Simple Euclidean arrangements with no (≥ 5)–gons are stretchable

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Abstract

It is shown that if a simple Euclidean arrangement of n pseudolines has no (≥ 5)–gons, then it has exactly $n-2$ triangles and $(n-2)(n-3)/2$ quadrilaterals. We also describe how to construct all such arrangements, and as a consequence we show that they are all stretchable.

1 Introduction

Our goal in this discussion is to analyze simple Euclidean arrangements of pseudolines in which every bounded cell is either a triangle or a quadrilateral.

We recall that a simple noncontractible closed curve in the projective plane $\mathbb{P}$ is a *pseudoline*, and an *arrangement of pseudolines* is a collection $\mathcal{B} = \{P_0, P_1, \ldots, P_n\}$ of pseudolines that intersect (necessarily cross) pairwise exactly once. Since $\mathbb{P} \setminus P_0$ is homeomorphic to the Euclidean plane $\mathbb{E}$, we may regard $\{P_1, \ldots, P_n\}$ as an *arrangement of pseudolines* in $\mathbb{E}$ (and regard $P_1, \ldots, P_n$ as *pseudolines* in $\mathbb{E}$). An arrangement is *simple* if no point belongs to more than two pseudolines.

The cell complex of an Euclidean arrangement in $\mathbb{P}$ has both bounded and unbounded cells. As in [4], we are only interested in bounded cells (whose interiors are the *faces*). Thus it is clear what is meant by a *triangle*, a *quadrilateral*, or, in general, an *n–gon* of the arrangement. In this work we are interested in arrangements in which every bounded cell is either a triangle or a quadrilateral; for obvious reasons we say that such an arrangement is (≥ 5)-*gon—free*.

Our first observation is that there are exponentially many nonisomorphic (≥ 5)-gon–free arrangements. To see this, for each $n \geq 3$ let $\mathcal{T}_n$ denote the set of all rooted full binary trees on $2n-1$ vertices (consequently n leaves), in which every leaf is labeled with a 1, and the label of each internal vertex is the sum of the labels of its two children. Thus the root of each tree in $\mathcal{T}_n$ is labelled n. The key observation is that, for each $n \geq 3$, there is a set $\mathbf{A}_{n+1}$ of pairwise nonisomorphic (≥ 5)-gon–free arrangements of $n+1$ pseudolines, and a two–to–one mapping from the set $\mathcal{T}_n$ to $\mathbf{A}_{n+1}$. The correspondence is illustrated in Figure 1. The mapping is two–to–one because the arrangements obtained from a tree and its reflection (around the root) are equivalent. On the other hand, if T' is not the reflection around the root of T, then the arrangements induced by T' and T are not equivalent: this is easily checked if we view the arrangement from the perspective

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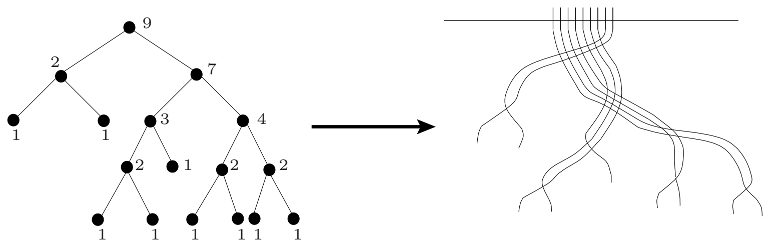

Figure 1: How to associate a (≥ 5)-gon–free arrangement with $n+1$ pseudolines to each labelled tree in $\mathcal{T}_n$. Here we show a labelled tree in $\mathcal{T}_9$ and its associated (≥ 5)-gon–free arrangement with 10 pseudolines.

of the topmost horizontal line. A standard counting argument shows that $|\mathcal{T}_n|$ grows exponentially with n, and so the claim follows.

One of the most interesting and widely studied problems concerning arrangements of lines and pseudolines is the determination of upper and lower bounds for the number p_k of k–sided faces. An extensive amount of research in such problems followed Grünbaum's seminal work [7] on arrangements and spreads (see for instance [5, 11, 12]).

In [4], it was proved that in every simple Euclidean arrangement, $p_3 \geq n-2$. Here we show that if the arrangement is (≥ 5)-gon–free, then equality always holds.

