# THE OPTIMAL DRAWINGS OF $K_{5,n}$

CÉSAR HERNÁNDEZ-VÉLEZ, CAROLINA MEDINA, AND GELASIO SALAZAR

ABSTRACT. Zarankiewicz's Conjecture (ZC) states that the crossing number $\mathrm{cr}(K_{m,n})$ equals $Z(m,n) := \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. Since Kleitman's verification of ZC for $K_{5,n}$ (from which ZC for $K_{6,n}$ easily follows), very little progress has been made around ZC; the most notable exceptions involve computer-aided results. With the aim of gaining a more profound understanding of this notoriously difficult conjecture, we investigate the *optimal* (that is, crossing-minimal) drawings of $K_{5,n}$. The widely known natural drawings of $K_{m,n}$ (the so-called *Zarankiewicz drawings*) with $Z(m,n)$ crossings contain *antipodal* vertices, that is, pairs of degree-$m$ vertices such that their induced drawing of $K_{m,2}$ has no crossings. Antipodal vertices also play a major role in Kleitman's inductive proof that $\mathrm{cr}(K_{5,n}) = Z(5,n)$. We explore in depth the role of antipodal vertices in optimal drawings of $K_{5,n}$, for $n$ even. We prove that if $n \equiv 2 \pmod 4$, then every optimal drawing of $K_{5,n}$ has antipodal vertices. We also exhibit a two-parameter family of optimal drawings $D_{r,s}$ of $K_{5,4(r+s)}$ (for $r,s \geq 0$), with no antipodal vertices, and show that if $n \equiv 0 \pmod 4$, then every optimal drawing of $K_{5,n}$ without antipodal vertices is (vertex rotation) isomorphic to $D_{r,s}$ for some integers $r,s$. As a corollary, we show that if $n$ is even, then every optimal drawing of $K_{5,n}$ is the superimposition of Zarankiewicz drawings with a drawing isomorphic to $D_{r,s}$ for some nonnegative integers $r,s$.

## 1. INTRODUCTION.

We recall that the *crossing number* $\mathrm{cr}(G)$ of a graph $G$ is the minimum number of pairwise crossings of edges in a drawing of $G$ in the plane. A drawing of a graph is *good* if no adjacent edges cross, and no two edges cross each other more than once. It is trivial to show that every *optimal* (that is, crossing-minimal) drawing of a graph is good.

One of the most tantalizingly open crossing number questions was raised by Turán in 1944: what is the crossing number $\mathrm{cr}(K_{m,n})$ of the complete bipartite graph $K_{m,n}$? Zarankiewicz [8] described how to draw $K_{m,n}$ with

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exactly $Z(m,n)$ crossings, where

$$Z(m,n) := \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

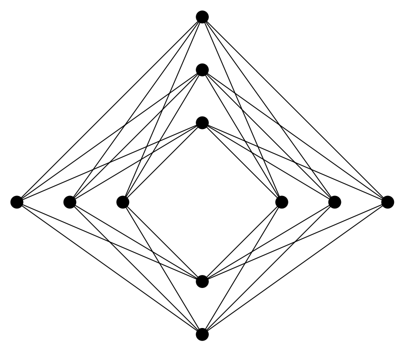

FIGURE 1. Drawing of $K_{5,6}$ with $Z(5,6) = 24$ crossings.

Zarankiewicz's construction is shown in Figure 1 for the case $m = 5, n = 6$. It is straightforward to generalize this drawing to a drawing of $K_{m,n}$ with $Z(m,n)$ crossings, for all positive integers $m$ and $n$, and so $\text{cr}(K_{m,n}) \leq Z(m,n)$. The drawings thus obtained are the *Zarankiewicz drawings* of $K_{m,n}$.

In [8], Zarankiewicz claimed to have proved that $\text{cr}(K_{m,n}) = Z(m,n)$ for all positive integers $m, n$. However, Kainen and Ringel independently found a flaw in Zarankiewicz's argument (see [5]), and the statement $\text{cr}(K_{m,n}) = Z(m,n)$ has become known as *Zarankiewicz's Conjecture*.

Very little of substance is known about $\text{cr}(K_{m,n})$. An elegant argument using $\text{cr}(K_{3,3}) = 1$ plus purely combinatorial arguments (namely, Turán's theorem on the maximum number of edges in a triangle-free graph) shows that $\text{cr}(K_{3,n}) = Z(3,n)$. An easy counting argument shows that $\text{cr}(K_{2s-1,n}) = Z(2s-1,n)$ (for any $s \geq 1$) implies that $\text{cr}(K_{2s,n}) = Z(2s,n)$. Thus it follows that $\text{cr}(K_{4,n}) = Z(4,n)$. Kleitman [6] proved that $\text{cr}(K_{5,n}) = Z(5,n)$. By our previous remark, this implies that $\text{cr}(K_{6,n}) = Z(6,n)$.

After Kleitman's theorem, most progress around Zarankiewicz's Conjecture consists of computer-aided results. Woodall [7] verified Zarankiewicz's Conjecture for $K_{7,7}$ and $K_{7,9}$. De Klerk et al. [2] used semidefinite programming techniques to show that $\lim_{n\to\infty} \text{cr}(K_{7,n})/Z(7,n) \geq 0.968$. Also using semidefinite programming and deeper algebraic techniques, De Klerk et al. [4] proved that $\lim_{n\to\infty} \text{cr}(K_{9,n})/Z(9,n) \geq 0.966$. In a related result, De Klerk and Pasechnik [3] recently showed that the 2-page crossing number $\nu_2(K_{7,n})$ of $K_{7,n}$ satisfies $\lim_{n\to\infty} \text{cr}(K_{7,n})/Z(7,n) = 1$.

We finally mention that recently Christian et al. [1] proved that deciding Zarankiewicz's Conjecture is a finite problem for each fixed $m$.

To give a brief description of our results, let us color the 5 degree-$n$ vertices of $K_{5,n}$ *black*, and color the $n$ degree-5 vertices *white*. Two white vertices are *antipodal* in a drawing $D$ of $K_{5,n}$ if the drawing of the $K_{5,2}$ they induce has no crossings. A drawing is *antipodal-free* if it has no antipodal vertices.

Antipodal pairs are evident in Zarankiewicz's drawings (moreover, the set of white vertices can be decomposed into two classes, such that any two white vertices in distinct classes are antipodal). Antipodal pairs are also crucial in the inductive step of Kleitman's proof, which does not concern itself with the different ways (if more than one) to achieve $Z(5,n)$ crossings with a drawing of $K_{5,n}$.

Given their preeminence in Zarankiewicz's Conjecture, we set out to investigate the role of antipodal pairs in the optimal drawings of $K_{5,n}$. Our main result (Theorem 1) characterizes optimal drawings of $K_{5,n}$, for even $n$, as follows. First, if $n \equiv 2 \pmod 4$, then all optimal drawings of $K_{5,n}$ have antipodal pairs. Second, if $n \equiv 0 \pmod 4$, then every antipodal-free optimal drawing of $K_{5,n}$ is isomorphic (we review vertex rotation isomorphism in Section 2) to a drawing in a two-parameter family $D_{r,s}$ of drawings we have fully characterized. As a consequence of these facts, we show (Theorem 2) that if $n$ is even, then every optimal drawing of $K_{5,n}$ can be obtained by starting with $D_{r,s}$, for some nonnegative (possibly zero) integers $r$ and $s$, and then superimposing Zarankiewicz drawings.

The rest of this paper is organized as follows. In Section 2 we review the concept of vertex rotation, which is central to the criterion to decide when two drawings are isomorphic. In Section 3 we describe the two-parameter family of optimal, antipodal-free drawings $D_{r,s}$ (for integers $r, s \geq 0$) of $K_{5,4(r+s)}$. In Section 4 we state our main results. Theorem 1 claims that (i) if $n \equiv 2 \pmod 4$, then every optimal drawing of $K_{5,n}$ has antipodal vertices; and that (ii) if $n \equiv 0 \pmod 4$, then every antipodal-free optimal drawing of $K_{5,n}$ is isomorphic to $D_{r,s}$ for some integers $r, s$ such that $4(r+s) = n$. In Theorem 2 we state the decomposition of optimal drawings of $K_{5,n}$, along the lines of the previous paragraph. The proof of Theorem 2 is also given in this section; the rest of the paper is devoted to the proof of Theorem 1. In Section 5 we introduce the concept of a *clean* drawing. Loosely speaking, a drawing is clean if its white vertices can be naturally partitioned into *bags*, so that vertices in the same bag have the same (crossing number wise) properties. In Section 6 we introduce *keys*, which are labelled graphs that capture the essential (crossing number wise) information of a clean drawing. This abstraction (and the related concept of *core*) will prove to be extremely useful for the proof of Theorem 1. In Section 7 we investigate which labelled graphs can be the key of a relevant (clean, optimal, antipodal-free) drawing. Cores are certain more manageable subgraphs of keys, that retain all the (crossing number wise) useful information of a key. We devote Sections 8, 9, 10, and 11 to the task of completely characterizing which graphs can be the core of an antipodal-free optimal drawing. The information in these sections is then put together in Section 12, where we show that the core of every optimal drawing is isomorphic either to the 4-cycle or to the graph $\overline{C}_6$ obtained by adding to the 6-cycle a diametral edge. The proof of Theorem 1, given in Section 13, is an easy consequence of this full characterization of cores.

## 2. Rotations and isomorphic drawings.

To help comprehension, throughout this paper we color the 5 degree-$n$ vertices in $K_{5,n}$ *black*, and the $n$ degree-5 vertices *white*. We label the black vertices $0, 1, 2, 3, 4$. Unless otherwise stated, we label the white vertices $a_0, a_1, \ldots, a_{n-1}$. We adopt the notation $[n] := \{0, 1, \ldots, n-1\}$.

Given vertices $a_i, a_j$ with $i, j \in [n]$, we let $S(a_i)$ denote the *star* centered at $a_i$, that is, the subgraph (isomorphic to $K_{5,1}$) induced by $a_i$ and the vertices $0, 1, 2, 3, 4$. If $D$ is a drawing of $K_{5,n}$, we let $\mathrm{cr}_D(a_i, a_j)$ denote the number of crossings in $D$ that involve an edge of $S(a_i)$ and an edge of $S(a_j)$, and we let $\mathrm{cr}_D(a_i) := \sum_{k\in[n],k\neq i} \mathrm{cr}_D(a_i, a_k)$. Formalizing the definition from Section 1, $a_i$ and $a_j$ are *antipodal* (*in* $D$) if $\mathrm{cr}_D(a_i, a_j) = 0$.

The *rotation* $\mathrm{rot}_D(a_i)$ of a white vertex $a_i$ in a drawing $D$ is the cyclic permutation that records the (cyclic) counterclockwise order in which the edges leave $a_i$. We use the notation 01234 for permutations, and (01234) for cyclic permutations. For instance, the rotation $\mathrm{rot}_D(a_3)$ of the vertex $a_3$ in the drawing $D$ in Figure 2 is (02431): following a counterclockwise order, if we start with the edge leaving from $a_3$ to 0, then we encounter the edge leaving to 2, then the edge leaving to 4, then the edge leaving to 3, and then the edge leaving to 1. We emphasize that a rotation is a cyclic permutation; that is, (02431), (24310), (43102), (31024), and (10243) denote (are) the same rotation. We let $\Pi$ denote the set of all cyclic permutations of $0, 1, 2, 3, 4$. Clearly, $|\Pi| = 5!/5 = 4! = 24$. The *rotation* $\mathrm{rot}_D(i)$ of a black vertex $i$ is defined analogously: for each $i \in [5]$, $\mathrm{rot}_D(i)$ is a cyclic permutation of $a_0, a_1, \ldots, a_{n-1}$.

The *rotation multiset* $\mathrm{Rot}_M(D)$ *of* $D$ is the multiset (that is, repetitions are allowed) containing the $n$ rotations $\mathrm{rot}_D(a_i)$, for $i = 0, 1, \ldots, n-1$. The *rotation set* $\mathrm{Rot}(D)$ of $D$ is the underlying set (that is, no repetitions allowed) of $\mathrm{Rot}_M(D)$. Thus, in the example of Figure 2, $\mathrm{Rot}_M(D) =$ [(04321), (04321), (01234), (02431)] (we use square brackets for multisets), and $\mathrm{Rot}(D) = \{(04321), (01234), (02431)\}$.

