An improved bound for the crossing number of $C_m \times C_n$: a self–contained proof using mostly combinatorial arguments

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September 9, 2002

Abstract. We present some surprisingly elementary arguments to prove that for every $\epsilon > 0$, if m is sufficiently large, then the crossing number of the Cartesian product $C_m \times C_n$ is at least $(0.8 - \epsilon)mn$, for every $n \geq m$. The self–contained proof we give involves only one (rather elementary) geometrical result. The rest of the proof involves purely combinatorial arguments.

1 Introduction

The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane.

A long standing conjecture is that the crossing number of $C_m \times C_n$ is $(m-2)n$, for all m, n satisfying $n \geq m \geq 3$ [3]. Recently, Glebsky and Salazar [2] proved that this conjecture holds for values of n sufficiently large compared to m (roughly, for $n \geq m^2$). The general conjecture remains open.

Progress toward a possible proof of the conjecture has come slowly. General lower bounds within a constant multiplicative factor of the conjectured value for $\mathrm{cr}(C_m \times C_n)$ were only proved recently by Shahrokhi et al. [6] (they proved $\mathrm{cr}(C_m \times C_n) \geq (1/9)mn$) and by Juarez and Salazar [4] (they proved the bound $\mathrm{cr}(C_m \times C_n) \geq (1/2)(m-2)n$).

It is not difficult to exhibit drawings of $C_m \times C_n$ with exactly $(m-2)n$ crossings, which is thus an upper bound (see for instance [3] or [5]). Lower bounds (of crossing numbers of graphs in general) are a lot more difficult to compute.

Our aim in this paper is to present some surprisingly elementary arguments that yield a remarkably good lower bound for $\mathrm{cr}(C_m \times C_n)$.

Theorem 1 *For every $\epsilon > 0$, there is an N_ϵ with the following property. For all integers m, n such that $n \geq m \geq N_\epsilon$, $cr(C_m \times C_n) \geq (0.8 - \epsilon)mn$.*

Since this holds for every $n \geq m$, this bound is better than the best general lower bound previously known (namely $\mathrm{cr}(C_m \times C_n) \geq (1/2)(m-2)n$), as long as m is sufficiently large.

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Recall that, for positive integers m, n, the *Cartesian product* of the cycles of sizes m and n is a 4–regular graph with mn vertices, labeled $v_{i,j}$, $i = 0, \dots, m-1$, $j = 0, \dots, n-1$. Vertices $v_{i-1,j}$ and $v_{i+1,j}$ are joined to $v_{i,j}$ by *red* edges, and $v_{i,j-1}$ and $v_{i,j+1}$ are joined to $v_{i,j}$ by the *blue* edges $b_{i,j}$ and $b_{i,j+1}$ respectively. Indices are read modulo m and n, respectively. The edge set of $C_m \times C_n$ is thus naturally partitioned into m edge sets of *blue* n–cycles and n edge sets of *red* m–cycles.

The only geometrical result needed in the proof of Theorem 1 (namely Lemma 3) is proved in Section 2. This is a rather elementary observation on drawings of subgraphs of $C_m \times C_n$ induced by three consecutive red cycles. The rest of the paper deals with combinatorial arguments.

Theorem 1 is reduced to a straightforward induction by the following result: in every drawing of $C_m \times C_n$, some monochromatic cycle has a large number of crossings.

Lemma 2 *Let $\epsilon > 0$ be given. Then there is an integer M_ϵ with the following property. In every drawing of $C_m \times C_n$ with $n \geq m \geq M_\epsilon$, there is either a red cycle with at least $(0.8 - \epsilon)m$ crossings or a blue cycle with at least $(0.8 - \epsilon)n$ crossings.*

To prove Lemma 2, we need to establish a couple of relationships between the number of blue–blue crossings, the average number of bichromatic crossings in a red cycle, and the size of a maximal pairwise noncrossing family of red cycles. These statements (namely Lemmas 4 and 5) are given in Section 3.

The proof of Lemma 2 is in Section 4. As we mentioned above, with the use of this result the proof of Theorem 1 reduces to a straightforward induction. This is the content of Section 5. Section 6 contains some concluding remarks.

2 Analysis of crossings in $(3, s)$–configurations

Our aim in this section is to analyze crossings in subgraphs of Cartesian products of cycles induced by three consecutive red cycles.