Theorem 1 *In every simple Euclidean (≥ 5)-gon–free arrangement of n pseudolines, $p_3 = n-2$, and $p_4 = (n-2)(n-3)/2$.*

We also obtain a procedure with which every (≥ 5)-gon–free simple Euclidean arrangement can be recursively constructed from the (unique up to isomorphism) simple Euclidean arrangement with 3 pseudolines. The central concept in this regard are twin pseudolines. Consider a simple Euclidean arrangement of pseudolines $\mathcal{B} = \{P_1, P_2, \ldots, P_n\}$, and let $P_i, P_j \in \mathcal{B}$. Then the intersection point $p_{i,j}$ between P_i and P_j divides P_i (respectively P_j) into two subarcs, say A_i and B_i (respectively A_j and B_j). We say that P_i and P_j are *twin* pseudolines (see Fig. 2) if the following hold:

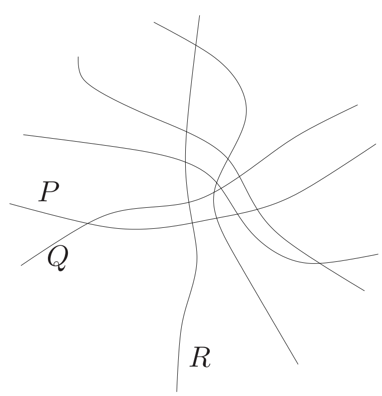

Figure 2: A simple Euclidean (≥ 5)-gon–free arrangement with 6 pseudolines. Pseudolines P and Q are twin. The triangle defined by P, Q, and R, is P–critical and Q–critical, but not R–critical.

(i) One of A_i and B_i (say B_i, without any loss of generality) does not contain any intersection point of P_i with the pseudolines in $\mathcal{B} \setminus \{P_i, P_j\}$. Similarly, one of A_j and B_j (say B_j, without any loss of generality) does not contain any intersection point of P_j with the pseudolines in $\mathcal{B} \setminus \{P_i, P_j\}$.

(ii) As we traverse each of A_i and A_j, starting at $p_{i,j}$, we intersect the pseudolines in $\mathcal{B} \setminus \{P_i, P_j\}$ in the exact same order.

Twin pseudolines are essential for the understanding of (≥ 5)-gon–free arrangements, as the following statement reveals.

Theorem 2 *In every simple (≥ 5)-gon–free Euclidean arrangement with at least 4 pseudolines there are distinct pseudolines Q_1, Q_2, Q_3, Q_4 such that Q_1 and Q_2 are twin, and Q_3 and Q_4 are twin.*

Remark 3 *Theorem 2 is best possible in every sense: (i) for each $n \geq 4$ there is a simple (≥ 5)-gon–free Euclidean arrangement $\mathcal{A}$ with n pseudolines, with exactly two pairs of twin pseudolines; and (ii) for each $n \geq 5$ there is a simple Euclidean arrangement $\mathcal{A}$ with n pseudolines, only one (≥ 5)–gon, and such that $\mathcal{A}$ has no twin pseudolines.*

The constructions that prove Remark 3 are given in Figure 3. Start by drawing P_1, P_2, P_3, P_4 as shown. Now suppose that for some $r \geq 4$, $\{P_1, P_2, \ldots, P_r\}$ has been constructed. If r is even, then add P_{r+1} so that P_{r-1} and P_{r+1} are twin, in such a way that as we traverse P_{r+1} from right to left, the first curve in $\{P_1, P_2, \ldots, P_r\}$ we intersect is P_{r-1}. If r is odd, then add P_{r+1} so that P_r and P_{r+1} are twin, in such a way that as we traverse P_{r+1} from left to right, the first curve in $\{P_1, P_2, \ldots, P_r\}$ we intersect is P_r. The simple Euclidean arrangement $\{P_1, P_2, \ldots, P_r\}$ obtained at each step is (≥ 5)-gon–free, and has exactly two pairs of twin pseudolines, namely $\{P_1, P_2\}$ and $\{P_{r-1}, P_{r+1}\}$ if r is even, and $\{P_1, P_2\}$ and $\{P_r, P_{r+1}\}$ if r is odd. Now suppose that $\{P_1, P_2, \ldots, P_k\}$ has been constructed for some $k \geq 4$, and add a pseudoline Q (dashed pseudoline in this figure) as illustrated. Then the simple Euclidean arrangement $\{P_1, P_2, \ldots, P_k, Q\}$ has no twin pseudolines at all, and it has exactly one (≥ 5)–gon, namely the 5–gon bounded by P_1, P_2, P_{k-1}, P_k, and Q.