Two multisets $M, M'$ of rotations are *equivalent* (we write $M \cong M'$) if one of them can be obtained from the other by a relabelling (formally, a self-bijection) of $0, 1, 2, 3, 4$. Two drawings $D, D'$ of $K_{5,n}$ are *isomorphic* if $\mathrm{Rot}_M(D) \cong \mathrm{Rot}_M(D')$. Loosely speaking, two drawings $D, D'$ of $K_{5,n}$ are isomorphic if $0, 1, 2, 3, 4$ and $a_0, a_1, \ldots, a_{n-1}$ can be relabelled (say in $D'$), if necessary, so that $\mathrm{rot}_D(a_i) = \mathrm{rot}_{D'}(a_i)$ for every $i \in [n]$.

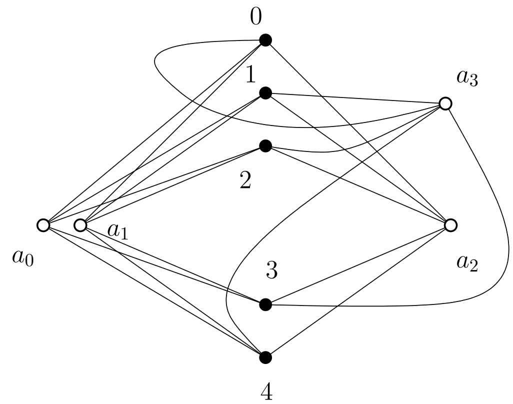


FIGURE 2. A drawing $D$ of $K_{5,4}$ with $\mathrm{rot}_D(a_0) = \mathrm{rot}_D(a_1) = (04321), \mathrm{rot}_D(a_2) = (01234)$, and $\mathrm{rot}_D(a_3) = (02431)$. Thus the pair $a_0, a_2$ (as well as the pair $a_1, a_2$) is antipodal.

Our ultimate interest lies in optimal drawings (of $K_{5,n}$). It is not difficult to see (we will prove this later) that if $D$ is an optimal drawing and $a_i, a_j, a_k, a_\ell$ are vertices such that $\mathrm{rot}_D(a_i) = \mathrm{rot}_D(a_j)$ and $\mathrm{rot}_D(a_k) = \mathrm{rot}_D(a_\ell)$, then $\mathrm{cr}_D(a_i, a_k) = \mathrm{cr}_D(a_j, a_\ell)$. Thus an optimal drawing of $K_{5,n}$ is adequately described by choosing a representative vertex of each rotation, and giving the information of how many vertices there are for each rotation. This supports the pertinence of focusing on the rotations as the criteria for isomorphism.

## 3. An antipodal-free drawing of $K_{5,4(r+s)}$

In this section we describe an antipodal-free drawing $D_{r,s}$ of $K_{5,4(r+s)}$, for each pair $r, s$ of nonnegative integers.

The construction is based on the drawing $D^*$ of $K_{5,6}$ in Figure 3. As shown, the rotations in $D^*$ of the white vertices are $\mathrm{rot}_{D^*}(a_0) = (01234)$, $\mathrm{rot}_{D^*}(a_1) = (04231), \mathrm{rot}_{D^*}(a_2) = (01342), \mathrm{rot}_{D^*}(a_3) = (04312), \mathrm{rot}_{D^*}(a_4) = (01432)$, $\mathrm{rot}_{D^*}(a_5) = (02314)$.

It is immediately checked that $D^*$ is antipodal-free. Note that $D^*$ itself is not optimal, as it has $25 = Z(5, 6) + 1$ crossings.

Suppose first that both $r$ and $s$ are positive. To obtain $D_{r,s}$, we add $4(r+s)-6$ white vertices to $D^*$. Now $r-1$ of these vertices are drawn very close to $a_1$, and $r-1$ are drawn very close to $a_2$; $s-1$ vertices are drawn very close to $a_4$, and $s-1$ are drawn very close to $a_5$; finally, $r+s-1$ vertices are drawn very close to $a_0$, and $r+s-1$ are drawn very close to $a_3$. It is intuitively clear what is meant by having $a_i$ drawn "very close" to $a_j$. Formally, we require that: (i) $a_i$ and $a_j$ have the same rotation; (ii) $\mathrm{cr}_{D_{r,s}}(a_i, a_j) = 4$; and (iii) for any other vertex $a_k$, $\mathrm{cr}_{D_{r,s}}(a_i, a_k) = \mathrm{cr}_{D_{r,s}}(a_j, a_k)$. These properties are easily satisfied by having the added vertex $a_i$ drawn sufficiently close to

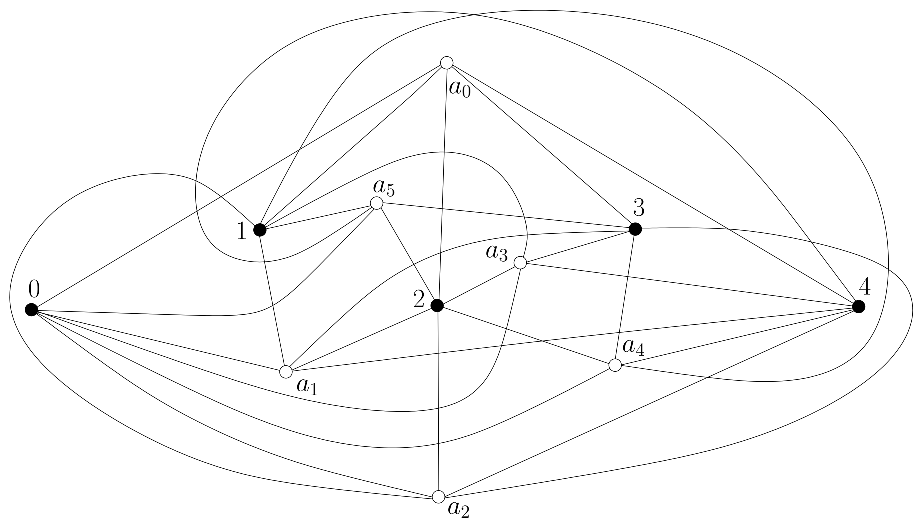


FIGURE 3. This antipodal-free drawing $D^*$ of $K_{5,6}$ is the base of the construction of the optimal antipodal-free drawing $D_{r,s}$ of $K_{5,4(r+s)}$ for all $r, s$. It is easily verified that $\mathrm{rot}_{D^*}(a_0) = (01234)$, $\mathrm{rot}_{D^*}(a_1) = (04231)$, $\mathrm{rot}_{D^*}(a_2) = (01342)$, $\mathrm{rot}_{D^*}(a_3) = (04312)$, $\mathrm{rot}_{D^*}(a_4) = (01432)$, $\mathrm{rot}_{D^*}(a_5) = (02314)$.

$a_j$, so that the edges incident with $a_i$ follow very closely the edges incident with $a_j$.

If one of $r$ or $s$ is 0, then we make the obvious adjustments. That is, (i) if $r = 0$, then we remove $a_1$ and $a_2$, and for each $i = 0, 3, 4, 5$, we draw $s - 1$ new vertices very close to $a_i$; and (ii) if $s = 0$, then we remove $a_4$ and $a_5$, and for each $i = 0, 1, 2, 3$, we draw $r - 1$ new vertices very close to $a_i$. (In the extreme case $r = s = 0$, we remove all the white vertices from $D^*$, and are left with an obviously optimal drawing of $K_{5,0}$).

For each $i = 0, 1, 2, 3, 4, 5$, the *bag* $[a_i]$ of $a_i$ is the set that consists of the vertices drawn very close to $a_i$, plus $a_i$ itself.

Note that each of $[a_0]$ and $[a_3]$ has $r + s$ vertices, each of $[a_1]$ and $[a_2]$ has $r$ vertices, and each of $[a_4]$ and $[a_5]$ has $s$ vertices.

An illustration of the construction for $r = 2$ and $s = 1$ is given in Figure 4, where the gray vertices are the ones added to $D^*$.

**Claim.** *For every pair $r, s$ of nonnegative integers, $D_{r,s}$ is an antipodal-free optimal drawing of $K_{5,4(r+s)}$.*

*Proof.* First we note that since $D^*$ is antipodal-free, it follows immediately that $D_{r,s}$ is also antipodal-free. Thus we only need to prove optimality.

An elementary calculation gives the number of crossings in $D_{r,s}$. For instance, take a vertex $u$ in $[a_0]$. Now $\mathrm{cr}_{D_{r,s}}(u, v)$ equals (i) 4 if $v \in [a_0], v \neq u$; (ii) 1 if $v \in [a_1]$; (iii) 2 if $v \in [a_2]$; (iv) 1 if $v \in [a_3]$; (v) 1 if $v \in [a_4]$; and (vi) 2 if $v \in [a_5]$. Since $|[a_0]| = r+s$, $|[a_1]| = r$, $|[a_2]| = r$, $|[a_3]| = r+s$, $|[a_4]| = s$,

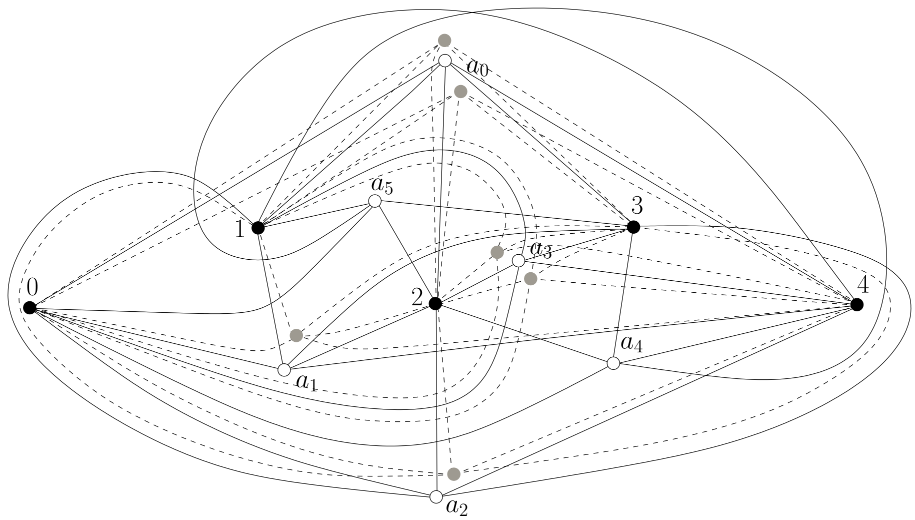


FIGURE 4. The antipodal-free drawing $D_{2,1}$. To obtain this optimal drawing of $K_{5,12} = K_{5,4(2+1)}$, we start with the drawing in Figure 3 and add two vertices very close to $a_0$, two vertices very close to $a_3$, one vertex very close to $a_1$, and one vertex very close to $a_2$. Since $s-1=0$, no vertices are added very close to either $a_4$ or $a_5$. The added vertices are colored gray in this drawing.

and $|[a_5]| = s$, it follows that $\mathrm{cr}_{D_{r,s}}(u) = 4(r+s-1)+r+2r+(r+s)+s+2s = 4(2r+2s-1)$.

A totally analogous argument shows that, actually, $\mathrm{cr}_{D_{r,s}}(w) = 4(2r+2s-1)$ for *every* white vertex $w$. Since there are $4(r+s)$ white vertices in total, it follows that $\mathrm{cr}(D_{r,s}) = (1/2)(4(r+s))\big(4(2r+2s-1)\big) = \big(4(r+s)\big)\big(4(r+s)-2\big) = Z(5, 4(r+s))$. □

## 4. Main results: the optimal drawings of $K_{5,n}$, for $n$ even.

We now state our main results.

**Theorem 1.** *Let $n$ be a positive even integer.*

(1) *If $n \equiv 2 \pmod 4$, then all optimal drawings of $K_{5,n}$ have antipodal vertices.*
(2) *If $n \equiv 0 \pmod 4$, then every antipodal-free optimal drawing of $K_{5,n}$ is isomorphic to $D_{r,s}$ (described in Section 3) for some integers $r, s$ such that $4(r+s) = n$.*

Before moving on to the proof of Theorem 1 (the rest of the paper is devoted to this proof), we will show that it implies a decomposition of all the optimal drawings of $K_{5,n}$, for $n$ even.