A set $\mathcal{S}$ of closed curves is *nonseparating* if, for every curve $S \in \mathcal{S}$, no two curves in $\mathcal{S} \setminus S$ are contained in different components of $\mathbb{R}^2 \setminus S$.

Let s be a positive integer. A $(3, s)$*–configuration* is a pair $(\mathcal{C}, \mathcal{P})$, where $\mathcal{C} = \{C_0, C_1, C_2\}$ is an ordered collection of s–cycles, and $\mathcal{P}$ is a collection of 2–paths starting at C_0, ending at C_2, and with middle vertex in C_1. For each $P \in \mathcal{P}$, the edge of P incident with C_0 and C_1 (respectively C_1 and C_2) is the *initial* (respectively *final*) edge of P.

A *clean* drawing of a $(3, s)$–configuration is a drawing such that (i) $\mathcal{C}$ is nonseparating; (ii) the cycles in $\mathcal{C}$ are pairwise noncrossing; (iii) no initial edge crosses $C_1 \cup C_2$; and (iv) no final edge crosses $C_0 \cup C_1$. A crossing in a clean drawing is *good* if it occurs either (i) between two initial edges or (ii) between an initial edge and a final edge not in the same 2–path.

Lemma 3 *Every clean drawing of a $(3, s)$–configuration has at least $s(s-2)/4$ good crossings.*

Proof. Let $\mathcal{D}$ be a clean drawing of a $(3, s)$–configuration $(\{C_0, C_1, C_2\}, \mathcal{P})$. Since if $s < 3$ then there is nothing to prove, we assume $s \geq 3$. Since $\binom{\lfloor s/2 \rfloor}{2} + \binom{\lceil s/2 \rceil}{2} \geq (1/4)s(s-2)$ for every integer s, it suffices to show that $\mathcal{D}$ has at least $\binom{\lfloor s/2 \rfloor}{2} + \binom{\lceil s/2 \rceil}{2}$ good crossings.

Let G denote the graph with vertex set $\mathcal{P}$, such that two elements P, Q in $\mathcal{P}$ are adjacent iff there is a good crossing that involves an edge in P and an edge in Q. An application of Turán's theorem to G yields that if G has no independent set of size three, then $\mathcal{D}$ has at least $\binom{\lfloor s/2 \rfloor}{2} + \binom{\lceil s/2 \rceil}{2}$ crossings, as required. Therefore, it suffices to show that, for any collection of three 2–paths in $\mathcal{P}$, there is a good crossing that involves two 2–paths in the collection.

Let P_0, P_1, P_2 be distinct 2–paths in $\mathcal{P}$. For each i, label the initial (respectively final) edge of P_i by e_i (respectively f_i).

Since $\mathcal{D}$ is a clean drawing then, for each i, e_i crosses neither C_1 nor C_2. By restricting each e_i to its (unique) subsegment from C_0 to C_1 that does not cross C_0, if necessary, we may also assume that, for each i, e_i does not cross C_0 either. Similarly, we can assume that, for each i, f_i crosses neither C_0 nor C_1 nor C_2.

Suppose that C_0 has a self–crossing (that is, two edges of C_0 cross each other at a point x). Then, by rerouting the portions of the edges in a small neighbourhood of x, we can eliminate the self–crossing. By repeating this procedure if necessary, and applying it to C_1 and C_2 if necessary, we can assume that no C_i has a self–crossing.

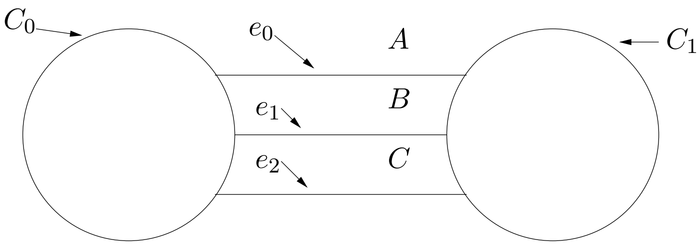

FIGURE 1. Since C_3 does not cross $C_0 \cup C_1 \cup e_0 \cup e_1 \cup e_2$, then C_3 is drawn inside one of the components labelled A, B, and C. No matter in which such component C_3 is drawn, it is evident that some final edge f_i will cross some initial edge e_j.