Theorem 2 not only gives a procedure to generate all simple (≥ 5)-gon–free Euclidean arrangements, but also has a neat consequence in the realm of stretchability.

An *arrangement of lines* in $\mathbb{E}$ is a collection of straight lines, no two of them parallel. Thus every arrangement of lines is an arrangement of pseudolines. On the other hand, not every arrangement of pseudolines is *stretchable*, that is, equivalent to an arrangement of lines (recall that two arrangements are *equivalent* if they generate isomorphic cell decompositions of $\mathbb{E}$). Every arrangement of 8 pseudolines is stretchable [6], but there is a simple non–stretchable arrangement of 9 pseudolines [10] (unique up to isomorphism; see [8]). Stretchability questions are typically difficult: deciding stretchability is NP–hard [14], even for simple arrangements [2].

The concept of stretchability is particularly relevant because of the close connection between arrangements of pseudolines and rank 3 oriented matroids: on this ground, the problem of stretchability of arrangements is equivalent to the problem of realizability for oriented matroids (see [1, 9]).

The following consequence of Theorem 2 settles the issue of stretchability for (≥ 5)-gon–free arrangements.

Theorem 4 *Every simple (≥ 5)-gon–free Euclidean arrangement is stretchable.*

Theorems 1, 2, and 4, are proved in Sections 2, 3, and 4, respectively.

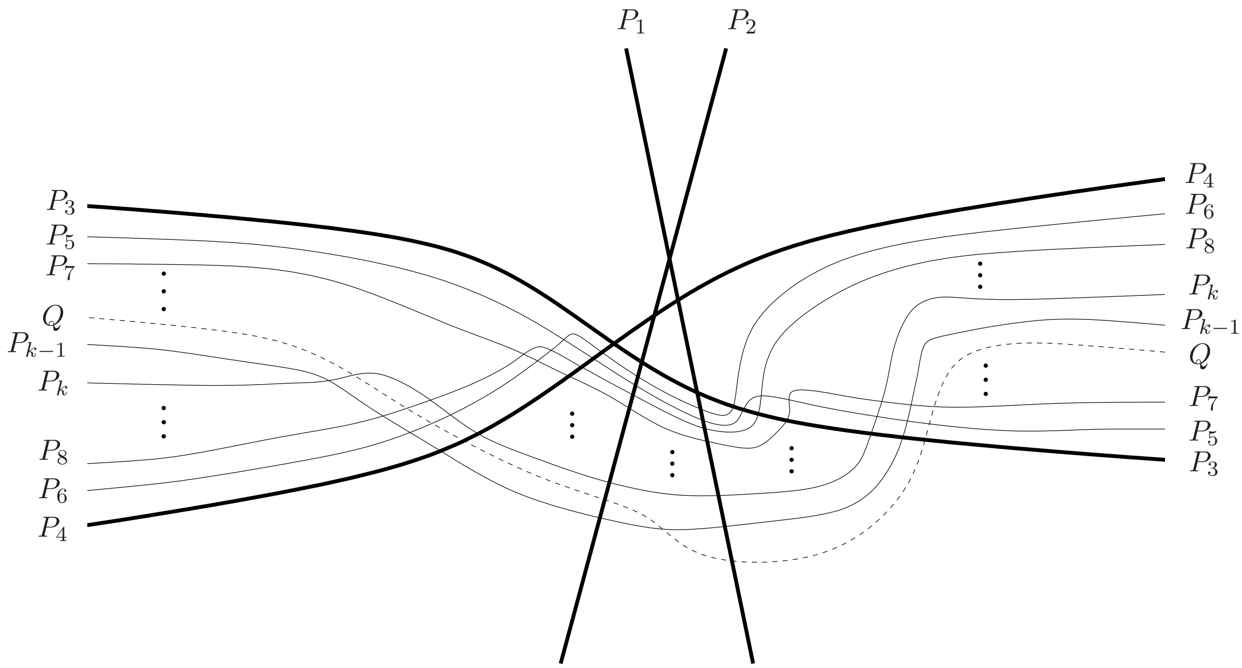

Figure 3: This construction shows that Theorem 2 is best possible (see Remark 3).

