The Crossing Number of $C_6 \times C_n$

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Abstract. It is proved that the crossing number of $C_6 \times C_n$ is $4n$ for every $n \geq 6$. This is in agreement with the general conjecture that the crossing number of $C_m \times C_n$ is $(m-2)n$, for $3 \leq m \leq n$.

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1. Introduction

Harary et al. [5] conjectured that the crossing number of $C_m \times C_n$ is $(m-2)n$, for all m, n satisfying $3 \leq m \leq n$. This has been verified for $m = 3, 4$, and 5 [8, 4, 3, 7, 6], and for the special cases $m = n = 6$ [1] and $m = n = 7$ [2]. Our goal in this article is to prove the following.

Main Theorem. *The crossing number of $C_6 \times C_n$ is $4n$, for every $n \geq 6$.*

The *crossing number* $cr(G)$ of a graph G is the minimum number of pairwise crossings of edges in a drawing of G in the plane. It is well–known [12] that the crossing number of a graph is attained by a *good* drawing, a drawing in which no edge crosses itself, no adjacent edges cross, and no two edges cross each other more than once.

It is easy to exhibit drawings of $C_m \times C_n$ with exactly $(m-2)n$ crossings, for every m, n satisfying $3 \leq m \leq n$ (see [5]). Thus, the difficult part of the Main Theorem is the inequality $cr(C_6 \times C_n) \geq 4n$. We prove this by induction on n, as in [8, 3, 5]. The strategy is as follows. The base case is $n = 6$, proved in [1]. Let $\mathcal{D}$ be an optimal drawing of $C_6 \times C_n$, where $n \geq 7$, and suppose that the statement of the Main Theorem holds for $C_6 \times C_{n-1}$. We show that if two 6–cycles cross each other in $\mathcal{D}$, then there is an optimal drawing $\mathcal{D}'$ (not necessarily different from $\mathcal{D}$) of $C_6 \times C_n$ in which a 6–cycle is crossed at least four times; thus in this case the inequality $cr(C_6 \times C_n) \geq 4n$ follows easily from the induction hypothesis. On the other hand, if the n 6–cycles are pairwise disjoint, then it follows from Theorem 1 in [9] that there are at least $4n$ crossings in $\mathcal{D}$ (Theorem 1 in [9] establishes that if $n \geq m \geq 3$, then every drawing of $C_m \times C_n$ with the n m–cycles pairwise disjoint has at least $(m-2)n$ crossings).

This paper is organized as follows. In Section 2 we show that the Main Theorem follows from Theorem 1 in [9] (which we state as Theorem 2) and our Theorem 1. In Section 3 we establish a technical lemma used in the proof of Theorem 1, and in Section 4 we prove Theorem 1 for one particular case. Section 5 contains the proof of Theorem 1, and in Section 6 we make some final remarks.

2. The Main Theorem follows from Theorem 1

As in [6], it is convenient for our subsequent work to color the edges in $C_6 \times C_n$ red and blue, so that the edges of the n 6–cycles are red and the edges of the 6 n–cycles are blue.

We often make no distinction between a cycle and its corresponding closed curve in a drawing of $C_6 \times C_n$, if no confusion arises. However, if we say that a cycle C is *crossed* in a drawing $\mathcal{D}$ of $C_6 \times C_n$, it must be understood that an *edge* of C is crossed in $\mathcal{D}$. If $\mathcal{D}$ is the only drawing considered in a discussion, we omit reference to $\mathcal{D}$ and simply speak of the crossings of a cycle or of an edge.

An *arc* A is a homeomorph of $[0, 1]$ contained in a drawing. As with cycles, we say that an arc A in a drawing $\mathcal{D}$ is *crossed* at the point p if an edge crosses A at p, and omit reference to $\mathcal{D}$ if no confusion arises. If q and r are the end points of A, then $A \setminus \{q, r\}$ is the *interior* of A.

An *optimal* drawing of a graph G is a drawing whose number of crossings equals $cr(G)$. An optimal drawing of $C_6 \times C_n$ is *really optimal* if the number of red–red crossings is least among all optimal drawings of $C_6 \times C_n$.