If two e_i's cross each other then we are done. Thus, we may assume the e_i's are pairwise disjoint. Thus the drawing $\mathcal{D}$ restricted to $C_0 \cup C_1 \cup e_0 \cup e_1 \cup e_2$ is homeomorphic to the drawing in Figure 1. Since by assumption no edge in C_2 crosses an edge in $C_0 \cup C_1 \cup e_0 \cup e_1 \cup e_2$, it follows that C_2 is contained in one of the components A, B, C of $\mathbb{R}^2 \setminus (C_0 \cup C_1 \cup e_0 \cup e_1 \cup e_2)$. It is now evident that no matter which such component contains C_2, some f_i will have to cross an e_j in the drawing. Since such a crossing is good, we are done. ∎

3 Blue–blue crossings in drawings of $C_m \times C_n$

Let $\mathcal{D}$ be a drawing of $C_m \times C_n$. A blue–blue crossing in $\mathcal{D}$ is *proper* if the edges involved are not in the same blue cycle. We denote by $\mathrm{bc}(\mathcal{D})$ (respectively $\mathrm{blu}(\mathcal{D})$)

the total number of bichromatic (respectively, proper blue–blue) crossings in $\mathcal{D}$. For each red cycle R, let $\text{bc}(\mathcal{D}, R)$ denote the number of bichromatic crossings (in $\mathcal{D}$) that involve an edge in R. Thus, $\text{bc}(\mathcal{D}) = \sum \text{bc}(\mathcal{D}, R)$, where the sum is over all red cycles in $C_m \times C_n$. A drawing of $C_m \times C_n$ is *red–disjoint* if no two red cycles intersect each other.

If $\mathcal{R}$ is a collection of red cycles in a drawing $\mathcal{D}$ of $C_m \times C_n$, then we let $\overline{\text{bc}}(\mathcal{D}, \mathcal{R})$ denote the average number of bichromatic crossings in a cycle in $\mathcal{R}$.

Lemma 4 *Let $\mathcal{D}$ be a drawing of $C_m \times C_n$, where $n \geq m \geq 3$. Suppose that no red cycle has $0.8(m-2)$ or more crossings. Let $\mathcal{R}$ be a maximal pairwise noncrossing collection of red cycles. Then $|\mathcal{R}| \geq 2n/(2 + 0.8(m-2) - \overline{\text{bc}}(\mathcal{D}, \mathcal{R}))$.*

Proof. Since $\mathcal{R}$ is maximal, each red cycle not in $\mathcal{R}$ crosses a cycle in $\mathcal{R}$. Since no red cycle has $0.8(m-2)$ or more crossings, and red cycles that cross do so in at least two points, it follows that each cycle R in $\mathcal{R}$ crosses fewer than $(0.8(m-2) - \text{bc}(\mathcal{D}, R))/2$ red cycles.

For each $R \in \mathcal{R}$, let $\mathcal{C}(R)$ denote the collection of red cycles that cross R. By the observation above, $|\mathcal{C}(R)| \leq (0.8(m-2) - \text{bc}(\mathcal{D}, R))/2$. On the other hand, every red cycle is either in $\mathcal{R}$ or in $\mathcal{C}(R)$ for some R in $\mathcal{R}$. Therefore, $n \leq |\mathcal{R}| + \sum_{R \in \mathcal{R}} |\mathcal{C}(R)| \leq |\mathcal{R}| + \sum_{R \in \mathcal{R}} (0.8(m-2) - \text{bc}(\mathcal{D}, R))/2 = |\mathcal{R}|(1 + (0.8(m-2)/2 - \overline{\text{bc}}(\mathcal{D}, \mathcal{R})/2))$. Hence $|\mathcal{R}| \geq 2n/(2 + (0.8(m-2) - \overline{\text{bc}}(\mathcal{D}, \mathcal{R})))$. ∎

Lemma 5 *Let m, q be integers such that $q, m \geq 3$. Let $\mathcal{E}$ be a red–disjoint drawing of $C_m \times C_q$ such that no red cycle has $0.8(m-2)$ or more crossings. Let $\mathcal{R}$ denote the collection of all red cycles in $C_m \times C_q$. Then $\text{blu}(\mathcal{E}) \geq (q/4)(m - \overline{\text{bc}}(\mathcal{E}, \mathcal{R}))(m - \overline{\text{bc}}(\mathcal{E}, \mathcal{R}) - 2)$.*

Proof. Let $\mathcal{E}$ be a drawing of $C_m \times C_q$ as in the statement of the lemma. Label the red cycles $(v_{i,j})$, $0 \leq i \leq m-1$, by R_j, $0 \leq j \leq q-1$. Using this notation, the blue edges $b_{i,j}$ are incident to the red cycles R_{j-1} and R_j.