2 Simple Euclidean (≥ 5)-gon–free arrangements have exactly $n-2$ triangles: proof of Theorem 1

In [4], Felsner and Kriegel showed that every simple Euclidean arrangement $\mathcal{B}$ with n lines has at least $n-2$ triangles (that is, $p_3(\mathcal{B}) \geq n-2$). Our goal this section is to prove Theorem 1: this minimum number of triangles is attained in every (≥ 5)-gon–free arrangement.

We start with a straightforward, but essential, observation.

Observation 5 *If $\mathcal{B}$ is a simple Euclidean (≥ 5)-gon–free arrangement, then every subarrangement of $\mathcal{B}$ is also (≥ 5)-gon–free.*

Proof. If an arrangement has an r–gon D with $r \geq 5$, and we add a pseudoline P to it, then either P leaves D untouched, or divides D into two polygons, at least one of which has at least 5 sides. In either case, the augmentated arrangement also has an r–gon with $r \geq 5$. ∎

Proof of Theorem 1. By Theorem 2 in [4], every simple Euclidean arrangement has at least $n-2$ triangles. Thus we need to prove that the converse inequality $p_3 \leq n-2$ holds for (≥ 5)-gon–free arrangements. We note that this implies that $p_4 = (n-2)(n-3)/2$, since the number of bounded cells in every simple Euclidean arrangement is $(n-1)(n-2)/2$.

We proceed by induction on n. The statement is clearly true for the unique (up to isomorphism) simple Euclidean arrangement with 3 pseudolines. Now suppose it holds for $n = k \geq 3$, and consider a (≥ 5)-gon–free simple Euclidean arrangement $\mathcal{B} = \{P_1, P_2, \ldots, P_{k+1}\}$. As usual, let $p_3(\mathcal{B})$ denote the number of triangles in the arrangement $\mathcal{B}$. By inductive assumption, $p_3(\mathcal{B} \setminus \{P_{k+1}\}) = k-2$.

If P is a pseudoline in an arrangement, and T is a triangle with a side in P, then T is *P–critical* if the side of T that lies in P is also part of the (topological) boundary of an unbounded region. For instance, the triangle in Fig. 2 bounded by P, Q, and R, is both P–critical and Q–critical. In

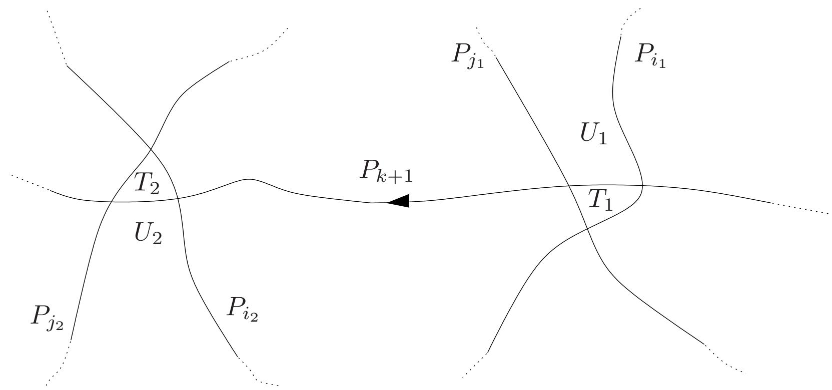

Figure 4: If P_{i_1} and P_{j_2} intersect each other to the right of P_{k+1}, then they will bound, together with P_{k+1}, the region U_1. If they intersect to the left of P_{k+1}, then they will bound, together with P_{k+1}, the region U_2.

every simple (≥ 5)-gon–free Euclidean arrangement with at least $n \geq 4$ pseudolines there is at least two pair of twin pseudolines.

We note that $p_3(\mathcal{B}) - p_3(\mathcal{B} \setminus \{P_{k+1}\})$ equals the number of P_{k+1}–critical triangles. This follows easily from the observation that if P_{k+1} goes through a quadrilateral in $\mathcal{B} \setminus \{P_{k+1}\}$, then it divides it into two quadrilaterals (otherwise a pentagon is created), and if it goes trough a triangle in $\mathcal{B} \setminus \{P_{k+1}\}$, it divides it into a triangle and a quadrilateral. It follows from this observation and the inductive hypothesis that in order to finish the proof it suffices to show that there is at most one P_{k+1}–critical triangle in $\mathcal{B}$.