We claim that the Main Theorem is a consequence of Theorems 1 and 2 below. We remark that Theorem 2 was proved in [9] (our Theorem 2 is precisely Theorem 1 in [9]).

Theorem 1. *Let $\mathcal{D}$ be a really optimal drawing of $C_6 \times C_n$. Suppose that two red cycles cross each other in $\mathcal{D}$. Then there is an optimal drawing of $C_6 \times C_n$ in which some red cycle has at least four crossings.*

Theorem 2 (Theorem 1 in [9]). *Let m, n be such that $n \geq m \geq 3$. Then every drawing of $C_m \times C_n$ such that either the n m–cycles are pairwise disjoint or the m n–cycles are pairwise disjoint has at least $(m-2)n$ crossings.* ∎

Proof of Main Theorem. First note that $C_6 \times C_n$ can be drawn with exactly $4n$ crossings (see for instance [5]). Therefore $cr(C_6 \times C_n) \leq 4n$. We prove the reverse inequality by induction on n. The base case, $cr(C_6 \times C_6) \geq 24$, is proved in [1]. Let $n \geq 7$ and suppose that $cr(C_6 \times C_{n-1}) \geq 4(n-1)$. Let $\mathcal{D}$ be a really optimal drawing of $C_6 \times C_n$. If no two red cycles cross each other, then the number of crossings $cr(\mathcal{D})$ in $\mathcal{D}$ is at least $4n$ by Theorem 2. On the other hand, by Theorem 1, if two red cycles cross each other, then a red cycle R has at least four crossings in an optimal drawing $\mathcal{D}'$. By the induction hypothesis, the drawing $\mathcal{D}''$ of $C_6 \times C_{n-1}$ obtained by deleting the edges of R from $\mathcal{D}'$ has at least $4(n-1)$ crossings. Hence $cr(C_6 \times C_n) = cr(\mathcal{D}') \geq cr(\mathcal{D}'') + 4 \geq 4(n-1) + 4 = 4n$. ∎

3. Red cycles with fewer than four crossings

Our first step towards the proof of Theorem 1 is a characterization of the drawings where a given red cycle has fewer than four crossings.

Lemma 3. *Let $\mathcal{D}$ be a drawing of $C_6 \times C_n$. Let R be a red cycle with fewer than four crossings. Suppose that there are different components $\mathcal{C}_1$ and $\mathcal{C}_2$ of $\mathbb{R}^2 \setminus R$ such that each of $\mathcal{C}_1$ and $\mathcal{C}_2$ contains at least one vertex. Then the following statements hold.*

- *(i) One of $\mathcal{C}_1$ and $\mathcal{C}_2$ contains exactly one vertex v.*
- *(ii) The component of $\mathbb{R}^2 \setminus R$ that contains v is intersected only by v and by the four edges incident with v.*
- *(iii) Both red edges incident with v cross R.*
- *(iv) One blue edge incident with v crosses R, and the other blue edge incident with v is incident with a vertex in R.*
- *(v) R has three crossings, none of which is a self–crossing.*

Proof. In order to obtain a contradiction, suppose that each $\mathcal{C}_i$ contains two different vertices u_i and v_i. Each vertex is in one red cycle and one blue cycle, and no two vertices have more than one monochromatic cycle in common. Therefore, for each i, at least three different monochromatic cycles $\{D_{i,1}, D_{i,2}, D_{i,3}\}$ intersect $\mathcal{C}_i$, and at least one of these cycles, say $D_{i,1}$, is red.

There is an $i \in \{1, 2\}$ such that each of $D_{i,1}, D_{i,2}$, and $D_{i,3}$, crosses R. For suppose there are $j, k \in \{1, 2, 3\}$ such that all the edges in $D_{1,j}$ are contained in $\mathcal{C}_1$ and all the edges in $D_{2,k}$ are contained in $\mathcal{C}_2$. Clearly $D_{1,j}$ and $D_{2,k}$ are of the same color, since every two cycles of different color have a common vertex. It follows that, possibly with the exception of R, each cycle of color different from that of $D_{1,j}$ and $D_{2,k}$ crosses R, since every such cycle has a common vertex with each of $D_{1,j}$ and $D_{2,k}$. Since there are at least six cycles of each color, this contradicts the hypothesis that R is crossed at most three times.