In Section 1 we defined, somewhat informally, a Zarankiewicz drawing. Let us now formally define these drawings using rotations (we focus on $K_{5,n}$, although the definition is obviously extended to $K_{m,n}$ for any $m$). For

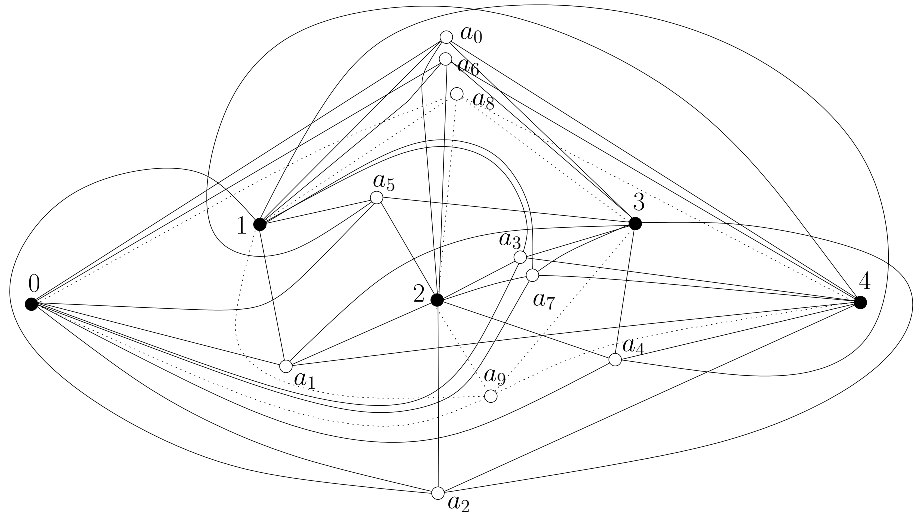


FIGURE 5. An optimal drawing of $K_{5,10}$ that is neither a Zarankiewicz drawing nor the superimposition of Zarankiewicz drawings. As predicted by Theorem 2, this is the superimposition of a Zarankiewicz drawing (the $K_{5,2}$ induced by $a_8, a_9$ and the five black vertices) plus a drawing $D_{r,s}$ (namely with $r = s = 1$).

a nonnegative integer $n$, a drawing $D$ of $K_{5,n}$ is a *Zarankiewicz drawing* if the white vertices can be partitioned into two sets, of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, so that vertices in different sets are antipodal in $D$, and vertices $a_i, a_j$ in the same set satisfy $\mathrm{cr}_D(a_i, a_j) = 4$ (see Figure 1 for a Zarankiewicz drawing of $K_{5,6}$). A quick calculation shows that every Zarankiewicz drawing of $K_{5,n}$ is an optimal drawing.

**Theorem 2** (Decomposition of optimal drawings of $K_{5,n}$, for $n$ even)**.** *Let $D$ be an optimal drawing of $K_{5,n}$, with $n$ even. Then the set of $n$ white vertices can be partitioned into two sets $A, B$ (one of which may be empty), with $|A| = 4t$ for some nonnegative integer $t$, such that: (i) the vertices in $B$ can be decomposed into $|B|/2$ antipodal pairs; and (ii) the drawing of $K_{5,4t}$ induced by $A$ is antipodal-free, and it is isomorphic to the drawing $D_{r,s}$ described in Section 3, for some integers $r, s$ such that $r + s = t$. Equivalently, either $D$ is the superimposition of Zarankiewicz drawings, or it can be obtained by superimposing Zarankiewicz drawings to the drawing $D_{r,s}$ described in Section 3, for some integers $r, s$ (see Figure 5).*

*Proof.* We proceed by induction on $n$. It is trivial to check that the two white vertices of every optimal drawing of $K_{5,2}$ are an antipodal pair, and so the statement holds in the base case $n = 2$. For the inductive step, we consider an even integer $n$, and assume that the statement is true for all $k < n$.

Let $D$ be an optimal drawing of $K_{5,n}$. If $D$ has no antipodal pairs, then the statement follows immediately from Theorem 1 (without even using the induction hypothesis). Thus we may assume that $D$ has at least one antipodal pair $a_i, a_j$. It suffices to show that the drawing $D'$ that results by removing $a_i$ and $a_j$ from $D$ is an optimal drawing of $K_{5,n-2}$, as then the result follows by the induction hypothesis. Clearly $\mathrm{cr}(D) = \mathrm{cr}(D') + \sum_{k\in[n]-\{i,j\}}(\mathrm{cr}_D(a_i, a_k) + \mathrm{cr}_D(a_j, a_k)) \geq \mathrm{cr}(D') + (n-2)Z(5,3) = \mathrm{cr}(D') + 4n-8$. Thus $\mathrm{cr}(D') \leq \mathrm{cr}(D) - 4n + 8 = Z(5,n) - 4n + 8$. An elementary calculation shows that $Z(5,n)-4n+8 = Z(5,n-2)$, so we obtain $\mathrm{cr}(D') \leq Z(5,n-2)$. Since $\mathrm{cr}(K_{5,n-2}) = Z(5,n-2)$, it follows that $\mathrm{cr}(D') = Z(5,n-2)$, that is, $D'$ is an optimal drawing of $K_{5,n-2}$. □

## 5. Clean drawings.

A good drawing of $K_{5,n}$ is *clean* if:

(1) for all distinct white vertices $a_i, a_j$ such that $\mathrm{rot}_D(a_i) = \mathrm{rot}_D(a_j)$, we have $\mathrm{cr}_D(a_i, a_j) = 4$;
(2) for all distinct white vertices $a_i, a_j, a_k, a_\ell$ such that $\mathrm{rot}_D(a_i) = \mathrm{rot}_D(a_j)$ and $\mathrm{rot}_D(a_k) = \mathrm{rot}_D(a_\ell)$, we have $\mathrm{cr}_D(a_i, a_k) = \mathrm{cr}_D(a_j, a_\ell)$; and
(3) for any distinct white vertices $a_i, a_k$, $\mathrm{cr}_D(a_i, a_k) \leq 4$.

**Proposition 3.** *Let $D$ be an optimal drawing of $K_{5,n}$. Then there is an optimal drawing $D'$, isomorphic to $D$, that is clean.*

*Proof.* For each white vertex $a_i$, define $d_i := \sum_{\{a_\ell \,|\, \mathrm{rot}_D(a_\ell)\neq \mathrm{rot}_D(a_i)\}} \mathrm{cr}_D(a_i, a_\ell)$. Let $\pi \in \mathrm{Rot}(D)$. Take a white vertex $a_i$ with $\mathrm{rot}_D(a_i) = \pi$, such that for all $j$ with $\mathrm{rot}_D(a_j) = \pi$ we have $d_i \leq d_j$. It is easy to see that we can move every vertex $a_j$ with $\mathrm{rot}_D(a_j) = \pi$ very close to $a_i$, so that $\mathrm{cr}_D(a_i, a_k) = \mathrm{cr}_D(a_j, a_k)$ for every white vertex $a_k \notin \{a_i, a_j\}$, and so that $\mathrm{cr}_D(a_i, a_j) = 4$. If we perform this procedure for every rotation in $\mathrm{Rot}(D)$, the result is an optimal drawing $D'$, isomorphic to $D$, that satisfies (1) and (2).

Now to prove that $D'$ also satisfies (3) we suppose, by way of contradiction, that there exist $a_i, a_k$ such that $\mathrm{cr}_D(a_i, a_k) > 4$. Define $d_i, d_k$ as in the previous paragraph. We may assume without loss of generality that $d_i \leq d_k$. Now let $D''$ be the drawing that results from moving $a_k$ very close to $a_i$, making it have the same rotation as $a_i$, and so that $\mathrm{cr}_{D''}(a_i, a_\ell) = \mathrm{cr}_{D''}(a_k, a_\ell)$ for every $\ell \notin \{i, k\}$, and $\mathrm{cr}_{D''}(a_i, a_k) = 4$. It is readily checked that $D''$ has fewer crossings than $D'$, contradicting the optimality of $D'$. □

**Remark 4.** *We are interested in classifying optimal drawings up to isomorphism (Theorem 1). In view of Proposition 3, we may assume that all drawings of $K_{5,n}$ under consideration are clean. We will work under this assumption for the rest of the paper.*

## 6. The key of a clean drawing.

We now associate to every clean drawing of $K_{5,n}$ an edge-labeled graph that (as we will see) captures all its relevant crossing number information.

Let $D$ be a clean drawing of $K_{5,n}$. The *key* $\Phi(D)$ of $D$ is the (edge-labeled) complete graph whose vertices are the elements of $\mathrm{Rot}(D)$, and where each edge is labeled according to the following rule: if $\pi, \pi' \in \mathrm{Rot}(D)$, with $\mathrm{rot}_D(a_i) = \pi$ and $\mathrm{rot}_D(a_j) = \pi'$, then the label of the edge joining $\pi$ and $\pi'$ is $\mathrm{cr}_D(a_i, a_j)$. It follows from the cleanness of $D$ that $\mathrm{cr}_D(a_i, a_j)$ does not depend on the choice of $a_i$ and $a_j$, and so $\Phi(D)$ is well-defined for every clean drawing $D$. Moreover, it also follows that every edge label in $\Phi(D)$ is in $\{0, 1, 2, 3, 4\}$. The *core* of $D$ is the subgraph $\Phi^1(D)$ of $\Phi(D)$ that consists of all the vertices of $\Phi(D)$ and the edges of $\Phi(D)$ with label 1. In Figure 6 we give a (clean and optimal) drawing $D$ of $K_{5,3}$, and illustrate its key and its core.

Our main interest is in antipodal-free drawings, that is, those drawings in which every edge label in $\Phi(D)$ is in $\{1, 2, 3, 4\}$. A key is 0*–free* (respectively, 4-*free*) if none of its edges has 0 (respectively, 4) as a label. A key is $\{0, 4\}$-*free* if it is both 0- and 4-free.

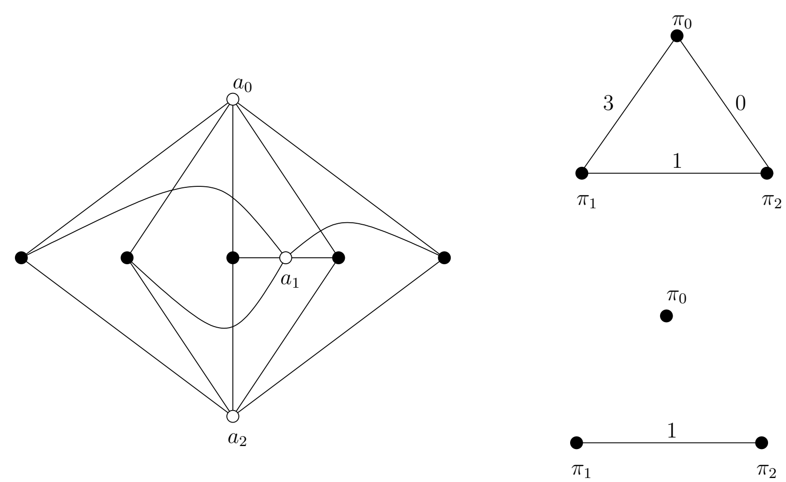


FIGURE 6. A drawing $D$ of $K_{5,3}$. By letting $\mathrm{rot}_D(a_0) = \pi_0, \mathrm{rot}_D(a_1) = \pi_1$, and $\mathrm{rot}_D(a_2) = \pi_2$, we obtain the key $\Phi(D)$ (right, above) and the core $\Phi^1(D)$ (right, below) of $D$.

The main step in our strategy to understand optimal drawings is to characterize which labelled graphs are the key of some optimal drawing. To this end, we introduce a system of linear equations associated to each key, as follows.

**Definition 5** (The system of linear equations of a key)**.** *Let $D$ be an optimal drawing of $K_{5,n}$, with $n$ even. Let the vertices of $\Phi(D)$ (that is, the elements of $\mathrm{Rot}(D)$) be labelled $\pi_0, \pi_1, \ldots, \pi_{m-1}$, and let $\lambda_{ij}$ denote the label of the edge $\pi_i\pi_j$, for all $i \neq j$. For each $i \in [m]$, the* linear equation $E(\pi_i, \Phi(D))$ for $\pi_i$ in $\Phi(D)$ *is the linear equation on the variables $t_0, t_1, \ldots, t_{m-1}$ given* *by*

$$E(\pi_i, \Phi(D)) \quad : \quad 2t_i + \sum_{j\in[m],\, j\neq i} (\lambda_{ij} - 2)t_j = 0.$$

*The set* $\{E(\pi_i, \Phi(D))\}_{i\in[m]}$ *is the* system of linear equations associated to $\Phi(D)$*, and is denoted* $\mathcal{L}(\Phi(D))$*.*

The characterization of when a labelled graph is the key of an optimal drawing is mainly based on the following crucial fact.