Seeking a contradiction, suppose that there are two P_{k+1}–critical triangles, say T_1 and T_2, in $\mathcal{B}$. As we traverse P_{k+1} in a given direction, we find the interior of each T_i either to our left or to our right. By changing the direction of traversal and/or exchanging T_1 and T_2 if necessary, we may assume that either (i) we find the interior of T_1 to our left, and then the interior of T_2 to our right; or (ii) we find both interiors to our right. We conclude the proof by showing that either possibility leads to a contradiction.

Suppose that (i) holds. Let U_1 (respectively U_2) denote the unbounded region whose topological boundary contains the side of T_1 (respectively T_2) contained in P_{k+1}. This is illustrated in Figure 4. We let P_{i_1}, P_{j_1} (respectively P_{i_2}, P_{j_2}) denote the pseudolines that, together with P_{k+1}, bound T_1 (respectively T_2), in the order that they intersect P_{k+1} as we traverse it in the given direction. Now regardless of where P_{i_1} and P_{j_2} intersect each other, they will bound, together with P_{k+1}, either U_1 or U_2, contradicting the unboundedness of these regions.

Suppose finally that (ii) holds. Define $U_1, U_2, P_{i_1}, P_{j_1}, P_{i_2}$, and P_{j_2} as in the previous paragraph. The situation is depicted in Figure 5. As we traverse P_{k+1}, the unbounded regions U_1 and U_2 lie to our left, and so P_{i_1} and P_{j_2} must intersect each other to the right of P_{k+1}. But then $P_{i_1}, P_{i_2}, P_{j_1}, P_{j_2}$, and P_{k+1}, form a pentagon, contradicting Observation 5. ∎

3 Simple Euclidean (≥ 5)-gon–free arrangements have twin pseudolines: proof of Theorem 2

First we show that there is at least one pair of twin pseudolines.

We proceed by induction on n. It is readily checked that the statement holds for the unique (up to isomorphism) simple Euclidean arrangement with 4 pseudolines. Thus we assume it holds for

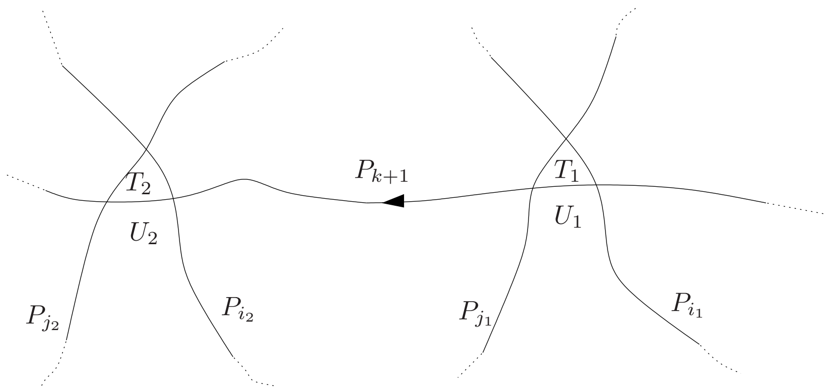

Figure 5: In this situation, P_{i_1} and P_{j_2} must intersect each other to the right of P_{k+1}, since otherwise U_1 and U_2 would not be unbounded. On the other hand, if they do intersect to the right of P_{k+1}, then they form a pentagon, together with P_{i_2} and P_{j_1}.

$n = k \geq 4$, and consider a simple Euclidean (≥ 5)-gon–free arrangement $\mathcal{B} = \{P_1, P_2, \ldots, P_{k+1}\}$.

By the inductive hypothesis, $\mathcal{B} \setminus \{P_{k+1}\}$ has a pair of twin pseudolines, say (without any loss of generality) P_1 and P_2. Moreover, we may also assume without any loss of generality that as we traverse P_1 (and P_2 as well, since they are twin), we meet $P_3, P_4, P_5, \ldots, P_k$ in this order. Thus, for each i, $3 \leq i \leq k-1$, P_1 and P_2 form a *basic* quadrilateral with P_i and P_{i+1}, and P_1, P_2, and P_3 form a triangle. Thus the layout of $P_1, P_2, \ldots, P_k$ is as illustrated in Figure 6.