Thus we can assume without any loss of generality that each of $D_{1,1}, D_{1,2}$, and $D_{1,3}$, crosses R. Since $D_{1,1}$ is red, it has no vertices in common with R. Therefore $D_{1,1}$ crosses R at least twice. Since both $D_{1,2}$ and $D_{1,3}$ cross R, it follows that R has at least four crossings in total, contradicting the assumption that R has fewer than four crossings. Therefore we conclude that one of $\mathcal{C}_1$ and $\mathcal{C}_2$ contains exactly one vertex.

Suppose that $\mathcal{C}_1$ has exactly one vertex v. It is clear that $\mathcal{C}_1$ is intersected by the edges incident with v. Since there are no other vertices in $\mathcal{C}_1$, both red edges incident with v must cross R. Since R has fewer than four crossings, at most one blue edge incident with v crosses R. On the other hand, at most one vertex in R is adjacent to any given vertex not in R. It follows that one blue edge incident with v crosses R and the other one joins v with a vertex in R. Therefore R has exactly three crossings with edges incident with v. Since v is not in R, it follows that none of these crossings is a self–crossing. ∎

4. Self–crossing red cycles

Our aim in this section is to prove Theorem 1 for the case where one of R_1 and R_2 has a self–crossing.

Proposition 4. *Let $\mathcal{D}$ be a really optimal drawing of $C_6 \times C_n$. Suppose that the red cycles R_1 and R_2 cross in $\mathcal{D}$, and suppose that one of R_1 and R_2 crosses itself in $\mathcal{D}$. Then either R_1 or R_2 has at least four crossings in $\mathcal{D}$.*

Proof. By symmetry we can assume that R_1 has a self–crossing. By the Jordan Curve Theorem, R_1 and R_2 cross each other an even number of times. Thus, if they cross in more than two points then we are done, and so we assume that they cross each other in exactly two points p and q. If one of R_1 and R_2 self–crosses more than once then it has at least four crossings in total, and so we can also assume that neither R_1 nor R_2 crosses itself more than once.

The points p and q divide R_i into two curves A_i and B_i, for each i. One of A_1 and B_1, say A_1, is simple, and the other one has a self–crossing. The curve B_1 contains at least two vertices, since otherwise the good–drawing condition for $\mathcal{D}$ would be violated.

Since the interiors of the curves A_2 and B_2 are contained in different components of $\mathbb{R}^2 \setminus R_1$, it follows from statement (v) of Lemma 3 that either one of A_2 and B_2 does not contain any vertex, or R_1 has at least four crossings. Since in the latter case we are done, we assume that A_2 does not contain any vertex.

Suppose that A_1 contains more than one vertex. Since B_1 contains at least two vertices, and A_1 and B_1 are contained in different components of $\mathbb{R}^2 \setminus R_2$, it follows from statement (i) of Lemma 3 that R_2 has at least four crossings. Since in this case we are done, we assume that A_1 contains at most one vertex. If A_1 does not contain any vertex, then A_1 and A_2 are contained in edges that cross each other more than once. Since this violates the good–drawing property of $\mathcal{D}$, we conclude that A_1 contains exactly one vertex v_1.

We modify $\mathcal{D}$ to obtain a drawing $\mathcal{D}''$ of $C_6 \times C_n$ in the following way. Let p'' and q'' be points in B_2, contained in small neighbourhoods of p and q respectively. Substitute A_2 by an arc A_2'' very close to A_1, so that the end points of A_2'' are p'' and q''. It is easy to see that we can draw A_2'' close enough to A_1, so that an edge e crosses A_2'' only if either e crosses A_1 or e is a blue edge incident with v_1. Let $\mathcal{D}''$ be the drawing thus obtained. Clearly, $\mathcal{D}''$ is a drawing of $C_6 \times C_n$, and R_1 and R_2 do not cross each other in $\mathcal{D}''$.