**Proposition 6.** *Let $D$ be an optimal drawing of $K_{5,n}$, with $n$ even. Then the system of linear equations $\mathcal{L}(\Phi(D))$ associated to $\Phi(D)$ has a positive integral solution $(t_0, t_1, \ldots, t_{m-1})$ such that $t_0 + t_1 + \cdots + t_{m-1} = n$.*

*Proof.* First we show that if $D$ is an optimal drawing of $K_{5,n}$ with $n$ even, then for every $i = 0, 1, \ldots, n-1$, we have $\mathrm{cr}_D(a_i) = 2n - 4$. To this end, suppose that $\mathrm{cr}_D(a_i) > 2n - 4$ for some $i$. Since $D$ is optimal, $\mathrm{cr}(D) = Z(5, n) = n(n-2)$, and so the drawing $D'$ of $K_{5,n-1}$ that results by removing $a_i$ from $D$ has fewer than $n(n-2) - (2n-4) = n^2 - 4n + 4 = (n-2)^2 = Z(5, n-1)$ crossings, contradicting that $\mathrm{cr}(K_{5,n-1}) = Z(5, n-1)$. Thus $\mathrm{cr}_D(a_i) \leq 2n - 4$ for every $i$. Now suppose that $\mathrm{cr}_D(a_i) < 2n - 4$ for some $i$. Then $\mathrm{cr}(D) = (1/2)\sum_{j\in[n]} \mathrm{cr}_D(a_j) < (1/2)(2n-4)n = n(n-2)$, contradicting that $\mathrm{cr}(K_{5,n}) = Z(5, n) = n(n-2)$. Thus for every $i \in [n]$ we have $\mathrm{cr}_D(a_i) = 2n - 4$, as claimed.

Now let $\pi_0, \pi_1, \ldots, \pi_{m-1}$ be the elements of $\mathrm{Rot}(D)$ (that is, the vertices of $\Phi(D)$), and for each $i, j \in [m], i \neq j$, let $\lambda_{ij}$ denote the label of the edge $\pi_i\pi_j$ in $\Phi(D)$. For each $i \in [m]$, let $t_i$ be the number of vertices with rotation $\pi_i$ in $D$. Then (using that $D$ is clean) for every $i \in [m]$ and every white vertex $a_k$ with $\mathrm{rot}_D(a_k) = \pi_i$ we have $\mathrm{cr}_D(a_k) = 4(t_i - 1) + \sum_{j\in[m], j\neq i} \lambda_{ij}t_j$. Now from the previous paragraph for each $a_k$ we have $\mathrm{cr}_D(a_k) = 2n - 4$. Using that $n = \sum_{j\in[m]} t_j$, we obtain $4(t_i - 1) + \sum_{j\in[m], j\neq i} \lambda_{ij}t_j = 2\sum_{j\in[m]} t_j - 4$. Equivalently, $2t_i + \sum_{j\in[m], j\neq i}(\lambda_{ij} - 2)t_j = 0$, for every $i \in [m]$. Thus $(t_0, t_1, \ldots, t_{m-1})$ is a positive integral solution of $\mathcal{L}(\Phi(D))$. □

## 7. Properties of the key of a clean drawing.

We start with an easy, yet crucial, observation.

**Proposition 7.** *Let $D$ be an optimal drawing of $K_{5,n}$. Then, for any three distinct white vertices $a_i, a_j, a_k$, $\mathrm{cr}_D(a_i, a_j) + \mathrm{cr}_D(a_j, a_k) + \mathrm{cr}_D(a_i, a_k)$ is an even number greater than or equal to 4.*

*Proof.* This follows since $\mathrm{cr}(K_{5,3}) = Z(5, 3) = 4$ and (see for instance [6]) every good drawing of $K_{5,3}$ has an even number of crossings. □

The following is an equivalent form of this statement, in the setting of keys.

**Proposition 8.** *Let $D$ be a clean drawing of $K_{5,n}$, and let $\pi_0, \pi_1, \pi_2$ be vertices of $\Phi(D)$. Let $\lambda_{ij}$ be the label of the edge $\pi_i\pi_j$, for $i, j \in \{0, 1, 2\}, i \neq j$. Then $\lambda_{01} + \lambda_{12} + \lambda_{02}$ is an even number greater than or equal to 4.* □

Let $\gamma, \kappa$ be cyclic permutations on the same set of symbols. A *route* from $\gamma$ to $\kappa$ is a set of distinct transpositions, which may be ordered into some sequence such that the successive application of (all) the transpositions in this sequence takes $\gamma$ to $\kappa$. For instance, if $\gamma = (abcd)$ and $\kappa = (acdb)$, then $\{(bd), (bc)\}$ is a route from $\gamma$ to $\kappa$: if we apply first $(bc)$ to $\gamma$, and then $(bd)$ to the resulting cyclic permutation, we obtain $\kappa$.

The *size* $|P|$ of a route $P$ is its number of transpositions. An *antiroute* from $\gamma$ to $\kappa$ is a route from $\gamma$ to the reverse cyclic permutation $\overline{\kappa}$ of $\kappa$. Note that if $P$ is a route (respectively, antiroute) from $\gamma$ to $\kappa$, then $P$ is also a route (respectively, antiroute) from $\kappa$ to $\gamma$. The *antidistance* between two cyclic permutations is the smallest size of an antiroute between them.

The following is an easy consequence of (the proof of) Theorem 5 in [7].

**Lemma 9.** *Let $D$ be a good drawing of $K_{5,2}$, with white vertices $a_0, a_1$. Then there is an antiroute from $\mathrm{rot}_D(a_0)$ to $\mathrm{rot}_D(a_1)$ of size $\mathrm{cr}_D(a_0, a_1)$.* □

The following statement is implicitly proved in the discussion after the proof of [7, Theorem 5].

**Lemma 10.** *Let $D$ be a clean drawing of $K_{5,r}$ with white vertices $a_0, a_1, \ldots, a_{r-1}$, and let $\pi_i := \mathrm{rot}_D(a_i)$. Suppose that $\pi_i \neq \pi_j$ whenever $i \neq j$, and for all $i \neq j$ let $\lambda_{ij} := \mathrm{cr}_D(a_i, a_j)$. For $k = 0, 1, 2, 3, 4$, let $\gamma_k := \mathrm{rot}_D(k)$. Then there exist:*

(1) *for all $i, j \in [r]$ with $i \neq j$, an antiroute $P_{ij}$ from $\pi_i$ to $\pi_j$ of size $\lambda_{ij}$;*
(2) *for all $k, \ell \in [5]$ with $k \neq \ell$, an antiroute $Q_{k\ell}$ from $\gamma_k$ to $\gamma_\ell$;*

*such that the transposition $(a_i\, a_j)$ is in $Q_{k\ell}$ if and only if the transposition $(k\, \ell)$ is in $P_{ij}$.* □

We now use these powerful statements to prove that certain graphs cannot be the subgraphs of the key of a clean drawing.

**Proposition 11.** *The graph in Figure 7 is not the key of any clean drawing of $K_{5,n}$.*

*Proof.* Suppose by way of contradiction that the graph in Figure 7 is the key of some clean drawing of $K_{5,n}$. This implies in particular that there exists a drawing $D$ of $K_{5,4}$ with white vertices $a_0, a_1, a_2, a_3$ such that $\mathrm{rot}_D(a_i) = \pi_i$ for $i = 0, 1, 2, 3$, with $\pi_0 = (01234), \pi_1 = (01432), \pi_2 = (04312)$, and $\pi_3 = (03421)$, and $\mathrm{cr}_D(a_0, a_1) = \mathrm{cr}_D(a_0, a_2) = \mathrm{cr}_D(a_0, a_3) = 1$, and $\mathrm{cr}_D(a_1, a_2) = \mathrm{cr}_D(a_1, a_3) = \mathrm{cr}_D(a_2, a_3) = 2$.

The required contradiction is obtained by showing that there do not exist rotations $\mathrm{rot}_D(0), \mathrm{rot}_D(1), \mathrm{rot}_D(2), \mathrm{rot}_D(3), \mathrm{rot}_D(4)$, and antiroutes $P_{ij}, Q_{k\ell}$ that satisfy Lemma 10 (with the given values of $\mathrm{cr}_D(a_i, a_j)$ for $i, j \in \{0, 1, 2, 3\}$, $i \neq j$). We start by determining the possible antiroutes $P_{ij}$ (these depend

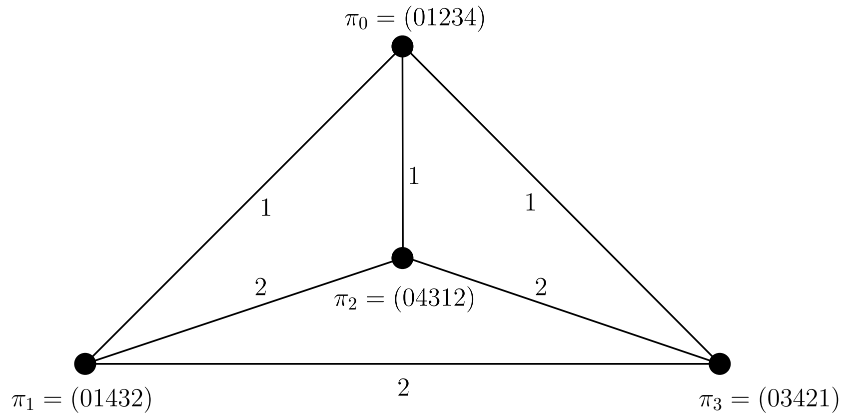


FIGURE 7. This cannot be the key of a clean drawing of $K_{5,n}$.

only on the information we already have). Then we investigate the possible antiroutes $Q_{k\ell}$ consistent with each choice of the antiroutes $P_{ij}$, and prove that, in all cases, every possible choice of $\mathrm{rot}_D(0), \mathrm{rot}_D(1), \mathrm{rot}_D(2)$, $\mathrm{rot}_D(3)$ and $\mathrm{rot}_D(4)$ leads to an inconsistency.

The following facts are easily verified: (i) the only antiroute from $\pi_0$ to $\pi_1$ of size 1 is $\{(01)\}$; (ii) the only antiroute from $\pi_0$ to $\pi_2$ of size 1 is $\{(12)\}$; (iii) the only antiroute from $\pi_0$ to $\pi_3$ of size 1 is $\{(34)\}$; (iv) the only antiroute of size 2 from $\pi_1$ to $\pi_2$ is $\{(02),(34)\}$; (v) there are two distinct antiroutes of size 2 from $\pi_2$ to $\pi_3$, namely $\{(01),(02)\}$ and $\{(03),(04)\}$; and (vi) there are two distinct antiroutes of size 2 from $\pi_1$ to $\pi_3$, namely $\{(02),(12)\}$ and $\{(23),(24)\}$.

Now for $i, j \in \{0,1,2,3\}, i \neq j$, let $P_{ij}$ be the antiroute guaranteed by Lemma 10. By the previous observations it follows that necessarily $P_{01} = \{(01)\}$, $P_{02} = \{(12)\}$, $P_{03} = \{(34)\}$, and $P_{12} = \{(02),(34)\}$. Also by the previous observations there are two choices for $P_{23}$, namely $\{(01),(02)\}$ and $\{(03),(04)\}$; and there are two choices for $P_{13}$, namely $\{(02),(12)\}$ and $\{(23),(24)\}$.

Thus $P_{01}, P_{02}, P_{03}, P_{12}$ are all determined:

$$P_{01} = \{(01)\}, P_{02} = \{(12)\}, P_{03} = \{(34)\}, P_{12} = \{(02),(34)\},$$

and there are four possible combinations of $P_{13}$ and $P_{23}$:

(a) $P_{23} = \{(01),(02)\}$ and $P_{13} = \{(02),(12)\}$.

In this case, by Lemma 10, we have $Q_{01} = \{(a_0a_1),(a_2a_3)\}, Q_{02} = \{(a_1a_2),(a_2a_3),(a_1a_3)\}, Q_{03} = \emptyset, Q_{04} = \emptyset, Q_{12} = \{(a_0a_2),(a_1a_3)\}$, $Q_{13} = \emptyset, Q_{14} = \emptyset, Q_{23} = \emptyset, Q_{24} = \emptyset$, and $Q_{34} = \{(a_0a_3),(a_1a_2)\}$.

(b) $P_{23} = \{(01),(02)\}$ and $P_{13} = \{(23),(24)\}$.

In this case, by Lemma 10, we have $Q_{01} = \{(a_0a_1), (a_2a_3)\}, Q_{02} = \{(a_1a_2), (a_2a_3)\}, Q_{03} = \emptyset, Q_{04} = \emptyset, Q_{12} = \{(a_0a_2)\}, Q_{13} = \emptyset, Q_{14} = \emptyset, Q_{23} = \{(a_1a_3)\}, Q_{24} = \{(a_1a_3)\}$, and $Q_{34} = \{(a_0a_3), (a_1a_2)\}$.

(c) $P_{23} = \{(03), (04)\}$ and $P_{13} = \{(02), (12)\}$.