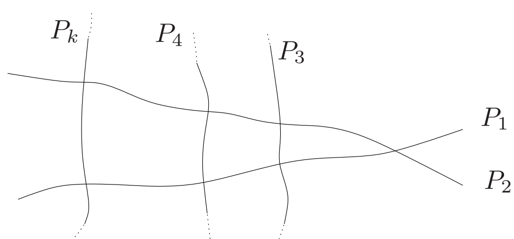

Figure 6: The pseudolines P_1 and P_2 are twin in the arrangement $\{P_1, P_2, \ldots, P_k\}$.

Suppose first that P_{k+1} crosses P_1 or P_2 in a basic quadrilateral. Then it must cross both P_1 and P_2 in the same quadrilateral (otherwise a pentagon would be formed, contradicting Observation 5), and so clearly P_1 and P_2 are also twin in $\mathcal{B}$. Thus for the rest of the proof we assume that P_{k+1} crosses neither P_1 nor P_2 in a basic quadrilateral.

Now suppose that P_{k+1} crosses the triangle defined by P_1, P_2, and P_3. The exchangeable role between P_1 and P_2 then allows us to assume that P_{k+1} crosses P_1 in this triangle. If P_{k+1} leaves the triangle by intersecting P_2, then we are clearly done, since then P_1 and P_2 are still twin in $\mathcal{B}$. Thus we assume that P_{k+1} leaves the triangle by intersecting with P_3. Now after intersecting P_3 (and thus entering the basic quadrilateral defined by P_1, P_2, P_3, and P_4), P_{k+1} must then intersect P_4, as otherwise a 5–gon would be formed, contradicting Observation 5. The same reasoning shows that P_{k+1} must intersect $P_4, P_5, \ldots, P_k$ in the given order. Thus P_{k+1} intersects P_2 either before intersecting P_1 or after intersecting P_k. It is straightforward to check that in either case P_2 and P_{k+1} are twin in $\mathcal{B}$. Thus for the rest of the proof we assume that P_{k+1} does not cross the triangle defined by P_1, P_2, and P_3.

The third case we analyze is when P_{k+1} intersects both P_1 and P_2 after they have already intersected P_k. That is, as we traverse P_1 we intersect the other pseudolines in the order $P_2, P_3, \ldots,$ P_k, P_{k+1}, and as we traverse P_2 we intersect the other pseudolines in the order $P_1, P_3, \ldots, P_k, P_{k+1}$. In this case P_1 and P_2 are clearly also twin in $\mathcal{B}$.

The remaining possibilities to be explored are that either (i) P_{k+1} intersects P_1 before P_2 intersects P_1; or (ii) P_{k+1} intersects P_2 before P_1 intersects P_2. Note that (i) and (ii) do not exclude each other. These conditions are equivalently described as follows: either (i) P_1 intersects $P_{k+1}, P_2, P_3, \ldots, P_k$ in the given order; or (ii) P_2 intersects $P_{k+1}, P_1, P_3, \ldots, P_k$ in the given order. Again the exchangeable roles of P_1 and P_2 allows us to assume that (i) applies. Also by exchanging P_1 and P_2 if necessary we may assume that as we traverse P_{k+1} in one of the two possible directions we intersect P_1 and the next pseudoline we intersect is P_i with $i \geq 3$. It is straightforward to check that if $i \neq 3$, then P_1, P_2, P_{k+1}, P_i, and P_3 form a pentagon (see Figure 7). Thus $i = 3$. The same reasoning shows that the next pseudoline that P_{k+1} intersects must be P_4, and so on. Thus P_{k+1} intersects $P_1, P_3, P_4, \ldots, P_k$ in this order, and it must intersect P_2 either before intersecting P_1 or after intersecting P_k. As illustrated in Figure 8, it is readily checked that in either case P_2 and P_{k+1} are twin in $\mathcal{B}$.

