On the Intersections of Systems of Curves*

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22 June 1998

Abstract. We show that the minimum number $i_m(n)$ of intersections in an n–family of simple closed curves whose elements pairwise intersect at least once and in less than m points satisfies the asymptotic behavior $\lim_{n\to\infty} i_m(n)/\binom{n}{2} = 2$.

* Revised manuscript for JCTB

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1. Introduction

In their work on intersections of curve systems [2], Richter and Thomassen raised the problem of evaluating $i(n)$, the minimum number of intersections in a collection of n simple closed curves in the plane, any two of which intersect, and where no point in the plane belongs to more than two curves in the collection. As is pointed out in their paper, it is easy to show that $\frac{3}{2} \leq \lim_{n\to\infty} i(n)/\binom{n}{2} \leq 2$. We prove that this limit is equal to two, if we add the condition that any two curves intersect each other in less than m points, where m is fixed but can be arbitrarily large. Thus if we let $i_m(n)$ to be the minimum number of intersections in a family of n simple closed curves which intersect each other at least once and in less than m points, we prove the following.

Theorem 1. *For each integer $m > 2$, $\lim_{n\to\infty} i_m(n)/\binom{n}{2} = 2$.*

The following argument shows that this limit exists and is not bigger than 2. Every collection of n curves has n $(n-1)$–subcollections, each of which has at least $i_m(n-1)$ intersections. Therefore $i_m(n) \geq (\frac{n}{n-2})i_m(n-1)$, and so $i_m(n)/\binom{n}{2} \geq i_m(n-1)/\binom{n-1}{2}$. Therefore $\{i_m(n)/\binom{n}{2}\}_{n=1}^{\infty}$ is a nondecreasing sequence. On the other hand, for each given n it is not difficult to produce drawings of n circles which pairwise intersect in exactly two points, and so $i_m(n)/\binom{n}{2} \leq 2$ for every $m > 2$.

This paper is organized as follows. In Section 2 we show that Theorem 1 follows from Lemmas 2 and 3. In Section 3 we give a proof of Lemma 2, and Section 4 contains a proof of Lemma 3. In Section 5 we make some final remarks.

2. Proof of Theorem 1

An *intersecting family* $\mathcal{C}$ is a set of simple closed curves in the plane such that any two curves in $\mathcal{C}$ have a point in common. A $(< m)$*–family* is a collection of simple closed curves in the plane which pairwise intersect in less than m points. An *intersecting* $(< m)$*–family* is both an intersecting family and a $(< m)$–family. We remark that in all these definitions we assume that no point of the plane is in more than two simple closed curves in the collection.

Two curves are *tangent* to each other if they intersect at only one point, at which they are tangent.

Theorem 1 is a consequence of the following two lemmas, whose proofs are presented in Sections 3 and 4 respectively.

Lemma 2. *Suppose that there exist $\epsilon > 0$ and $m > 2$ such that $\lim_{n\to\infty} i_m(n)/\binom{n}{2} < 2-\epsilon$. Then there exist $s > 0$ and $N > 0$ such that, for every $n \geq N$, there is an intersecting $(< m)$–family $\{C_1, C_2, \ldots, C_{2m}, C_{2m+1}, \ldots, C_n\}$ so that every curve in $\{C_1, \ldots, C_{2m}\}$ is tangent to every curve in the same subset of $\{C_{2m+1}, \ldots, C_n\}$ of size at least sn.*

Lemma 3. *For every integer $m > 2$ there exists a number $B(m)$ such that if $\mathcal{C}_1$ is an intersecting $(< m)$–family of size $2m$ and $\mathcal{C}_2$ is any $(< m)$–family such that every curve in $\mathcal{C}_2$ is tangent to every curve in $\mathcal{C}_1$, then $|\mathcal{C}_2| \leq B(m)$.*

Proof of Theorem 1. Seeking a contradiction, suppose that $\lim_{n\to\infty} i_m(n)/\binom{n}{2} \neq 2$. Then $\lim_{n\to\infty} i_m(n)/\binom{n}{2} < 2-\epsilon$ for some $\epsilon > 0$. Let s and N be as in Lemma 2 and let

