly difficult task.

This paper is organized as follows. In Section 2 we introduce $(m,n)_M$–meshes, which are the collections of closed curves underlying in the M–drawings of $C_m \times C_n$. These objects are stripped of any graph–theoretic structure; however, as Richter and Thomassen have shown [10], their study can yield important results involving the number of crossings in drawings of $C_m \times C_n$. In Section 3 we establish our main topological tool, a technical lemma on collections of curves in the plane. In Section 4 we prove the Main Theorem, which turns out to be a consequence of a more general result on $(m,n)_M$–meshes. Section 5 contains some final remarks.

2. $(m,n)_M$–MESHES

As Richter and Thomassen [10] showed in their work on intersections of systems of curves, the problem of determining the crossing number of $C_m \times C_n$ is closely related to the computation of the minimum number $i^*(m,n)$ of intersections in an (m,n)–mesh. An *(m,n)–mesh* $(\mathcal{C}_1, \mathcal{C}_2)$ is a pair of collections $\mathcal{C}_1$ and $\mathcal{C}_2$ of closed curves, where $|\mathcal{C}_1| = m$ and $|\mathcal{C}_2| = n$, such that each curve in $\mathcal{C}_1$ intersects each curve in $\mathcal{C}_2$ in at least one point. It is a trivial observation that $cr(C_m \times C_n) \geq i^*(m,n) - mn$. If the exact value of $i^*(m,n)$ is known for a pair (m,n), this inequality can be used to obtain a lower bound on the crossing number of $C_m \times C_n$. Moreover, if there is a drawing $\mathcal{D}$ of $C_m \times C_n$ with exactly $i^*(m,n) - mn$ crossings, this shows that the crossing number of $C_m \times C_n$ is attained in $\mathcal{D}$.

Based on this approach, Richter and Thomassen [10] showed that the crossing number of $C_5 \times C_5$ is 15, and following the same ideas Anderson et al. [2], [3] proved that the crossing numbers of $C_6 \times C_6$ and $C_7 \times C_7$ are 24 and 35 respectively. All these values are in agreement with the conjecture $cr(C_m \times C_n) = (m-2)n$, and it seems reasonable to expect that the same methods will help to calculate $cr(C_m \times C_n)$ for more values of m and n, although the amount of work involved in the calculation of $i^*(m,n)$ seems to increase considerably as m and n become larger.

In this paper we focus our attention on (m,n)–meshes $(\mathcal{C}_1, \mathcal{C}_2)$ such that each pair of curves in $\mathcal{C}_1$ intersect each other in at most M points, where M is any fixed integer. A mesh satisfying this property is an $(m,n)_M$*–mesh*. Let $i^*_M(m,n)$ denote the minimum number of intersections in an $(m,n)_M$–mesh. It is easy to check that if $(\mathcal{C}_1, \mathcal{C}_2)$ is an $(m,n)_M$–mesh whose total number of intersections equals $i^*_M(m,n)$, then all the curves in $\mathcal{C}_1 \cup \mathcal{C}_2$ are simple. Therefore we will restrict our attention to meshes that satisfy this property.

3. An Auxiliary Lemma

The following technical lemma is the main topological tool involved in the proof of the Main Theorem. If C and D are closed curves whose only intersection point is tangential, we say that C and D are *tangent* to each other.

Lemma 1. *Let I_1, I_2, I_3 be pairwise disjoint simple arcs in the plane, and let $\mathcal{C}$ be a collection of simple closed curves. Suppose that each curve in $\mathcal{C}$ is tangent to each I_j on the same side of I_j for each $j \in \{1,2,3\}$. Let m be any integer. Then there exists an integer $B(m)$ such that, whenever $|\mathcal{C}| \geq B(m)$, there exists a C in $\mathcal{C}$ having at least $m-2$ intersections with curves in $\mathcal{C} \setminus \{C\}$.*

Proof. It is easy to construct simple arcs J_1, J_2, J_3 such that $D_i = I_i \cup J_i$ is a simple closed curve for each i and the following conditions are satisfied:

(1) Each element in $\mathcal{C}$ is tangent to each D_i.
(2) No curve in $\mathcal{C}$ and no D_j is contained in the closed disc bounded by D_i if $i \neq j$.
(3) The curves D_i and D_j are disjoint if $i \neq j$.