Let α_j (respectively $\beta_j, \gamma_j, \delta_j$) denote the number of blue edges $b_{i,j}$ that cross R_{j-2} (respectively R_{j-1}, R_j, R_{j+1}). Let $\Omega_j = \alpha_{j+1} + \beta_{j+1} + \gamma_j + \delta_j$. Since each bichromatic crossing contributes in 1 to at most one element in $\{\alpha_j, \beta_j, \gamma_j, \delta_j\}_{j=0}^{q-1}$, it follows that $\text{bc}(\mathcal{E}) \geq \sum_{j=0}^{q-1} \Omega_j$.

We now show that $\text{blu}(\mathcal{E}) \geq (1/4) \sum_{j=0}^{q-1} \max\{0, (m - \Omega_j)(m - \Omega_j - 2)\}$.

To prove this, it suffices to show that, for each fixed j, the number of blue–blue crossings in $\mathcal{E}$ that involve an edge of the form $b_{i,j}$ and an edge either of the form $b_{i',j}$ or of the form $b_{i',j+1}$ is at least $(1/4)(m - \Omega_j)(m - \Omega_j - 2)$. If $\Omega_j \geq m-2$ then this is obviously true. Thus we only analyze those j for which $\Omega_j \leq m-3$.

For each $j \in \{0, \ldots, q-1\}$, define $\mathcal{I}_j = \{i \in \{0, \ldots, m-1\} | (b_{i,j}$ crosses neither R_j nor $R_{j+1})$ and $(b_{i,j+1}$ crosses neither R_{j-1} nor $R_j)\}$. It follows immediately that $|\mathcal{I}_j| \geq m - (\alpha_{j+1} + \beta_{j+1} + \gamma_j + \delta_j) = m - \Omega_j$.

Consider the subdrawing $\mathcal{E}_j$ of $\mathcal{E}$ induced by the red cycles R_{j-1}, R_j, R_{j+1}, and the 2–paths $v_{i,j-1}v_{i,j}v_{i,j+1}$ such that $i \in \mathcal{I}_j$. Clearly, $\mathcal{E}_j$ is a clean drawing of a $(3, |\mathcal{I}_j|)$–configuration. Moreover, the good crossings of $\mathcal{E}_j$ are precisely those that involve an edge of the form $b_{i,j}$ and an edge either of the form $b_{i',j}$ or of the form $b_{i',j+1}$ (with $i \neq i'$ in both cases).

Thus it follows from Lemma 3 that there are at least $|\mathcal{I}_j|(|\mathcal{I}_j| - 2)/4$ crossings in $\mathcal{E}$ that involve an edge of the form $b_{i,j}$ and an edge either of the form $b_{i',j}$ or of the form $b_{i',j+1}$ (with $i \neq i'$). Since every such crossing contributes to $\mathrm{blu}(\mathcal{E})$, we conclude $\mathrm{blu}(\mathcal{E}) \geq (1/4)\sum_{j=0}^{q-1} \max\{0, (m - \Omega_j)(m - \Omega_j - 2)\}$.

When is $(1/4)\sum_{j=0}^{q-1} \max\{0, (m - \Omega_j)(m - \Omega_j - 2)\}$ minimized, subject to $0 \leq \Omega_j \leq m - 3$ and $\mathrm{bc}(\mathcal{E}) \geq \sum_{j=0}^{q-1} \Omega_j$?