In this case, by Lemma 10, we have $Q_{01} = \{(a_0a_1)\}, Q_{02} = \{(a_1a_2), (a_1a_3)\}, Q_{03} = \{(a_2a_3)\}, Q_{04} = \{(a_2a_3)\}, Q_{12} = \{(a_0a_2), (a_1a_3)\}, Q_{13} = \emptyset, Q_{14} = \emptyset, Q_{23} = \emptyset, Q_{24} = \emptyset$, and $Q_{34} = \{(a_0a_3), (a_1a_2)\}$.

(d) $P_{23} = \{(03), (04)\}$ and $P_{13} = \{(23), (24)\}$.

In this case, by Lemma 10, we have $Q_{01} = \{(a_0a_1)\}, Q_{02} = \{(a_1a_2)\}, Q_{03} = \{(a_2a_3)\}, Q_{04} = \{(a_2a_3)\}, Q_{12} = \{(a_0a_2)\}, Q_{13} = \emptyset, Q_{14} = \emptyset, Q_{23} = \{(a_1a_3)\}, Q_{24} = \{(a_1a_3)\}$, and $Q_{34} = \{(a_0a_3), (a_1a_2)\}$.

We only analyze (that is, derive a contradiction from) (a). The cases (b), (c), and (d) are handled in a totally analogous manner.

Since $Q_{13} = Q_{14} = \emptyset$, it follows that $\mathrm{rot}_D(3)$ and $\mathrm{rot}_D(4)$ are both equal to the reverse of $\mathrm{rot}_D(1)$; in particular, $\mathrm{rot}_D(3) = \mathrm{rot}_D(4)$. Since $Q_{01} = \{(a_0a_1), (a_2a_3)\}$ and $Q_{12} = \{(a_0a_2), (a_1a_3)\}$, it follows that in $\mathrm{rot}_D(1)$: (i) $a_0$ and $a_1$ must be adjacent; (ii) $a_2$ and $a_3$ must be adjacent; (iii) $a_0$ and $a_2$ must be adjacent; and (iv) $a_1$ and $a_3$ must be adjacent. It follows immediately that $\mathrm{rot}_D(1)$ is either $(a_0a_2a_3a_1)$ or $(a_0a_1a_3a_2)$. Since $\mathrm{rot}_D(3)$ and $\mathrm{rot}_D(4)$ are both the reverse of $\mathrm{rot}_D(1)$, then each of $\mathrm{rot}_D(3)$ and $\mathrm{rot}_D(4)$ is either $(a_0a_1a_3a_2)$ or $(a_0a_2a_3a_1)$. However, since $Q_{34} = \{(a_0a_3), (a_1a_2)\}$, then one must reach the reverse of $\mathrm{rot}_D(4)$ from $\mathrm{rot}_D(3)$ by applying the transpositions $(a_0a_3)$ and $(a_1a_2)$ (in some order). Since neither of these transpositions may be applied to $(a_0a_1a_3a_2)$ or $(a_0a_2a_3a_1)$, we obtain the required contradiction. □

**Proposition 12.** *The graph in Figure 8 is not the key of any clean drawing of $K_{5,n}$.*

*Proof.* Suppose by way of contradiction that the graph in Figure 8 is the key of some clean drawing of $K_{5,n}$. Thus there exists a drawing $D$ of $K_{5,4}$ with white vertices $a_0, a_1, a_2, a_3$ such that $\mathrm{rot}_D(a_i) = \pi_i$ for $i = 0, 1, 2, 3$, with $\pi_0 = (01234), \pi_1 = (01432), \pi_2 = (03241)$, and $\pi_3 = (04231)$, and $\mathrm{cr}_D(a_0, a_1) = \mathrm{cr}_D(a_1, a_2) = \mathrm{cr}_D(a_2, a_3) = \mathrm{cr}_D(a_0, a_3) = 1$, and $\mathrm{cr}_D(a_0, a_2) = \mathrm{cr}_D(a_1a_3) = 2$. For $i, j \in \{0, 1, 2, 3\}, i \neq j$, let $P_{ij}$ be the antiroute guaranteed by Lemma 10. It is easy to verify that the only antiroute of size 1 from $\pi_0$ to $\pi_1$ is $\{(01)\}$, and so necessarily $P_{01} = \{(01)\}$. Analogous arguments show that necessarily $P_{23} = \{(01)\}$ and that $P_{12} = P_{03} = \{(23)\}$. It is also readily checked that there are two antiroutes of size 2 from $\pi_0$ to $\pi_2$, namely $\{(04), (14)\}$ and $\{(24), (34)\}$ (moreover, these are also the two antiroutes of size 2 from $\pi_1$ to $\pi_3$). Thus each of $P_{02}$ and $P_{13}$ is either $\{(04), (14)\}$ or $\{(24), (34)\}$.

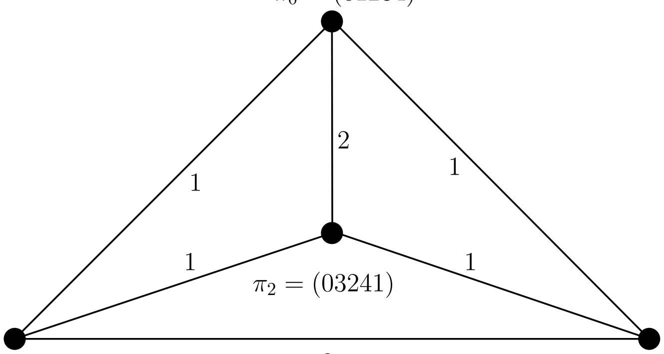


FIGURE 8. This cannot be the key of a clean drawing of $K_{5,n}$.

Thus $P_{01}, P_{03}, P_{12}$, and $P_{23}$ are all determined:

$$P_{01} = P_{23} = \{(01)\}, P_{03} = P_{12} = \{(23)\},$$

and there are four possible combinations of $P_{02}$ and $P_{13}$:

(a) $P_{02} = P_{13} = \{(04), (14)\}$.

In this case, by Lemma 10, $Q_{01} = \{(a_0a_1), (a_2a_3)\}, Q_{04} = \{(a_0a_2), (a_1a_3)\}$, $Q_{14} = \{(a_0a_2), (a_1a_3)\}, Q_{23} = \{(a_0a_3), (a_1a_2)\}$, and $Q_{02} = Q_{03} = Q_{12} = Q_{13} = Q_{24} = Q_{34} = \emptyset$.

(b) $P_{02} = \{(04), (14)\}$ and $P_{13} = \{(24), (34)\}$.

In this case, by Lemma 10, $Q_{01} = \{(a_0a_1), (a_2a_3)\}, Q_{04} = Q_{14} = \{(a_0a_2)\}$, $Q_{23} = \{(a_0a_3), (a_1a_2)\}, Q_{24} = Q_{34} = \{(a_1a_3)\}$, and $Q_{02} = Q_{03} = Q_{12} = Q_{13} = \emptyset$.

(c) $P_{02} = \{(24), (34)\}$ and $P_{13} = \{(04), (14)\}$.

In this case, by Lemma 10, $Q_{01} = \{(a_0a_1), (a_2a_3)\}$, $Q_{04} = Q_{14} = \{(a_1a_3)\}$, $Q_{23} = \{(a_0a_3), (a_1a_2)\}$, $Q_{24} = Q_{34} = \{(a_0a_2)\}$, and $Q_{02} = Q_{03} = Q_{12} = Q_{13} = \emptyset$.

(d) $P_{02} = P_{13} = \{(24), (34)\}$.

In this case, by Lemma 10, $Q_{01} = \{(a_0a_1), (a_2a_3)\}, Q_{23} = \{(a_0a_3), (a_1a_2)\}, Q_{24} = Q_{34} = \{(a_0a_2), (a_1a_3)\}$, and $Q_{02} = Q_{03} = Q_{04} = Q_{12} = Q_{13} = Q_{14} = \emptyset$.

We only analyze (that is, derive a contradiction from) (a). The cases (b), (c), and (d) are handled analogously.

Since $Q_{02} = Q_{03} = Q_{12} = Q_{13} = Q_{24} = Q_{34} = \emptyset$, it follows that $\text{rot}_D(2)$ and $\text{rot}_D(3)$ are equal to each other, and equal to the reverse of each of $\text{rot}_D(0)$, $\text{rot}_D(1)$, and $\text{rot}_D(4)$. Thus $\text{rot}_D(0) = \text{rot}_D(1) = \text{rot}_D(4)$. Since $Q_{01} = \{(a_0a_1), (a_2a_3)\}$ and $Q_{04} = \{(a_0a_2), (a_1a_3)\}$, it follows that in $\text{rot}_D(0)$: (i) $a_0$ and $a_1$ must be adjacent; (ii) $a_2$ and $a_3$ must be adjacent; (iii) $a_0$ and $a_2$ must be adjacent; and (iv) $a_1$ and $a_3$ must be adjacent. Thus $\text{rot}_D(0)$ is either $(a_0a_2a_3a_1)$ or $(a_0a_1a_3a_2)$. Now since $Q_{23} = \{(a_0a_3), (a_1a_2)\}$, it follows that in $\text{rot}_D(2)$ (and hence in its reverse $\text{rot}_D(0)$) we have that $a_0$ is adjacent to $a_3$, and that $a_1$ is adjacent to $a_2$. But this is impossible, since in neither $(a_0a_2a_3a_1)$ nor $(a_0a_1a_3a_2)$ any of these adjacencies occurs. □

## 8. Properties of cores. I. Forbidden subgraphs.

We recall that the *core* of a clean drawing $D$ of $K_{5,n}$ is the subgraph $\Phi^1(D)$ of $\Phi(D)$ that consists of all the vertices of $\Phi(D)$ and the edges of $\Phi(D)$ with label 1. Note that while $\Phi(D)$ is obviously connected, $\Phi^1(D)$ may be disconnected. As all edges of a core are labelled 1, we sometimes omit the reference to the edge labels altogether when working with $\Phi^1(D)$.

Our first result on the structure of cores is a workhorse for the next few sections.

**Claim 13.** *If $\pi_1, \pi_2$ and $\pi_3$ are distinct rotations for white vertices in a drawing of $K_{5,n}$, then there exists at most one rotation $\pi_0$ such that there is an antiroute of size 1 from $\pi_0$ to each of $\pi_1, \pi_2$, and $\pi_3$.*

*Proof.* By way of contradiction, suppose that there exist distinct vertices $\pi_0, \pi_1, \pi_2, \pi_3, \pi_4$ and antiroutes of size 1 from $\pi_i$ to $\pi_1, \pi_2$, and $\pi_3$, for $i = 0$ and 4. For $j = 1, 2, 3$ the antiroutes from $\pi_0$ and $\pi_4$ to $\pi_j$ induce a route $P_{04}(j)$ of size two from $\pi_0$ to $\pi_4$. Assume without loss of generality that $\pi_0 = (01234)$. Suppose that for some $j$, the transpositions in $P_{04}(j)$ involve (in total) four distinct elements in $\{0, 1, 2, 3, 4\}$. It is immediately checked that this implies that $P_{04}(j)$ is the only route of size 2 from $\pi_0$ to $\pi_4$, and that this in turn implies that at least two of $\pi_1, \pi_2$, and $\pi_3$ are equal to each other, a contradiction. Thus each of $P_{04}(1), P_{04}(2)$, and $P_{04}(3)$ involve fewer than four elements in $\{0, 1, 2, 3, 4\}$. None of these routes can involve only two elements (since they have size 2, and $\pi_0 \neq \pi_4$), and so we conclude that each of $P_{04}(1), P_{04}(2)$, and $P_{04}(3)$ involve exactly three elements in $\{0, 1, 2, 3, 4\}$. In particular, $P_{04}(1)$ must equal either $\{(k, k + 1), (k, k + 2)\}$ or $\{(k + 1, k + 2), (k, k + 2)\}$, for some $j \in \{0, 1, 2, 3, 4\}$ (operations are modulo 5; we note that we deviate from the usual notation and separate the elements of a transposition with a comma, for readability purposes). We derive a contradiction assuming that the first possibility holds; the other possibility is handled analogously. Relabelling $0, 1, 2, 3$, and 4, if needed, we may assume that $P_{04}(1) = \{(01), (02)\}$. Thus $\pi_4$ is $(03412)$. It is readily verified that the only routes of size 2 from $\pi_0 = (01234)$ to $\pi_4 = (03412)$ are $P_{04}(1) = \{(01), (02)\}$ and $\{(03), (04)\}$. This in turn immediately implies

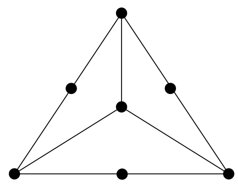

FIGURE 9. The graph obtained by subdividing exactly once each of the edges in a 3-cycle of $K_4$.

that the antiroutes of size 1 from $\pi_0$ to $\pi_1$, $\pi_2$, and $\pi_3$ are either $\{(01)\}$ or $\{(04)\}$, since the transpositions (02) and (03) cannot be applied to $\pi_0$. But then we arrive from $\pi_0$ to two elements in $\{\overline{\pi_1}, \overline{\pi_2}, \overline{\pi_3}\}$ by applying the same transposition; that is, $\pi_i = \pi_j$ for some $i, j \in \{1, 2, 3\}$, $i \neq j$, a contradiction. □

**Proposition 14.** *Let $D$ be an optimal drawing of $K_{5,n}$. Suppose that $\Phi(D)$ is $\{0,4\}$-free. Then:*

(1) *$\Phi^1(D)$ does not contain $K_{2,3}$ as a subgraph.*
(2) *$\Phi^1(D)$ has maximum degree at most 3.*
(3) *$\Phi^1(D)$ does not contain as a subgraph the graph obtained from $K_4$ by subdividing exactly once each of the edges in a 3-cycle (see Fig. 9).*

*Proof.* We start by noting that (1) follows immediately by Claim 13 and Lemma 9.