Let $B(m) = m + 3(\lceil \frac{m}{2} \rceil - 1)$. We show that, whenever $|\mathcal{C}| \geq B(m)$, there exists a C in $\mathcal{C}$ having at least $m-2$ intersections with curves in $\mathcal{C} \setminus \{C\}$.

Let $|\mathcal{C}| \geq m + 3(\lceil \frac{m}{2} \rceil - 1)$, and let us suppose that there is a subcollection $\mathcal{C}'$ of at least $\lceil \frac{m}{2} \rceil$ elements in $\mathcal{C}$ whose bounded discs contain the same curve D_i. It follows that any two elements in $\mathcal{C}'$ have at least two points in common, and so each element C in $\mathcal{C}'$ has at least $2(\lceil \frac{m}{2} \rceil - 1) \geq m-2$ intersections with the elements in $\mathcal{C}' \setminus C$. Since in such a case we are done, we assume that there exist at most $3(\lceil \frac{m}{2} \rceil - 1)$ elements in $\mathcal{C}$ whose bounded discs contain a D_i. Thus, there is a subcollection $\mathcal{C}''$ of at least m elements in $\mathcal{C}$ whose bounded discs do not contain any D_i.

If $|C \cap C'| \neq \emptyset$ for all C, C' in $\mathcal{C}''$, then the conclusion of the lemma follows, since then each curve in $\mathcal{C}''$ has at least $m-1$ intersections with the elements in $\mathcal{C}'' \setminus \{C\}$. Therefore we assume that there exist C, C' in $\mathcal{C}''$ such that $C \cap C' = \emptyset$.

There are exactly eight components in $R^2 \setminus (D_1 \cup D_2 \cup D_3 \cup C \cup C')$. Five of these components are the discs bounded by the five curves, and the boundary of each of the other three components has empty intersection with D_i for exactly one i. Let H_i be the component whose boundary does not intersect D_i, and let C'' belong to $\mathcal{C}'' \setminus \{C, C'\}$. Since C'' is tangent to D_1, D_2, and D_3, it must intersect H_i for two different values of i, and this can only happen if C'' intersects either C or C' in at least two points. Therefore either C or C' has at least $2\lceil \frac{|\mathcal{C}''|-2}{2} \rceil \geq m-2$ intersections with the elements in $\mathcal{C}'' \setminus \{C, C'\}$.$\square$

4. Proof of the Main Theorem

We start with a lemma about $(3,n)_M$–meshes.

Lemma 2. *Let $m \geq 3$ be fixed. Then there exists an integer $B'(m)$ such that, for any $(3, B'(m))_M$–mesh $(\mathcal{C}_1, \mathcal{C}_2)$ with each curve in $\mathcal{C}_2$ tangent to each curve in $\mathcal{C}_1$, some element C in $\mathcal{C}_2$ satisfies $\sum_{D \in \mathcal{C}_2 \setminus \{C\}} |C \cap D| \geq m-2$.*

Proof. Let $\mathcal{C}_1 = \{C_1, C_2, C_3\}$ and $\mathcal{C}_2$ be collections of curves such that $(\mathcal{C}_1, \mathcal{C}_2)$ is a $(|\mathcal{C}_1|, |\mathcal{C}_2|)_M$–mesh, and such that each curve in $\mathcal{C}_2$ is tangent to each curve in $\mathcal{C}_1$.

Our first goal is to show that each curve C_i in $\mathcal{C}_1$ can be partitioned into a collection $\mathcal{I}_i = \{I_{i,j}\}_{j=1}^{T(i)}$ of $T(i)$ homeomorphs of $I = [0,1]$ such that if $I_{i,j} \cap I_{k,l} \neq \emptyset$, then (i) $I_{i,j}$ and $I_{k,l}$ meet in exactly one common end and (ii) no curve in $\mathcal{C}_2$ touches both $I_{i,j}$ and $I_{k,l}$. Moreover, we claim that $T(i)$ can be chosen to be at most $12M$, for each i.