$n \geq N$ be such that $B(m) < sn$. By Lemma 2, there is an intersecting $(< m)$–family $\mathcal{C}_1$ of size $2m$ and a $(< m)$–family $\mathcal{C}_2$ of size at least sn such that every curve in $\mathcal{C}_2$ is tangent to every curve in $\mathcal{C}_1$. Since $sn > B(m)$, this contradicts Lemma 3. ∎

3. Proof of Lemma 2

Suppose that there exist $\epsilon > 0$ and $m > 2$ such that $\lim_{n\to\infty} i_m(n)/\binom{n}{2} < 2-\epsilon$. For each n, let $\mathcal{D}_n = \{C_1, \ldots, C_n\}$ be an intersecting $(< m)$–family, such that the total number $i_m(\mathcal{D}_n)$ of intersections among the curves in $\mathcal{D}_n$ is less than $(2-\epsilon)\binom{n}{2}$. The existence of $\mathcal{D}_n$ is guaranteed for every n, since the sequence $\{i_m(n)/\binom{n}{2}\}_{n=1}^{\infty}$ is nondecreasing. For each $\mathcal{D}_n$, let $G_n = (\mathcal{D}_n, E_n)$ be the graph with vertex set $\mathcal{D}_n$ such that the curves $C_i, C_j \in \mathcal{D}_n$ are joined by an edge in E_n if and only if they are tangent to each other. Clearly, to prove Lemma 2 it suffices to show that there exist $N > 0$ and $s > 0$ such that, for every $n \geq N$, $K_{2m,sn}$ is a subgraph of G_n.

Let s be such that $s < \epsilon^{2m}$. Since $i_m(\mathcal{D}_n) < (2-\epsilon)\binom{n}{2}$, it follows that $|E_n| \geq \epsilon\binom{n}{2}$ for every n. It follows from standard results in extremal graph theory (see for instance Theorem 1.3.2 and Eq. (1.3.6) in [1]) that every graph with at least $b = (1/2)((sn-1)^{1/2m}(n-2m+1)n^{1-1/2m} + (2m-1)n)$ edges contains $K_{2m,sn}$ as a subgraph. Thus we are done, since for n sufficiently large $\epsilon\binom{n}{2}$ is bigger than b. ∎

4. Proof of Lemma 3

Let $\mathcal{C}_1 = \{C_1, C_2, \ldots, C_{2m}\}$ be an intersecting $(< m)$–family, and let $\mathcal{C}_2 = \{C'_1, C'_2, \ldots, C'_r\}$ be a $(< m)$–family such that every curve in $\mathcal{C}_2$ is tangent to every curve in $\mathcal{C}_1$. The number of faces in the embedding of $\mathcal{C}_1$ is bounded by a function of m, and, since each $C'_i \in \mathcal{C}_2$ is contained in the closure of one of these faces, it is enough to show that the number of elements of $\mathcal{C}_2$ in a given face F cannot be arbitrarily large. The boundary $\partial(F)$ of F is a simple closed curve, and the intersection of $\partial(F)$ with each $C_i \in \mathcal{C}_1$ is a disjoint collection of simple closed arcs. It is not difficult to see that it is enough to prove that the number of elements in $\mathcal{C}_2$ which touch each element in a given collection $\{A_1, A_2, \ldots, A_{2m}\}$ of arcs cannot be arbitrarily large, where A_i is a subarc of C_i contained in $\partial(F)$. Let the collection $\{A_1, A_2, \ldots, A_{2m}\}$ be fixed. We can assume without any loss of generality that as we go along $\partial(F)$ we find these arcs in the given cyclic order. We will show the number of elements in $\mathcal{C}_2$ which touch each element in $\{A_1, \ldots, A_{2m}\}$ cannot be bigger than one, and this will finish the proof.