Suppose now by way of contradiction that $\Phi^1(D)$ has a vertex $\pi_0$ of degree at least 4. Thus $\Phi^1(D)$ has distinct vertices $\pi_1, \pi_2, \pi_3, \pi_4$ such that the edge joining $\pi_0$ to $\pi_i$ has label 1, for $i = 1, 2, 3, 4$. Thus, for $i = 1, 2, 3, 4$, there exists an antiroute from $\pi_0$ to $\pi_i$ of size 1. Without loss of generality we may assume $\pi_0 = (01234)$. The five cyclic rotations that have an antiroute of size 1 to $\pi_0$ are $(01432), (03214), (03421), (04312)$, and $(04231)$. By performing a relabelling $j \to j + 1$ on $\{0, 1, 2, 3, 4\}$ for some $j \in \{0, 1, 2, 3, 4\}$ (with operations modulo 5) if needed (note that the cyclic permutation $\pi_0 = (01234)$ is left unchanged in such a relabelling), we may assume without loss of generality that $\{\pi_1, \pi_2, \pi_3, \pi_4\} = \{(01432), (03214), (03421), (04312)\}$. By exchanging $\pi_1, \pi_2, \pi_3, \pi_4$ if needed, we may assume that $\pi_1 = (01432), \pi_2 = (04312)$, and $\pi_3 = (03421)$.

Since $\Phi(D)$ is $\{0, 4\}$-free, it follows by Proposition 8 that the edge joining $\pi_i$ to $\pi_j$ has label 2, for $i, j \in \{1, 2, 3\}, i \neq j$. Thus, for $i, j = 1, 2, 3, i \neq j$, there exists an antiroute from $\pi_i$ to $\pi_j$ of size 2. Thus $\Phi(D)$ contains as a subgraph the graph in Figure 7, contradicting Proposition 11. This proves (2).

We finally prove (3). Suppose by way of contradiction that $\Phi^1(D)$ contains as a subgraph the graph obtained from $K_4$ by subdividing once each of the edges in a 3-cycle (Fig. 9). Let $\rho_0$ be the "central vertex" in Fig. 9, that is, the only vertex in $\Phi^1(D)$ adjacent to three degree-3 vertices, and let $\rho_1, \rho_3, \rho_4$ denote these three vertices. An argument similar to the one in the second paragraph of this proof shows the following: if $\rho_0 = (01234)$ is a vertex adjacent to vertices $\rho_1, \rho_3, \rho_4$ in $\Phi^1(D)$, then we may assume (that is, perhaps after a relabelling of $0, 1, 2, 3, 4$), that $\rho_1 = (01432), \rho_3 = (04231)$, and $\rho_4 = (04312)$. Now let $\rho_2$ be the vertex adjacent to $\rho_1$ and $\rho_3$ in $\Phi^1(D)$. Thus it follows that in $\Phi(D)$, the edges joining $\rho_0$ and $\rho_1$, $\rho_0$ and $\rho_3$, $\rho_1$ and $\rho_2$, and $\rho_2$ and $\rho_3$ are labelled 1. By Proposition 8, the edge joining $\rho_1$ and $\rho_3$, as well as the edge joining $\rho_0$ and $\rho_2$ have even labels, which must be 2 since $\Phi(D)$ is $\{0, 4\}$-free. Now it is easy to verify that the only cyclic permutation other than $\rho_0$ which has antiroutes of size 1 to both $\rho_1$ and $\rho_3$ is $(03241)$. Thus $\rho_2$ must be $(03241)$. But then the subgraph of $\Phi(D)$ induced by $\rho_0, \rho_1, \rho_2$, and $\rho_3$ is isomorphic to the graph in Figure 8, contradicting Proposition 12. □

## 9. Properties of cores. II. Structural properties.

**Proposition 15.** *Let $D$ be an optimal drawing of $K_{5,n}$, with $n$ even. Suppose that $\Phi(D)$ is $\{0, 4\}$-free. Then:*

(1) *$\Phi^1(D)$ is bipartite.*
(2) *$\Phi^1(D)$ is connected.*

*Proof.* Suppose that $C = (\pi_0, \pi_1, \pi_2, \ldots, \pi_{r-1}, \pi_r, \pi_0)$ is an odd cycle in $\Phi^1(D)$. It follows from Proposition 8 that $\pi_0\pi_2$ must have an even label in $\Phi(D)$, since $\pi_0\pi_1$ and $\pi_1\pi_2$ are both labelled 1 in $\Phi(D)$; now this even label must be 2, since $\Phi(D)$ is $\{0, 4\}$-free. Similarly, since $\pi_2\pi_3$ and $\pi_3\pi_4$ are also labelled 1 in $\Phi(D)$, then $\pi_2\pi_4$ must also be labelled 2 in $\Phi(D)$. Now since both $\pi_0\pi_2$ and $\pi_2\pi_4$ have label 2 in $\Phi(D)$, it follows that $\pi_0\pi_4$ also has label 2 in $\Phi(D)$. By repeating this argument we find that $\pi_0\pi_j$ must have label 2 in $\Phi(D)$ for every even $j$. In particular, $\pi_0\pi_r$ must have label 2, contradicting that $\pi_0\pi_r$ is in $\Phi^1(D)$ (that is, that the label of $\pi_0\pi_r$ in $\Phi(D)$ is 1). Thus $\Phi^1(D)$ cannot have an odd cycle. This proves (1).

To prove (2) we assume, by way of contradiction, that $\Phi^1(D)$ is not connected.

We start by observing that $\Phi(D)$ must have at least one edge labelled 1. Indeed, otherwise every edge $\Phi(D)$ has label of at least 2, and so $\mathrm{cr}\,(D) \geq 2\binom{n}{2} = n(n-1) > Z(5, n)$, contradicting the optimality of $D$.

Thus there exists a component $H$ of $\Phi^1(D)$ with at least 2 vertices. Let $U$ be the set of white vertices whose rotation is a vertex in $H$, and let $V$ be all the other white vertices. Let $r := |U|$ and $s := |V|$. Note that

$$\begin{aligned} \mathrm{cr}\,(D) &= \sum_{\substack{a_i, a_j \in U, \\ a_i \neq a_j}} \mathrm{cr}_D(a_i, a_j) + \sum_{\substack{a_i, a_j \in V, \\ a_i \neq a_j}} \mathrm{cr}_D(a_i, a_j) + \sum_{a_i \in U, a_j \in V} \mathrm{cr}_D(a_i, a_j) \\ &\geq Z(5, r) + Z(5, s) + 2rs, \end{aligned} \tag{1}$$

since every vertex of $U$ is joined to every vertex of $V$ by an edge with a label 2 or greater.

We claim that, moreover, strict inequality must hold in (1). To see this, first we note that, since $H$ has at least 2 vertices, it follows that there exist white vertices $a_k, a_\ell$ whose rotations are in $H$ and such that $\mathrm{cr}_D(a_k, a_\ell) = 1$. Since by assumption $\Phi^1(D)$ is not connected, there is a vertex $\pi$ in $\Phi^1(D)$ not in $H$. Let $a_i$ be a white vertex such that $\mathrm{rot}_D(a_i) = \pi$. Now $\mathrm{cr}_D(a_k, a_i)$ and $\mathrm{cr}_D(a_\ell, a_i)$ are both at least 2. However, we cannot have $\mathrm{cr}_D(a_k, a_i)$ and $\mathrm{cr}_D(a_\ell, a_i)$ both equal to 2, since then $\mathrm{cr}_D(a_k, a_\ell) = 1$ would contradict Proposition 7. Thus either $\mathrm{cr}_D(a_k, a_i)$ or $\mathrm{cr}_D(a_\ell, a_i)$ is at least 3. This proves that Inequality (1) must be strict, that is,

$$\mathrm{cr}\,(D) > Z(5, r) + Z(5, s) + 2rs. \tag{2}$$

Suppose that $r$ (and consequently, also $s$) is even. In this case, since $Z(5, m) = m(m - 2)$ for even $m$, using (2) we obtain $\mathrm{cr}\,(D) > r(r - 2) + s(s-2)+2rs = (r+s)(r+s-2) = Z(5, r+s) = Z(5, n)$, contradicting the optimality of $D$.

Suppose finally that $r$ is odd (and so $s$ is odd, since $|U|+|V| = n$ is even). Using that $r$ and $s$ are odd, and that $Z(5, m) = (m - 1)^2$ for odd $m$, with (2) we obtain $\mathrm{cr}\,(D) > (r - 1)^2 + (s - 1)^2 + 2rs = (r + s)(r + s - 2) + 2 = Z(5, r + s) + 2 = Z(5, n) + 2$, again contradicting the optimality of $D$. This finishes the proof of (2). □

## 10. Properties of cores. III. Minimum degree.

**Proposition 16.** *Let $D$ be an optimal drawing of $K_{5,n}$, with $n$ even. Suppose that $\Phi(D)$ is $\{0, 4\}$-free. Let $\pi_0, \pi_1, \pi_2, \pi_3$ be a path in $\Phi^1(D)$. Suppose that in $\Phi^1(D)$, $\pi_1$ is the only vertex adjacent to both $\pi_0$ and $\pi_2$, and $\pi_2$ is the only vertex adjacent to both $\pi_1$ and $\pi_3$. Then:*

1. *every vertex in $\Phi^1(D)$ is adjacent (in $\Phi^1(D)$) to a vertex in $\{\pi_0, \pi_1, \pi_2, \pi_3\}$; and*
2. *$\pi_0$ and $\pi_3$ are adjacent in $\Phi^1(D)$.*

*Proof.* Let $\pi_0, \pi_1, \ldots, \pi_{r-1}$ be the vertices of $\Phi^1(D)$ (and of $\Phi(D)$ as well). For $i, j \in [r], i \neq j$, let $\lambda_{ij}$ denote the label of the edge that joins $\pi_i$ to $\pi_j$ in $\Phi(D)$. Recall that $\Phi^1(D)$ is bipartite (Proposition 15(1)). Since $\pi_0, \pi_1, \pi_2, \pi_3$ is a path in $\Phi(D)$, it follows that $\pi_0$ and $\pi_2$ are in the same chromatic class $A$, and $\pi_1$ and $\pi_3$ are in the same chromatic class $B$. Moreover, since $\Phi(D)$ is $\{0, 4\}$-free, it follows from Proposition 8 that $\lambda_{ij} = 2$ whenever $\pi_i$ and $\pi_j$ belong to the same chromatic class. Thus we have $\lambda_{02} = \lambda_{13} = 2$ and (since $\pi_0, \pi_1, \pi_2, \pi_3$ is a path in $\Phi^1(D)$) $\lambda_{01} = \lambda_{12} = \lambda_{23} = 1$. It follows that the equations of $\mathcal{L}(\Phi(D))$ corresponding to $\pi_0, \pi_1, \pi_2$, and $\pi_3$ are:

$$\begin{array}{lrcrcrcrcrcl}
E_0: & 2t_0 & - & t_1 & & & + & (\lambda_{03}-2)t_3 & + & \sum\limits_{j\in[r],j>3} (\lambda_{0j}-2)t_j & = & 0,\\
E_1: & -t_0 & + & 2t_1 & - & t_2 & & & + & \sum\limits_{j\in[r],j>3} (\lambda_{1j}-2)t_j & = & 0,\\
E_2: & & - & t_1 & + & 2t_2 & - & t_3 & + & \sum\limits_{j\in[r],j>3} (\lambda_{2j}-2)t_j & = & 0,\\
E_3: & (\lambda_{03}-2)t_0 & & & - & t_2 & + & 2t_3 & + & \sum\limits_{j\in[r],j>3} (\lambda_{3j}-2)t_j & = & 0,
\end{array}$$

where for simplicity we define $E_i := E(\pi_i, \Phi(D))$ for $i \in \{0,1,2,3\}$. Summing up these four linear equations we obtain

$$(3)\quad (\lambda_{03}-1)t_0 + (\lambda_{03}-1)t_3 + \sum_{j\in[r],j>3} (\lambda_{0j}+\lambda_{1j}+\lambda_{2j}+\lambda_{3j}-8)t_j = 0$$

We claim all the coefficients in (3) are nonnegative. First we note that since $\lambda_{03} \geq 1$, then the coefficients of $t_0$ and $t_3$ are indeed nonnegative. For the remaining coeficients, consider any vertex $\pi_j$ in $\Phi(D)$, with $j > 3$. Since $\Phi(D)$ is $\{0,4\}$-free, it follows that $\lambda_{ij} \geq 1$ for every $i \in \{0,1,2,3\}$.