First we note that each curve in $\mathcal{C}_1$ is crossed by the other two curves in $\mathcal{C}_1$ in at most $2M$ points. Thus each curve in $\mathcal{C}_1$ is divided into at most $2M$ *primary* intervals by these intersection points. It is possible, however, that a primary interval of a curve in $\mathcal{C}_1$ has both end points in common with a primary interval of a curve in $\mathcal{C}_1$. To avoid this conflict we subdivide each primary interval into two *secondary* intervals. It follows that each curve in $\mathcal{C}_1$ can be divided into at most $4M$ intervals to satisfy the condition that if $I_{i,j} \cap I_{k,l} \neq \emptyset$ then $I_{i,j}$ and $I_{k,l}$ meet in exactly one common end (if some curve in $\mathcal{C}_1$ is not intersected by any other curve in $\mathcal{C}_1$, then we subdivide it into at least three secondary intervals, to satisfy this last condition).

Now in order to satisfy the condition that if $I_{i,j} \cap I_{k,l} \neq \emptyset$, then no curve in $\mathcal{C}_2$ touches both $I_{i,j}$ and $I_{k,l}$, it clearly suffices to subdivide each of the at most $4M$ secondary intervals into at most three subintervals (namely a middle subinterval that contains all the intersections with the curves in $\mathcal{C}_2$ plus two subintervals, each of which has a common end with the middle one). Therefore each curve C_i in $\mathcal{C}_1$ can be subdivided in $T(i) \leq 12M$ intervals to meet the required conditions.

Let $B(m)$ be as in Lemma 1, and let $B'(m) = (8(12M)^3)B(m)$. Since there are at most $(2T(1))\,(2T(2))\,(2T(3)) \leq 8(12M)^3$ different ways of choosing one interval from each of $\mathcal{I}_1, \mathcal{I}_2$, and $\mathcal{I}_3$ together with a side for each of the intervals, it follows that there are intervals $I_{1,j_1}, I_{2,j_2}, I_{3,j_3}$, such that at least $B(m)$ curves in $\mathcal{C}_2$ are tangent to these intervals on the same side of each of I_{1,j_1}, I_{2,j_2}, and I_{3,j_3}. Thus, from Lemma 1, there exists a curve $C \in \mathcal{C}_2$ that has at least $m-2$ intersections with the elements in $\mathcal{C}_2 \setminus \{C\}$.$\square$

Our next step is to lift the condition, in the statement of Lemma 2, that each curve in $\mathcal{C}_2$ is tangent to each curve in $\mathcal{C}_1$. Moreover, we prove a statement that applies to $(m,n)_M$–meshes with m arbitrary, as opposed to Lemma 2, which deals exclusively with $(3,n)_M$–meshes.

Lemma 3. *Let $m \geq 3$ be fixed. Then there exists an integer $B''(m)$ such that, for any $(m, B''(m))_M$–mesh $(\mathcal{C}_1, \mathcal{C}_2)$, some curve C in $\mathcal{C}_2$ satisfies $\sum_{D \in \mathcal{C}_1 \cup (\mathcal{C}_2 \setminus \{C\})} |C \cap D| \geq 2m-2$.*

Proof. First we note that if a curve C in $\mathcal{C}_2$ is tangent to no more than two curves in $\mathcal{C}_1$, then C has at least $m+(m-2) = 2m-2$ intersections with the curves in $\mathcal{C}_1$. Since in this case the lemma follows, we can assume that each curve in $\mathcal{C}_2$ is tangent to at least three curves in $\mathcal{C}_1$.

Let $B''(m) = \binom{m}{3} B'(m)$, where $B'(m)$ is as in Lemma 2.

Let $(\mathcal{C}_1, \mathcal{C}_2)$ be an $(m, \binom{m}{3} B'(m))_M$–mesh. Then there is a collection of at least $B'(m)$ curves in $\mathcal{C}_2$ tangent to the same three curves in $\mathcal{C}_1$, and so by Lemma 2 there is a curve C in $\mathcal{C}_2$ such that $\sum_{D \in \mathcal{C}_2 \setminus \{C\}} |C \cap D| \geq m-2$. Since each curve in $\mathcal{C}_2$ intersects every curve in $\mathcal{C}_1$, C has at least m intersections with the curves in $\mathcal{C}_1$, and so the total number of intersections of C with the curves in $\mathcal{C}_1 \cup (\mathcal{C}_2 \setminus \{C\})$ is at least $2m-2$.□

We are now ready to prove an asymptotic result on $(m,n)_M$–meshes which is the heart of the proof of the Main Theorem.