Let $\mathcal{C}'_2$ be the subcollection of $\mathcal{C}_2$ which consists of all the curves which touch each arc in $\{A_1, \ldots, A_{2m}\}$ exactly once. Since every curve in $\mathcal{C}_2$ is simple, each curve in $\mathcal{C}'_2$ touches the arcs in the given cyclic order. Suppose that C'_i, C'_j are different curves in $\mathcal{C}'_2$, and let $I_{i,k}, I_{j,k}$ be the arcs of C'_i and C'_j respectively which go from A_k to A_{k+1}, and let $a_{l,k}$ be the point where C'_l meets A_k for $l \in \{i, j\}$. If we go along $\partial(F)$ in one of its two possible orientations, we find these end points in one of two possible cyclic orders, namely $(a_{i,k}, a_{j,k}, a_{i,k+1}, a_{j,k+1})$ and $(a_{i,k}, a_{j,k}, a_{j,k+1}, a_{i,k+1})$. In the first case $I_{i,k} \cap I_{j,k} \neq \emptyset$, and in the second case either both $I_{i,k-1} \cap I_{j,k}$ and $I_{i,k+1} \cap I_{j,k}$ are nonempty or both $I_{i,k} \cap I_{j,k-1}$ and $I_{i,k} \cap I_{j,k+1}$ are nonempty. Hence for each k either $I_{i,k} \cap C'_j \neq \emptyset$ or $I_{j,k} \cap C'_i \neq \emptyset$. This implies that $|C'_i \cap C'_j| \geq m$, a contradiction. Thus there is at most one element in $\mathcal{C}'_2$. ∎

5. Comments

It follows from Theorem 1 that if we denote by $i_C(n)$ the minimum number of intersections in a collection of n pairwise nondisjoint circles, then $\lim_{n\to\infty} i_C/\binom{n}{2} = 2$. This can be extended to $\lim_{n\to\infty} i_{\mathcal{F}}/\binom{n}{2} = 2$, where $i_{\mathcal{F}}(n)$ denotes the minimum number of intersections in a collection of n pairwise nondisjoint curves, each of which is similar to some element in a fixed finite family $\mathcal{F}$ of simple closed curves, where each curve in $\mathcal{F}$ is the boundary of a closed convex set.

The convexity condition for the elements in the family $\mathcal{F}$ above is enough to guarantee that there is a fixed number $m(\mathcal{F})$ such that any two curves similar to elements in $\mathcal{F}$ have at most $m(\mathcal{F})$ intersections, and so Theorem 1 can be applied. It is easy to construct a simple closed curve C similar to two curves that intersect in an infinite number of points; therefore some condition, such as the convexity condition above, must be imposed on the curves in $\mathcal{F}$ in order to guarantee that the number of intersections between two curves similar to curves in $\mathcal{F}$ cannot be arbitrarily large. This raises an interesting question, from the case where $\mathcal{F}$ consists of a single curve: Is it possible to characterize the simple closed curves C such that the maximum number of intersections between two curves similar to C is finite? Sufficient conditions, such as the convexity condition in the above paragraph, are easy to obtain, but it seems to be very difficult to give a complete characterization of the curves that satisfy this property.

Theorem 1 also gives a condition that has to be satisfied if $\lim_{n\to\infty} i(n)/\binom{n}{2} < 2$. Suppose the limit is $L < 2$. Since $\{i(n)/\binom{n}{2}\}_{n=1}^{\infty}$ is a nondecreasing sequence (the proof of this is identical to the proof in Section 1 that $\{i_m(n)/\binom{n}{2}\}_{n=1}^{\infty}$ is nondecreasing), it follows that for any integer m_0 there is a collection $\mathcal{C}_{m_0}$ such that $i(\mathcal{C}_{m_0}) \leq L\binom{|\mathcal{C}_{m_0}|}{2}$, and such that two curves in $\mathcal{C}_{m_0}$ intersect each other in at least m_0 points. This seemingly unnatural condition reinforces our conjecture that $\lim_{n\to\infty} i(n)/\binom{n}{2} = 2$.

Acknowledgements

I thank Bruce Richter for carefully reading the manuscript and for his numerous suggestions on improving the presentation of this paper. I also thank two anonymous referees for helpful suggestions. This research was done while the author was a CONACYT fellow at Carleton University.

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