Since $\Phi^1(D)$ is bipartite, it follows that $\pi_j$ cannot be adjacent (in $\Phi^1(D)$) to two elements in $\{\pi_0,\pi_1,\pi_2,\pi_3\}$ whose indices have distinct parity. Now it follows by hypothesis that $\pi_j$ cannot be adjacent to both $\pi_0$ and $\pi_2$, or to $\pi_1$ and $\pi_3$. Thus $\pi_j$ is adjacent to at most one of $\pi_0,\pi_1,\pi_2$ and $\pi_3$ in $\Phi^1(D)$. Using this, and the fact that $\pi_j$ has the same chromatic class as exactly two of these vertices, it follows that at least one element in $\{\lambda_{0j},\lambda_{1j},\lambda_{2j},\lambda_{3j}\}$ is 3, and at least two elements are 2. Thus it follows that $(\lambda_{0j}+\lambda_{1j}+\lambda_{2j}+\lambda_{3j}-8) \geq 0$.

Therefore (3) implies that $(\lambda_{03}-1)t_0 + (\lambda_{03}-1)t_3 \leq 0$. Recall that $\lambda_{03}$ is either 1 or 3. If $\lambda_{03} = 3$, then we have $2t_0 + 2t_3 \leq 0$, which contradicts (Proposition 6) that $\mathcal{L}(\Phi(D))$ has a positive integral solution. We conclude that $\lambda_{03} = 1$, that is, $\pi_0$ and $\pi_3$ are adjacent in $\Phi^1(D)$. This proves (2).

We also note that since $\lambda_{03} = 1$, (3) implies that

$$(4)\quad \sum_{j\in[r],j>3} (\lambda_{0j}+\lambda_{1j}+\lambda_{2j}+\lambda_{3j}-8)t_j = 0.$$

By way of contradiction suppose there is a vertex $\pi_4$ adjacent to none of $\pi_0,\pi_1,\pi_2,\pi_3$ in $\Phi^1(D)$. Then each of $\lambda_{04},\lambda_{14},\lambda_{24},\lambda_{34}$ is at least 2. Using Proposition 8 and that $\Phi(D)$ is $\{0,4\}$-free, it follows that two of these $\lambda$s are 2, and the other two are 3. Therefore $(\lambda_{04}+\lambda_{14}+\lambda_{24}+\lambda_{34}-8) = 2$. Using (4) we obtain

$$(5)\quad 2t_4 + \sum_{j\in[r],j>4} (\lambda_{0j}+\lambda_{1j}+\lambda_{2j}+\lambda_{3j}-8)t_j = 0.$$

We recall that $\lambda_{0j}+\lambda_{1j}+\lambda_{2j}+\lambda_{3j}-8 \geq 0$ for every $j > 3$. Using this and (5), it follows that $2t_4 \leq 0$. But this contradicts that $\mathcal{L}(\Phi(D))$ has a positive integral solution. □

**Proposition 17.** *Let $D$ be an optimal drawing of $K_{5,n}$, with $n$ even. Suppose that $\Phi(D)$ is $\{0,4\}$-free. Then $\Phi^1(D)$ has minimum degree at least 2.*

*Proof.* By way of contradiction, suppose that $\Phi^1(D)$ has a vertex of degree 0 or 1.

Suppose first that $\Phi^1(D)$ has a vertex of degree 0. Then the connectedness of $\Phi^1(D)$ implies that this is the only vertex in $\Phi^1(D)$ (and, consequently, the only vertex in $\Phi(D)$). Thus all vertices of $D$ have the same rotation. Since if $a_i, a_j$ have the same rotation in a drawing $D'$ then $\mathrm{cr}_{D'}(a_i,a_j) = 4$, it follows that $\mathrm{cr}(D) \ge 4\binom{n}{2} = 2n(n-1)$. Since $Z(5,n) = n(n-2)$ and $D$ is optimal, we must have $2n(n-1) \le n(n-2)$, but this inequality does not hold for any positive integer $n$.

Thus we may assume that $\Phi^1(D)$ has a vertex of degree 1.

Let $\pi_0, \pi_1, \ldots, \pi_{m-1}$ denote the vertices of $\Phi^1(D)$. Without any loss of generality we may assume that $\pi_0$ has degree 1 in $\Phi^1(D)$. For $i, j \in [m]$, let $\lambda_{ij}$ denote the label of the edge $\pi_i\pi_j$.

We divide the rest of the proof into two cases.

Case 1. *$\Phi^1(D)$ has a path with 4 vertices starting at $\pi_0$.*

Without loss of generality, let $\pi_0, \pi_1, \pi_2, \pi_3$ be this path. Since $\pi_0$ is a leaf, it follows that $\pi_1$ is the only vertex of $\Phi^1(D)$ adjacent to both $\pi_0$ and $\pi_2$. We note that then there must be a vertex in $\Phi^1(D)$ (say $\pi_4$, without loss of generality) adjacent to both $\pi_1$ and $\pi_3$, as otherwise it would follow by Proposition 16(2) that $\pi_0$ is adjacent to $\pi_3$, contradicting that $\pi_0$ is a leaf. Thus $(\pi_1, \pi_2, \pi_3, \pi_4, \pi_1)$ is a cycle.

For $i, j \in [5]$, let $\lambda_{ij}$ denote the label of $\pi_i\pi_j$ in $\Phi(D)$. Since the edges $\pi_0\pi_1, \pi_1\pi_2, \pi_2\pi_3, \pi_3\pi_4$ and $\pi_1\pi_4$ are all in $\Phi^1(D)$, it follows that $\lambda_{01} = \lambda_{12} = \lambda_{23} = \lambda_{34} = \lambda_{14} = 1$. Now since $\Phi(D)$ is $\{0,4\}$-free, using Proposition 8 it follows that $\lambda_{02} = \lambda_{04} = \lambda_{24} = \lambda_{13} = 2$ and (since $\pi_0\pi_3$ is not in $\Phi^1(D)$) that $\lambda_{03} = 3$.

Subcase 1.1. *$\pi_0, \pi_1, \pi_2, \pi_3, \pi_4$ are all the vertices in $\Phi^1(D)$.*

In this case the linear system $\mathcal{L}(\Phi(D))$ reads:

$$
\begin{array}{rcrcrcrcrcrcl}
E_0 & : & 2t_0 & - & t_1 & & & + & t_3 & & & = & 0, \\
E_1 & : & -t_0 & + & 2t_1 & - & t_2 & & & - & t_4 & = & 0, \\
E_2 & : & & - & t_1 & + & 2t_2 & - & t_3 & & & = & 0, \\
E_3 & : & t_0 & & & - & t_2 & + & 2t_3 & - & t_4 & = & 0, \\
E_4 & : & & - & t_1 & & & - & t_3 & + & 2t_4 & = & 0,
\end{array}
$$

where for brevity we let $E_i := E(\pi_i, \Phi(D))$ for $i \in [5]$.

Subtracting $E_4$ from $E_2$, we obtain that $t_2 = t_4$. Adding the equations $E_0, E_1, E_2$, and using $t_2 = t_4$, we obtain $t_0 = 0$. Thus the system $\mathcal{L}(\Phi(D))$ has no positive integral solution, contradicting (by Proposition 6) the optimality of $D$.

Subcase 1.2. *$\Phi^1(D)$ has a vertex $\pi_5 \notin \{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4\}$.*

The connectedness of $\Phi^1(D)$ implies that $\pi_5$ is adjacent to $\pi_i$ for some $i \in \{0,1,2,3,4\}$. Since $\pi_0$ is a leaf only adjacent to $\pi_1$, then $i \ne 0$. Since $\pi_1$ already has degree 3 in $\Phi^1(D)$, it follows from Proposition 14(2) that $i \ne 1$. Thus $i$ is either $2, 3$ or $4$. Since the roles of 2 and 4 are symmetric, we may conclude that $\pi_5$ is adjacent to either $\pi_2$ or to $\pi_3$.

Suppose first that $\pi_5$ is adjacent to $\pi_3$ in $\Phi^1(D)$.

In this case $\lambda_{35} = 1$. Using Proposition 8, that $\Phi(D)$ is $\{0,4\}$-free, that $\pi_0$ is only adjacent to $\pi_1$, and Claim 13, we obtain $\lambda_{05} = \lambda_{25} = \lambda_{45} = 2$ and that $\lambda_{15} = 3$. Thus in this case the 0-th and the 5-th equations of the system $\mathcal{L}(\Phi(D))$ read:

$$\begin{array}{lcccccccccccc} E_0 & : & 2t_0 & - & t_1 & + & t_3 & & & + & \sum_{j\in[m],j>5} (\lambda_{0j}-2)t_j & = & 0. \\ E_5 & : & & + & t_1 & - & t_3 & + & 2t_5 & + & \sum_{j\in[m],j>5} (\lambda_{5j}-2)t_j & = & 0. \end{array}$$

where for brevity we let $E_i := E(\pi_i, \Phi(D))$ for $i = 0$ and 5.

Adding these equations, we get

$$2t_0 + 2t_5 + \sum_{j\in[m],j>5} (\lambda_{0j} + \lambda_{5j} - 4)t_j = 0. \tag{6}$$

We now argue that $\lambda_{0j} + \lambda_{5j} - 4 \geq 0$ whenever $j > 5$. To see this, note that $\pi_0$ and $\pi_5$ are in the same chromatic class. If $\pi_j$ is in the same chromatic class, then, since $\Phi(D)$ is $\{0,4\}$-free, it follows that $\lambda_{0j}$ and $\lambda_{5j}$ are both 2, and so $\lambda_{0j} + \lambda_{5j} - 4 \geq 0$, as claimed. If $\pi_j$ is in the other chromatic class, then both $\lambda_{0j}$ and $\lambda_{5j}$ are odd. Since $\pi_0$ is a leaf whose only adjacent vertex is $\pi_1$, it follows that $\lambda_{0j} = 3$. On the other hand, $\lambda_{5j}$ is either 1 or 3. In particular, $\lambda_{5j} \geq 1$, and thus also in this case $\lambda_{0j} + \lambda_{5j} - 4 \geq 0$, as claimed. It follows from this observation and (6) that

$$2t_0 + 2t_5 \leq 0,$$

and so the system $\mathcal{L}(\Phi(D))$ has no positive integral solution, contradicting Proposition 6.

Suppose finally that $\pi_5$ is adjacent to $\pi_2$ in $\Phi^1(D)$.

Consider then the path $\pi_0, \pi_1, \pi_2, \pi_5$. Since $\pi_0$ is a leaf, it follows that $\pi_1$ is the only vertex adjacent to both $\pi_0$ and $\pi_2$. Now note that $\pi_2$ is the only vertex adjacent to both $\pi_1$ and $\pi_5$, since by Proposition 14(2) $\pi_1$ cannot be incident to any vertex other than $\pi_0, \pi_2$, and $\pi_4$. Thus Proposition 16 applies, and so we must have that $\pi_0$ and $\pi_5$ are adjacent in $\Phi^1(D)$. But this is impossible, since the only vertex in $\Phi^1(D)$ adjacent to the leaf $\pi_0$ is $\pi_1$.

Case 2. *$\Phi^1(D)$ has no path with 4 vertices starting at $\pi_0$.*

We recall that $\pi_0$ is a leaf in $\Phi^1(D)$. Let $\pi_1$ be the vertex adjacent to $\pi_0$.

Suppose first that $\pi_0$ and $\pi_1$ are the only vertices in $\Phi^1(D)$. Then $\mathcal{L}(\Phi(D))$ consists of only two equations, namely $2t_1 - t_0 = 0$ and $2t_0 - t_1 = 0$. This system obviously has no positive integral solutions, contradicting Proposition 6.