Theorem 4. *Let M be fixed. Then, for any $m \geq 3$,*

$$\lim_{n \to \infty} \frac{i_M^*(m,n) - mn}{(m-2)n} = 1.$$

Proof. Let $B''(m)$ be as in Lemma 3. If $(\mathcal{C}_1, \mathcal{C}_2)$ is an $(m,n)_M$–mesh with $n > B''(m)$, then an easy induction on n, using Lemma 3, shows that there are at least $mB''(m)+(2m-2)(n-B''(m))$ intersections in $(\mathcal{C}_1, \mathcal{C}_2)$. Therefore, for $n > B''(m)$, $i_M^*(m,n) \geq mB''(m) + (2m-2)(n-B''(m)) = mn+(m-2)(n-B''(m))$, and so $(i_M^*(m,n)-mn)/((m-2)n) \geq 1 - B''(m)/n$. On the other hand, the drawings constructed to show that $C_m \times C_n \leq (m-2)n$ also show that $i_M^*(m,n) \leq mn + (m-2)n$, and so $(i_M^*(m,n) - mn)/((m-2)n) \leq 1$. Therefore $(1 - B''(m)/n) \leq (i_M^*(m,n) - mn)/((m-2)n) \leq 1$ for every $n > B''(m)$. Since $B''(m)$ is fixed, $1 - B''(m)/n$ approaches 1 as n approaches infinity, and so Theorem 4 follows.□

Proof of the Main Theorem. Since $cr_M(C_m \times C_n) \geq i_M^*(m,n) - mn$, it follows from Theorem 4 that $cr_M(C_m \times C_n)/((m-2)n)$ cannot converge to a number smaller than one. On the other hand, there are drawings of $C_m \times C_n$ with exactly $(m-2)n$ crossings such that the m n–cycles are pairwise disjoint (actually, these drawings prompted Harary et al. [7] to conjecture that $cr(C_m \times C_n) = (m-2)n$). Therefore $cr_M(C_m \times C_n) \leq (m-2)n$ for every M, and so the Main Theorem follows.□

We remark that Theorem 4 is a stronger statement than the Main Theorem. Indeed, as we have seen, the Main Theorem follows easily from Theorem 4; however, Theorem 4 cannot be derived from the Main Theorem. Thus, once again, the Richter and Thomassen approach to the problem is the way to go: in order to get insight into the asymptotical behaviour of $cr_M(C_m \times C_n)$, ignore the graph–theoretical structure of the M–drawings of $C_m \times C_n$.

5. Comments

As we mentioned in Section 1, the general conjecture $cr(C_m \times C_n) = (m-2)n$ remains open for all but a finite number of values of m. The Main Theorem reinforces that conjecture, as does the following result, which is a straightforward consequence of Lemma 3.

Theorem 5. *Let $m \geq 3$ be a given integer, and suppose that there exists an infinite subsequence $\{n_1, n_2, \ldots\}$ of $\{1, 2, \ldots\}$ such that $cr(C_m \times C_{n_j}) < (m-2)n_j$ for each n_j. Then either*

(i) there exists an integer $r(m)$ such that $cr(C_m \times C_{n_j}) \geq (m-2)n_j - r(m)$ for each n_j, or

(ii) given any integer s there exists an n_j such that there is a drawing $\mathcal{D}$ of $C_m \times C_{n_j}$, where the crossing number is attained, such that the drawings induced by $\mathcal{D}$ of two of the m n_j–cycles intersect each other in more than s points.

Basically, Theorem 5 says that if the crossing number of $C_m \times C_n$ is less than as conjectured, then either the actual and the conjectured crossing numbers are not very different or else some crossing numbers are attained in drawings where two n–cycles intersect each other an arbitrarily large number of times. This last condition is rather unappealing, and this increases our optimism in the correctness of the given conjecture.

The techniques introduced by Richter and Thomassen to compute $cr(C_m \times C_n)$ from $i^*(m,n)$ have been crucial for determining certain crossing numbers. However, as noted in their paper [10], these methods cannot be used to obtain the crossing number of $C_m \times C_n$ for arbitrary values of m and n. This is illustrated by the discrepancy between $i^*(3,n) - 3n$ and $cr(C_3 \times C_n)$: for large values of n, $i^*(3,n) - 3n$ is approximately $n/2$, and the crossing number of $C_3 \times C_n$ is equal to n for every $n \geq 3$ [11]. We remark that this is not in contradiction with the asymptotic behaviour of $cr_M(C_m \times C_n)/((m-2)n)$ given by the Main Theorem, since not every (m,n)–mesh is an $(m,n)_M$–mesh.

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