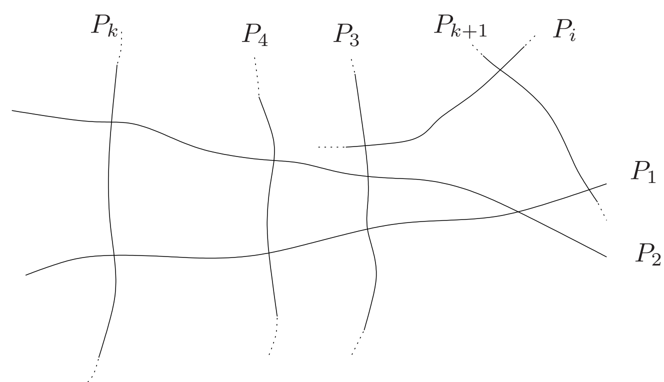

Figure 7: If P_{k+1} intersects P_i before intersecting P_3, then $P_{k+1}, P_i, P_3, P_2, P_1$ form a pentagon.

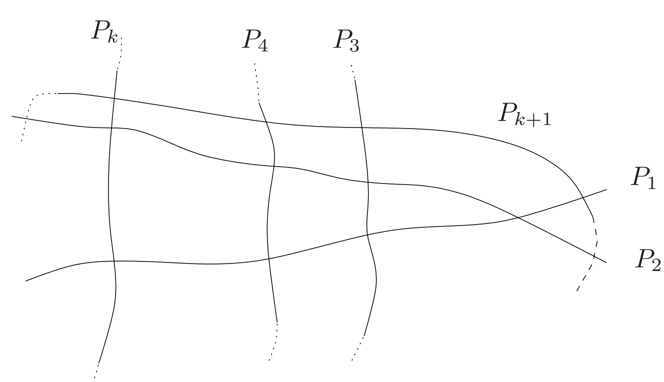

Figure 8: If P_{k+1} intersects $P_1, P_3, P_4, \ldots, P_k$ in this order, then P_{k+1} could intersect P_2 as in the dashed extension or as in the dotted extension. In either case, P_2 and P_{k+1} are twin.

We have thus proved that there is at least one pair of twin pseudolines, as claimed at the beginning of the proof.

We finally show that there are at least two distinct pairs of twin pseudolines, as claimed in Theorem 2. Again we proceed by induction on n, and the base case $n = 4$ is easily checked. Thus we assume the statement holds for $n = k \geq 4$, and consider a simple Euclidean (≥ 5)-gon–free arrangement $\mathcal{B} = \{P_1, P_2, \ldots, P_{k+1}\}$. We assume without any loss of generality that P_1 and P_2 are twin pseudolines in $\mathcal{B}$.

By the inductive hypothesis, $\mathcal{B}\backslash\{P_1\}$ has two disjoint pairs of twin pseudolines. Then at least one of these pairs, say $\{P_i, P_j\}$, is disjoint from $\{P_2\}$. Since P_i and P_j are twin pseudolines in $\mathcal{B} \setminus \{P_1\}$, then as we traverse P_2 and we intersect P_i, immediately after that (in $\mathcal{B} \setminus \{P_1\}$) we intersect P_j. On the other hand, P_1 and P_2 are twin in $\mathcal{B}$, and so P_1, P_2, P_i, and P_j must form a basic quadrilateral in $\mathcal{B}$. Therefore P_i and P_j are also twin pseudolines in $\mathcal{B}$. Since $\{P_1, P_2\} \cap \{P_i, P_j\} = \emptyset$, this completes the proof. ∎

4 Simple Euclidean (≥ 5)-gon–free arrangements are stretchable: proof of Theorem 4

We proceed inductively. The unique (up to isomorphism) simple Euclidean (≥ 5)-gon–free arrangement with 3 pseudolines is clearly stretchable. Now fix $k \geq 3$, and suppose that every simple Euclidean (≥ 5)-gon–free arrangement with k pseudolines is stretchable, and let $\mathcal{B} = \{P_1, P_2, \ldots, P_{k+1}\}$ be a simple Euclidean (≥ 5)-gon–free arrangement. By Theorem 2, there is a pair of twin pseudolines in $\mathcal{B}$, say P_k and P_{k+1} without any loss of generality.

By the inductive hypothesis, $\mathcal{B} \setminus \{P_{k+1}\}$ is stretchable, so it can be equivalently drawn with $P_1, P_2, \ldots, P_k$ as straight lines. It now suffices to observe that since P_k and P_{k+1} are twin, then P_{k+1} may be added as a straight line, if it is drawn sufficiently close to P_k. ∎

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