The arc A_2'' must be crossed at least twice, since otherwise $\mathcal{D}''$ would have fewer crossings than $\mathcal{D}$, contradicting the optimality of $\mathcal{D}$. If A_2'' is crossed by an edge that also crosses A_1, then R_1 has at least four crossings in total. Since in this case we are done, we assume that A_2'' is crossed only by the two blue edges incident with v_1. Then $\mathcal{D}$ and $\mathcal{D}''$ have the same number of crossings. On the other hand, $\mathcal{D}''$ has fewer red–red crossings than $\mathcal{D}$, since the two crossings of A_2'' are blue–red crossings. This contradicts the real–optimality of $\mathcal{D}$, since $\mathcal{D}''$ is also optimal. ∎

5. Proof of Theorem 1

We prove Theorem 1 in two steps. In the first step we obtain a detailed picture of what the drawing $\mathcal{D}$ must look like if neither R_1 nor R_2 has at least four crossings. In the second step we show, using Claim 7, that under these conditions we can guarantee the existence of an optimal drawing $\mathcal{D}'$ in which some red cycle has at least four crossings.

Proof of Theorem 1. Let $\mathcal{D}$ be a really optimal drawing of $C_6 \times C_n$, and let R_1, R_2 be red cycles that cross each other in $\mathcal{D}$. Let us suppose that both R_1 and R_2 are crossed fewer than four times in $\mathcal{D}$. As explained above, we divide this proof in two steps.

Step 1. In this step we obtain a detailed picture of the properties of the drawing $\mathcal{D}$ that follow from the assumption that both R_1 and R_2 have fewer than four crossings.

By Proposition 4, neither R_1 nor R_2 has a self–crossing. Since R_1 and R_2 cross each other in an even number of points, it follows that they cross each other in exactly two points p and q. Let A_i and B_i be the arcs with end points p and q contained in R_i, for each $i \in \{1, 2\}$.

Claim 5. *Each of A_1, A_2, B_1, and B_2 contains at least one vertex.*

Proof. At least one of A_i and B_i contains a vertex for each i, since each R_i contains six vertices. If two arcs in $\{A_1, A_2, B_1, B_2\}$ contain no vertices, then the edges that contain these two arcs cross each other more than once. Since this would violate the good–drawing condition for $\mathcal{D}$, it follows that at most one of A_1, A_2, B_1, and B_2 contains no vertices.

Suppose that one of these four arcs, say A_1, contains no vertices. We show that this implies that there is an optimal drawing $\mathcal{D}''$ of $C_6 \times C_n$ with fewer red–red crossings than $\mathcal{D}$, contradicting the real–optimality of $\mathcal{D}$.

Since A_1 is the only arc in $\{A_1, A_2, B_1, B_2\}$ that contains no vertices, each of A_2 and B_2 contains at least one vertex. Since the interiors of A_2 and B_2 are contained in distinct components of $\mathbb{R}^2 \setminus R_1$, it follows from Statement (i) in Lemma 3 that one of A_2 and B_2, say A_2, contains exactly one vertex v_2.

Now we obtain from $\mathcal{D}$ a drawing $\mathcal{D}''$ in the following way. Let p'' and q'' be points in B_1 very close to p and q respectively. Substitute A_1 by an arc A_1'' very close to A_2, so that p'' and q'' are the ends of A_1''. It is easy to see that we can draw A_1'' close enough to A_2, so that an edge e crosses A_1'' only if e crosses A_2 or if it is incident with v_2. Let $\mathcal{D}''$ be the drawing thus obtained. Since $\mathcal{D}''$ is also a drawing of $C_6 \times C_n$, it follows from the optimality of $\mathcal{D}$ that A_1'' is crossed at least twice.

Let $\mathcal{C}_{v_2}$ be the component of $\mathbb{R}^2 \setminus R_1$ (in $\mathcal{D}$) that contains v_2. It follows from Statement (ii) in Lemma 3 that the only edges that intersect $\mathcal{C}_{v_2}$ are the four edges incident with v_2. Since every edge that crosses A_2 intersects $\mathcal{C}_{v_2}$, it follows that A_2 is not crossed by any edge. Therefore, since A_1'' is crossed at least twice, it follows that A_1'' is crossed once by each blue edge incident with v_2, and that A_1'' is not crossed by any other edge. Thus $\mathcal{D}''$ is also optimal. However, $\mathcal{D}''$ has fewer red–red crossings than $\mathcal{D}$, contradicting the real–optimality of $\mathcal{D}$. ∎

By Claim 5, each of A_1 and B_1 contains at least one vertex. Since the interiors of A_1 and B_1 are contained in different components of $\mathbb{R}^2 \setminus R_2$, it follows from Statement (i) in Lemma 3 that one of these arcs, say A_1, contains exactly one vertex v_1. By an analogous argument we can assume that A_2 contains exactly one vertex v_2.