An elementary Lagrange multipliers manipulation shows that the minimum is attained when the Ω_j's are equal to each other, and their sum is maximum possible. That is, the minimum is attained when $\Omega_j = \mathrm{bc}(\mathcal{E})/q$ for every $j \in \{0, \ldots, q-1\}$. Therefore $\mathrm{blu}(\mathcal{E}) \geq (q/4)(m - \mathrm{bc}(\mathcal{E})/q)(m - \mathrm{bc}(\mathcal{E})/q - 2) = (q/4)(m - \overline{\mathrm{bc}}(\mathcal{E}, \mathcal{R}))(m - \overline{\mathrm{bc}}(\mathcal{E}, \mathcal{R}) - 2$. ∎

4 Proof of Lemma 2

Let $\epsilon > 0$ be given. First we prove that there is an integer m_ϵ such that every drawing of $C_m \times C_n$ with $n \geq m \geq m_\epsilon$, has either a red cycle with at least $(0.8 - \epsilon/2)(m - 2)$ crossings or a blue cycle with at least $(0.8 - \epsilon/2)n$ crossings.

Let $\mathcal{D}$ be a drawing of $C_m \times C_n$ such that $n \geq m \geq 3$. Let $\mathcal{R}$ be a maximal collection of pairwise noncrossing red cycles in $\mathcal{D}$. Suppose that no red cycle has $(0.8 - \epsilon/2)(m - 2)$ or more crossings. We will show that if m is sufficiently large, then $2\mathrm{blu}(\mathcal{D}) \geq (0.8 - \epsilon/2)mn$. Since proper blue–blue crossings occur between different blue cycles, by the pigeon–hole principle this implies that there is a blue cycle with at least $(0.8 - \epsilon/2)n$ crossings, as required.

Let $|\mathcal{R}| = q$, and let $\mathcal{E}$ denote the drawing of $C_m \times C_q$ induced (from $\mathcal{D}$) by the blue cycles plus the cycles in $\mathcal{R}$. Note that $\overline{\mathrm{bc}}(\mathcal{E}, \mathcal{R}) = \overline{\mathrm{bc}}(\mathcal{D}, \mathcal{R})$, and $\mathrm{blu}(\mathcal{D}) = \mathrm{blu}(\mathcal{E})$.

By Lemma 5, $2\mathrm{blu}(\mathcal{E}) \geq (q/2)(m - \overline{\mathrm{bc}}(\mathcal{E}, \mathcal{R}))(m - \overline{\mathrm{bc}}(\mathcal{E}, \mathcal{R}) - 2)$. By Lemma 4, $q \geq 2n/(2 + 0.8(m - 2) - \overline{\mathrm{bc}}(\mathcal{E}, \mathcal{R}))$. Therefore, $2\mathrm{blu}(\mathcal{E}) \geq n \cdot (m - \overline{\mathrm{bc}}(\mathcal{E}, \mathcal{R}))(m - \overline{\mathrm{bc}}(\mathcal{E}, \mathcal{R}) - 2)/(0.8m - \overline{\mathrm{bc}}(\mathcal{E}, \mathcal{R}) + 0.4)$

An elementary calculus argument shows that there is an m_ϵ such that, if $m \geq m_\epsilon$, then $(m - \overline{\mathrm{bc}}(\mathcal{E}, \mathcal{R}))(m - \overline{\mathrm{bc}}(\mathcal{E}, \mathcal{R}) - 2)/(0.8m - \overline{\mathrm{bc}}(\mathcal{E}, \mathcal{R}) + 0.4) \geq (0.8 - \epsilon/2)m$. Thus, if $m \geq m_\epsilon$, then $2\mathrm{blu}(\mathcal{E}) = 2\mathrm{blu}(\mathcal{D}) \geq (0.8 - \epsilon/2)mn$. We have then proved that $\mathcal{D}$ has either a red cycle with at least $(0.8 - \epsilon/2)(m - 2)$ crossings or a blue cycle with at least $(0.8 - \epsilon/2)n$ crossings.

Finally, define $M_\epsilon = \max\{m_\epsilon, (16/5\epsilon) - 2\}$. Suppose that $n \geq m \geq M_\epsilon$. Since $M_\epsilon \geq m_\epsilon$, then every drawing of $C_m \times C_n$ has either a red cycle with at least $(0.8 - \epsilon/2)(m - 2)$ crossings or a blue cycle with at least $(0.8 - \epsilon/2)n$ crossings. On the other hand, (i) $M_\epsilon \geq (16/5\epsilon) - 2$ guarantees that $(0.8 - \epsilon/2)(m - 2) \geq (0.8 - \epsilon)m$; and trivially (ii) $(0.8 - \epsilon/2)n \geq (0.8 - \epsilon)n$. Thus, every drawing of $C_m \times C_n$ has either a red cycle with at least $(0.8 - \epsilon)m$ crossings or a blue cycle with at least $(0.8 - \epsilon)n$ crossings. ∎

5 Proof of Theorem 1

Let $\epsilon > 0$ be given. Let $M_{\epsilon/2}$ be as in Lemma 2. First we show that $\mathrm{cr}(C_m \times C_n) \geq (0.8 - \epsilon/2)(m - M_{\epsilon/2})n$, for all m, n such that $n \geq m \geq M_{\epsilon/2}$.