Let $\mathcal{D}_{R_1 \cup R_2}$ be the drawing of R_1 and R_2 induced by $\mathcal{D}$. We denote by F_A, F_B, F_{12}, and F_{21} the faces in $\mathcal{D}_{R_1 \cup R_2}$ bounded by the pairs of arcs $\{A_1, A_2\}, \{B_1, B_2\}, \{A_1, B_2\}$, and $\{A_2, B_1\}$ respectively.

Let $\mathcal{C}_{v_1}$ be the component of $\mathbb{R}^2 \setminus R_2$ that contains v_1. Since by assumption R_2 has fewer than four crossings, it follows from Statement (i) of Lemma 3 that v_1 is the only vertex contained in $\mathcal{C}_{v_1}$. An analogous argument shows that v_2 is the only vertex contained in $\mathcal{C}_{v_2}$, where $\mathcal{C}_{v_2}$ denotes the component of $\mathbb{R}^2 \setminus R_1$ that contains v_2. We note that $\mathcal{C}_{v_1}$ consists of the union of the faces F_A and F_{12} with the interior of the arc A_1. Similarly, $\mathcal{C}_{v_2}$ consists of the union of F_A and F_{21} with the interior of the arc A_2.

Claim 6. *No edge intersects F_A.*

Proof. We show that if an edge intersects F_A, then we can modify $\mathcal{D}$ to obtain an optimal drawing $\mathcal{D}''$ with fewer red–red crossings than $\mathcal{D}$, contradicting the real–optimality of $\mathcal{D}$.

Suppose that F_A is intersected by some edge. Since F_A is contained in both $\mathcal{C}_{v_2}$ and $\mathcal{C}_{v_1}$, it follows from Statement (ii) of Lemma 3 that the only edges that can intersect F_A are the blue edges incident with *both* v_1 and v_2. Thus, v_1 and v_2 must be joined by a blue edge e that intersects F_A, and no edge other than e intersects F_A.

Since e is incident with both v_1 and v_2, and A_1 and A_2 form the boundary of F_A, it follows from the good–drawing property of $\mathcal{D}$ that e is contained in F_A. By Statement (iv) of Lemma 3, the other blue edge e_1 incident with v_1 crosses R_2 at a point q_2, and the other blue edge e_2 incident with v_2 crosses R_1 at a point q_1. Now we explain how to modify $\mathcal{D}$ to obtain the drawing $\mathcal{D}''$.

Let p_{B_1} and q_{B_1} be points in B_1 contained in small neighbourhoods of p and q respectively. Similarly, let p_{B_2} and q_{B_2} be points in B_2 contained in small neighbourhoods of p and q respectively. Delete the small subarcs of B_1 going from p_{B_1} to p and from q_{B_1} to q, and delete the small subarcs of B_2 going from p_{B_2} to p and from q_{B_2} to q. Also delete A_1, A_2, e, and the pieces of e_1 and e_2 contained in F_{12} and F_{21} respectively.

Join p_{B_1} and q_{B_1} by an arc A_1'' very close to A_2 contained in F_{21}, and join p_{B_2} and q_{B_2} by an arc A_2'' very close to A_1 contained in F_{12}. Let v_1'' and v_2'' be (new) vertices contained in A_1'' and A_2'' respectively. Join v_1'' and v_2'' by an edge e''. Join v_1'' to q_2 by an arc a_1'', so that the only edge that crosses a_1'' is an edge in A_2''. Similarly, join v_2'' to q_1 by an arc a_2'', so that the only edge that crosses a_2'' is an edge in A_1''. Let $\mathcal{D}''$ be the drawing thus obtained. It is trivial to check that $\mathcal{D}''$ is indeed a drawing of $C_6 \times C_n$.