We proceed by induction on $m + n$. As base case we use the whole collection $C_m \times C_n$ with $m = M_{\epsilon/2}$, for which there is nothing to prove.

For the inductive step, we suppose the inequality holds for all m, n such that $n \geq m \geq M_{\epsilon/2}$ and $m + n \leq k$ (where $k \geq 2M_{\epsilon/2}$). Thus, we need to prove the inequality for all integers m, n such that $n \geq m > M_{\epsilon/2}$ and $m + n = k + 1$.

So, let m, n be such that $n \geq m > M_{\epsilon/2}$, and $m+n = k+1$, and let $\mathcal{D}$ be a drawing of $C_m \times C_n$. By Lemma 2, there is either a blue cycle B with at least $(0.8 - \epsilon/2)n$ crossings or a red cycle R with at least $(0.8 - \epsilon/2)m$ crossings.

In the first case, let $\mathcal{D}'$ be the drawing of $C_{m-1} \times C_n$ that results by removing B from $\mathcal{D}$. By the induction hypothesis, $\mathcal{D}'$ has at least $(0.8 - \epsilon/2)((m-1) - M_{\epsilon/2})n$ crossings, and so $\mathcal{D}$ has at least $(0.8-\epsilon/2)((m-1)-M_{\epsilon/2})n+(0.8-\epsilon/2)n = (0.8-\epsilon/2)(m-M_{\epsilon/2})n$ crossings, as required.

Suppose now that the second case applies. Note that we can assume $n > m$, as otherwise ($n = m$) blue and red cycles are interchangeable, and we could assume that the first case applies. Now let $\mathcal{D}''$ be the drawing of $C_m \times C_{n-1}$ that results by removing R from $\mathcal{D}$. By the induction hypothesis, $\mathcal{D}''$ has at least $(0.8 - \epsilon/2)(m - M_{\epsilon/2})(n - 1)$ crossings, and so $\mathcal{D}$ has at least $(0.8 - \epsilon/2)(m - M_{\epsilon/2})(n - 1) + (0.8 - \epsilon/2)m > (0.8 - \epsilon/2)(m - M_{\epsilon/2})n$ crossings, as required.

Thus we have shown that $\mathrm{cr}(C_m \times C_n) \geq (0.8 - \epsilon/2)(m - M_{\epsilon/2})n$, for all m, n such that $n \geq m \geq M_{\epsilon/2}$.

Finally, define $N_\epsilon = M_{\epsilon/2}((8/5\epsilon) - 1)$. A straightforward manipulation shows that for every $m \geq N_\epsilon$, $(0.8 - \epsilon/2)(m - M_{\epsilon/2})n \geq (0.8 - \epsilon)mn$. Thus, if $n \geq m \geq N_\epsilon$, then $\mathrm{cr}(C_m \times C_n) \geq (0.8 - \epsilon)mn$, as claimed. ∎

6 Concluding Remarks

As we mentioned in the Introduction, although the conjecture $\mathrm{cr}(C_m \times C_n) = (m-2)n$ was recently confirmed for values of n large compared to m, the general conjecture remains open. One of the main goals in this paper is to show that some rather elementary arguments, based on only one geometrical result, suffice to give a lower bound within a constant factor of the conjectured value.

We naturally hope that our approach will foster fresh, novel attempts to settle the conjecture.

Recently, Adamsson and Richter [1] developed a theory of arrangements, and analyzed some of its implications for the conjecture $\mathrm{cr}(C_m \times C_n) = (m-2)n$. Among other strong and interesting results (such as settling the conjecture for $n \geq m = 7$), they proved that every drawing of $C_m \times C_n$ in which the union of the blue cycles has four or more components, has at least $(m - 2)n$ crossings. Adamsson and Richter recently lowered the requirement from four to three components (private communication).

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