It is not difficult to see that the only crossings in $\mathcal{D}$ that are not present in $\mathcal{D}''$ are p and q. Similarly, it is not difficult to check that the only crossings in $\mathcal{D}''$ that are not present in $\mathcal{D}$ are the point r_1 where a_1'' crosses A_2'' and the point r_2 where a_2'' crosses A_1''. Hence $\mathcal{D}''$ has the same number of crossings as $\mathcal{D}$. On the other hand, $\mathcal{D}''$ has two fewer red–red crossings than $\mathcal{D}$, since p and q are red–red crossings and r_1 and r_2 are red–blue crossings. This violates the real–optimality of $\mathcal{D}$. ∎

By Statement (ii) in Lemma 3, the only edges that intersect $F_A \cup F_{12}$ are the blue edges incident with v_1, and by Claim 6 none of these edges intersects F_A. Therefore, the only edges that intersect F_{12} are a blue edge B_{v_1,w_2} joining v_1 to a vertex w_2 in R_2, and a blue edge B_{v_1} incident with v_1 that crosses R_2 at a point t_2. A similar argument shows that the only edges that intersect F_{21} are the blue edge B_{v_2,w_1} joining v_2 to a vertex w_1 in R_1, and the blue edge B_{v_2} incident with v_2 that crosses R_1 at a point t_1.

By the definition of $C_6 \times C_n$, if two vertices in different red cycles R and R' are adjacent, then every vertex in R is adjacent to a vertex in R'. Since the vertex v_1 in R_1 is adjacent to the vertex w_2 in R_2, it follows that every vertex in R_1 is adjacent to a vertex in R_2. In particular, since v_1 and v_2 are the only vertices in A_1 and A_2 respectively, each vertex in B_1 different from w_1 is adjacent to a vertex in B_2 different from w_2.

Step 2. The goal in this step is to show how to modify $\mathcal{D}$ to obtain an optimal drawing $\mathcal{D}'$ in which some red cycle has at least four crossings. The next result is crucial for the construction of $\mathcal{D}'$.

Claim 7. *Let D_{v_1} and D_{v_2} be the distinct blue cycles that contain v_1 and v_2 respectively. Then there is an edge e_{u_1,u_2} joining vertices u_1 and u_2 in B_1 and B_2 respectively, such that e_{u_1,u_2} crosses at least two edges not in $D_{v_1} \cup D_{v_2}$.*

We defer the proof of Claim 7 for the moment, and use this result to finish the proof of Theorem 1.

Let u_1, u_2, and e_{u_1,u_2} be as in Claim 7. By the remark at the end of Step 1, $u_1 \neq w_1$ and $u_2 \neq w_2$. Let b_i denote the subarc of B_i going from w_i to t_i, for each $i \in \{1,2\}$. It is straightforward to check that if u_1 is in b_1, then u_1 and u_2 are in different components of D_{v_2}, and so e_{u_1,u_2} must cross an edge in D_{v_2}. A similar argument shows that if u_2 is in b_2, then e_{u_1,u_2} crosses an edge in D_{v_1}.

Now re–draw e_{u_1,u_2} in the following way to obtain a drawing $\mathcal{D}'$. Let e_{u_1,u_2} pass through the faces F_{12}, F_A, and F_{21}, so that e_{u_1,u_2} crosses each A_i exactly once. It is not difficult to check that if u_1 is not in b_1, then e_{u_1,u_2} can be drawn without crossing D_{v_2}. Similarly, if u_2 is not in b_2, then e_{u_1,u_2} can be drawn without crossing D_{v_1}.

We say that a crossing of e_{u_1,u_2} (in either $\mathcal{D}$ or $\mathcal{D}'$) is *of type I* if it involves e_{u_1,u_2} and an edge in $D_{v_1} \cup D_{v_2}$. If a crossing of e_{u_1,u_2} is not of type I, then we say it is *of type II*.

Now we show that (a) the number of crossings of type I in $\mathcal{D}'$ is not bigger than the number of crossings of type I in $\mathcal{D}$, and (b) the number of crossings of type II in $\mathcal{D}'$ is not bigger than the number of crossings of type II in $\mathcal{D}$. This finishes the proof of Theorem 1, since it follows that $\mathcal{D}'$ is also optimal, and each of R_1 and R_2 has four crossings in $\mathcal{D}'$.

The edge e_{u_1,u_2} crosses D_{v_2} in $\mathcal{D}'$ only if u_1 is in b_1. On the other hand, if u_1 is in b_1, then e_{u_1,u_2} crosses D_{v_2} in $\mathcal{D}$. Therefore e_{u_1,u_2} crosses D_{v_2} in $\mathcal{D}'$ only if e_{u_1,u_2} crosses D_{v_2} in $\mathcal{D}$. An analogous argument shows that e_{u_1,u_2} crosses D_{v_1} in $\mathcal{D}'$ only if e_{u_1,u_2} crosses D_{v_1} in $\mathcal{D}$. Hence (a) follows.

To prove (b), we note that there are exactly two crossings of type II in $\mathcal{D}'$, namely the points where e_{u_1,u_2} crosses A_1 and A_2. On the other hand, by Claim 7 there are at least two crossings of type II in $\mathcal{D}$. Therefore the number of crossings of type II in $\mathcal{D}'$ is not bigger than the number of crossings of type II in $\mathcal{D}$. ∎

Proof of Claim 7. Since R_1 contains six vertices and A_1 has only one vertex, B_1 contains exactly five vertices. Let y_1, u_1 and z_1 be vertices in B_1 distinct from w_1, ordered in such a way that as we go from p to q along B_1 we find y_1, u_1, and z_1 in this order. Let y_2, u_2, and z_2 be the vertices in B_2 adjacent to y_1, u_1, and z_2 respectively. By the remark at the end of Step 1, none of y_2, u_2, and z_2 is equal to w_2.

Let e_{u_1,u_2} be the edge that joins u_1 and u_2. To finish the proof of Claim 7, we show that e_{u_1,u_2} crosses at least two edges in neither D_{v_1} nor D_{v_2}.

Let D_{y_1}, D_{z_1} be the blue cycles containing y_1 and z_1 respectively. If D_{y_1} crosses e_{u_1,u_2}, then it does so at least twice, since D_{y_1} crosses neither R_1 nor R_2. Similarly, if D_{z_1} crosses e_{u_1,u_2} then it does so in at least two points. Since in either case Claim 7 follows, we assume that e_{u_1,u_2} crosses neither D_{y_1} nor D_{z_1}.

Every red cycle has a common vertex with each of D_{y_1} and D_{z_1}. In particular, each of R_3, R_4, R_5, and R_6 has a common vertex with each of D_{y_1} and D_{z_1}. It is easy to check that it follows that each of R_3, R_4, R_5, and R_6 crosses e_{u_1,u_2}, since neither R_1 nor R_2 is crossed by a red cycle in $\{R_3, R_4, R_5, R_6\}$, and e_{u_1,u_2} crosses neither D_{y_1} nor D_{z_1}. Thus in this case e_{u_1,u_2} crosses at least four red cycles, and so Claim 7 follows. ∎

6. Comments

Computing the exact crossing number of $C_m \times C_n$ has proved to be a very difficult task. However, in [10] it is shown that, if we specify in advance a b so that no two n–cycles intersect in more than b points, then $\lim_{n\to\infty} cr(C_m \times C_n)/(m-2)n = 1$. The general conjecture is also supported by Theorem 2 and by this work.

The best general lower bound known for the crossing number of $C_m \times C_n$ appears in [11], where it is proved that $cr(C_m \times C_n) \geq (m-2)n/3$.

Anderson et al. [2] have proved that the crossing number of $C_7 \times C_7$ is 35. This is also in agreement with the general conjecture for $cr(C_m \times C_n)$. It seems reasonable to expect that this result, together with the techniques developed above, could be used to calculate $cr(C_7 \times C_n)$. However, our experience suggests that such a proof would involve a lot more case analysis than the one we have presented to prove that $cr(C_6 \times C_n) = 4n$.

The crossing number of $C_m \times C_n$ remains unknown for all other values of m and n.

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