We may then assume that there is an additional vertex $\pi_2$ in $\Phi^1(D)$. By connectedness of $\Phi^1(D)$, and since $\pi_0$ is a leaf, it follows that $\pi_2$ is adjacent to $\pi_1$.

If $\pi_0, \pi_1, \pi_2$ are the only vertices $\Phi(D)$, then the system $\mathcal{L}(\Phi(D))$ consists of the three equations $2t_0 - t_1 = 0$, $-t_0 + 2t_1 - t_2 = 0$, and and $-t_1 + 2t_2 = 0$.

Adding these equations we obtain $t_0+t_2 = 0$. Thus also in this case $\mathcal{L}(\Phi(D))$ does not have a positive integral solution, again contradicting Proposition 6.

Thus there must exist an additional vertex $\pi_3$ in $\Phi^1(D)$. Since $\pi_0$ is a leaf, and by assumption (we are working in Case 2) there is no path with 4 vertices starting at $\pi_0$, it follows that $\pi_3$ must be adjacent to $\pi_1$. We already know that $\lambda_{01} = \lambda_{12} = \lambda_{13} = 1$. Since $\Phi(D)$ is $\{0,4\}$ free, it follows from Proposition 8 that $\lambda_{02} = \lambda_{03} = \lambda_{23} = 2$. Thus in this case $\mathcal{L}(\Phi(D))$ consists of the equations $2t_0 - t_1 = 0$, $-t_0 + 2t_1 - t_2 - t_3 = 0$, $-t_1 + 2t_2 = 0$, and $-t_1 + 2t_3 = 0$. It is an elementary exercise to show that these equations do not have a simultaneous positive integral solution, and so in this case we also obtain a contradiction to Proposition 6. □

## 11. Properties of cores. IV. Girth and maximum size.

**Proposition 18.** *Let $D$ be an optimal drawing of $K_{5,n}$, with $n$ even. Suppose that $\Phi(D)$ is $\{0,4\}$-free. Then:*

(1) *$\Phi^1(D)$ has girth 4.*
(2) *If $v$ is a degree-2 vertex in $\Phi^1(D)$, then $v$ is in a 4-cycle in $\Phi^1(D)$.*
(3) *$\Phi^1(D)$ has at most 7 vertices.*

*Proof.* By Proposition 17, the minimum degree of $\Phi^1(D)$ is at least 2. Since $\Phi^1(D)$ is simple and bipartite, it immediately follows that the girth of $\Phi^1(D)$ is a positive number greater than or equal to 4. Let $\pi_0, \pi_1, \pi_2, \pi_3$ be a path in $\Phi^1(D)$. If there is a vertex other than $\pi_1$ adjacent to both $\pi_0$ and $\pi_2$, or a vertex other than $\pi_2$ adjacent to both $\pi_1$ and $\pi_3$, then $\Phi^1(D)$ clearly has a 4-cycle, and we are done. Otherwise, it follows from Proposition 16(2) that $\pi_0$ is adjacent to $\pi_3$, and so $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$ is a 4-cycle. Thus (1) follows.

Now let $\pi_1$ be a degree-2 vertex in $\Phi^1(D)$. Since $\Phi^1(D)$ has minimum degree at least 2, using (1) it obviously follows that there exists a path $\pi_0, \pi_1, \pi_2, \pi_3$ in $\Phi^1(D)$. If there is a vertex adjacent to both $\pi_0$ and $\pi_2$ other than $\pi_1$, then $\pi_1$ is obviously contained in a 4-cycle. In such a case we are done, so suppose that this is not the case. Since $\pi_1$ is only adjacent to $\pi_0$ and $\pi_2$, using that the degree of $\pi_1$ is 2 it follows that no vertex other than $\pi_2$ is adjacent to both $\pi_1$ and $\pi_3$. Thus it follows from Proposition 16(2) that $\pi_0$ and $\pi_3$ are adjacent in $\Phi^1(D)$. Thus $\pi_1$ is contained in the 4-cycle $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$, and (2) follows.

Let $C = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$ be a 4-cycle in $\Phi^1(D)$; the existence of $C$ is guaranteed from (1). By Proposition 14(1) $\Phi^1(D)$ contains no subgraph isomorphic to $K_{2,3}$, and so, in $\Phi^1(D)$, no vertex other than $\pi_1$ or $\pi_3$ is adjacent to both $\pi_0$ and $\pi_2$, and no vertex other than $\pi_2$ or $\pi_0$ is adjacent to both $\pi_1$ and $\pi_3$. Thus Proposition 16 applies. Using Proposition 14(2) and Proposition 16(1), we obtain that $\Phi^1(D)$ has at most 4 vertices other than $\pi_0, \pi_1, \pi_2$, and $\pi_3$; that is, $\Phi^1(D)$ has at most 8 vertices in total; moreover, if $\Phi^1(D)$ has exactly 8 vertices, then every vertex of $C$ has degree 3. Since $C$ was an arbitrary 4-cycle, we have actually proved that if $\Phi^1(D)$ has 8 vertices, then every vertex contained in a 4-cycle must have degree 3. In view of (2), this implies that if $\Phi^1(D)$ has 8 vertices, then it must be cubic.

Now the unique (up to isomorphism) cubic connected bipartite graph on 8 vertices is the 3-cube. Since the 3-cube contains as a subgraph the graph in Figure 9, it follows that $\Phi^1(D)$ cannot have exactly 8 vertices. □

## 12. The possible cores of an antipodal-free optimal drawing.

Our goal in this section is to establish Lemma 21, which states that the core of every antipodal-free optimal drawing of $K_{5,n}$ is isomorphic to either a 4-cycle or to the graph $\overline{C}_6$ obtained from the 6-cycle by adding an edge joining two diametrically oposed vertices (see Figure 10).

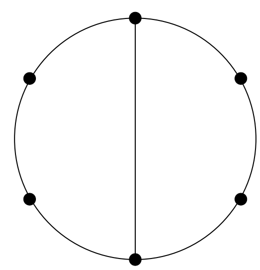

Figure 10. The graph $\overline{C}_6$.

We first show this for the particular case in which $\Phi(D)$ is not only antipodal-free (that is, 0-free), but also 4-free:

**Proposition 19.** *Let $D$ be an optimal drawing of $K_{5,n}$, with $n$ even. If $\Phi(D)$ is $\{0,4\}$-free, then $\Phi^1(D)$ is isomorphic to the 4-cycle or to $\overline{C}_6$.*

*Proof.* By way of contradiction, suppose that $\Phi^1(D)$ is isomorphic to neither a 4-cycle nor to $\overline{C}_6$. Recall that $\Phi^1(D)$ has minimum degree at least 2 (Proposition 17). We divide the proof into two cases, depending on whether or not $\Phi^1(D)$ has degree-2 vertices.

Case 1. $\Phi^1(D)$ *has at least one degree-2 vertex.*

By Proposition 18(3), $\Phi^1(D)$ has at most 7 vertices. If all the vertices in $\Phi^1(D)$ have degree 2, then (since $\Phi^1(D)$ is simple and, by Proposition 15(2), connected) $\Phi^1(D)$ is a cycle. By Proposition 18(1), in this case $\Phi^1(D)$ is a 4-cycle, contradicting our assumption at the beginning of the proof.

Thus we may assume that $\Phi^1(D)$ has at least one degree-3 vertex. Let $H$ be the graph obtained by suppressing the degree-2 vertices from $\Phi^1(D)$. We call the vertices of $\Phi^1(D)$ that correspond to the vertices in $H$ (that is, the degree-3 vertices of $\Phi^1(D)$) the *nodes* of $\Phi^1(D)$.

It follows from elementary graph theory that $\Phi^1(D)$ has an even number of nodes. Since $\Phi^1(D)$ has at most 7 vertices, it follows that $\Phi^1(D)$ has either $2, 4$, or 6 nodes.

SUBCASE 1.1. *$\Phi^1(D)$ has 6 nodes.*

Up to isomorphism, there are only two cubic simple graphs on 6 nodes, namely $K_{3,3}$ and the triangular prism $T_3$ (this is the simple cubic graph with a matching whose removal leaves two disjoint 3-cycles). Now $T_3$ has two vertex disjoint 3-cycles, and so in order to turn it into a bipartite graph, we must subdivide at least 2 edges, that is, add at least two vertices to $T_3$. Since $\Phi^1(D)$ has at most 7 vertices, it follows that $H$ cannot be isomorphic to $T_3$.

Suppose finally that $H$ is isomorphic to $K_{3,3}$. Since no bipartite graph on 7 vertices is a subdivision of $K_{3,3}$, it follows that $\Phi^1(D)$ must be itself isomorphic to $K_{3,3}$. Since $K_{3,3}$ obviously contains $K_{2,3}$ as a subgraph, this contradicts Proposition 14(1).

SUBCASE 1.2. *$\Phi^1(D)$ has 4 nodes.*

In this case $H$ must be isomorphic to $K_4$, the only cubic graph on four vertices. It is readily seen that there are only two ways to turn $K_4$ into a bipartite graph using at most three edge subdivisions. One way is to subdivide once each of the edges in a 3-cycle of $K_4$, and the other way is to subdivide (once) two nonadjacent edges (in the latter case, we obtain a graph that has a subgraph isomorphic to $K_{2,3}$). By Proposition 14, neither of these graphs can be the core of $D$.

SUBCASE 1.3. *$\Phi^1(D)$ has 2 nodes.*

In this case $H$ must consist of two vertices joined by three parallel edges. Since $\Phi^1(D)$ is bipartite it follows that each of these edges must be subdivided the same number of times modulo 2 (subdividing an edge 0 times being a possibility). Moreover, since $\Phi^1(D)$ is simple at least two edges must be subdivided at least once each.

Now no edge may be subdivided more than twice, as in this case the result would be a graph with a degree-2 vertex belonging to no 4-cycle, contradicting Proposition 18(2).

Suppose now that some edge of $H$ is subdivided exactly twice. Then, since $\Phi^1(D)$ has at most 7 vertices, it follows that two edges of $H$ are subdivided exactly twice, and the other edge of $H$ is not subdivided. Thus it follows that in this case $\Phi^1(D)$ is isomorphic to $\overline{C}_6$, contradicting our initial assumption.

Suppose finally that no edge of $H$ is subdivided more than once. Since $\Phi^1(D)$ is bipartite, it follows that every edge of $H$ must be subdivided exactly once. Thus $\Phi^1(D)$ is isomorphic to $K_{2,3}$, contradicting Proposition 14(1).

Case 2. $\Phi^1(D)$ *has no degree-*2 *vertices.*

In this case, $\Phi^1(D)$ is cubic. By Proposition 15, $\Phi^1(D)$ is bipartite and connected. By Proposition 18(3), $\Phi^1(D)$ has at most 7 vertices. By elementary graph theory, since $\Phi^1(D)$ is cubic, then it has an even number of vertices. Since $\Phi^1(D)$ is simple, it follows that $\Phi^1(D)$ has either 4 or 6 vertices.

Now there are no simple cubic bipartite graphs on 4 vertices, so $\Phi^1(D)$ must have 6 vertices. Up to isomorphism, the only cubic bipartite graph on 6 vertices is $K_{3,3}$. But $\Phi^1(D)$ cannot be isomorphic to $K_{3,3}$, since by Proposition 14(1) $\Phi^1(D)$ does not contain a subgraph isomorphic to $K_{2,3}$.

□

**Proposition 20.** *Let $D$ be an antipodal-free, optimal drawing of $K_{5,n}$, with $n$ even. Then $\Phi(D)$ is* 4*-free.*

*Proof.* By way of contradiction, suppose that $\Phi(D)$ is not 4-free. Then there exist distinct rotations $\pi, \pi'$, and white vertices $a_i, a_j$ such that $\text{rot}_D(a_i) = \pi$ and $\text{rot}_D(a_j) = \pi'$, and $\text{cr}_D(a_i, a_j) = 4$.

Without loss of generality, suppose that $\text{cr}_D(a_i) \leq \text{cr}_D(a_j)$. We move, one by one, every vertex $a_j$ with rotation $\pi'$ very close to $a_i$, so that in the resulting drawing $D'$ we have $\text{cr}_{D'}(a_j, a_k) = \text{cr}_{D'}(a_i, a_k)$ for every vertex $k \notin \{i, j\}$. It is readily checked that the resulting drawing $D'$ is also optimal, and $\Phi(D')$ has one fewer edge with label 4 than $\Phi(D)$. By repeating this process as many times as needed, we arrive to a drawing $D^o$ such that $\Phi(D^o)$ has exactly one edge with label 4 (if $\Phi(D)$ has exactly one edge with label 4 to begin with, then we let $D^o = D$). Denote by $\pi_0, \pi_1$ the vertices of $\Phi(D^o)$ whose joining edge has label 4.

If we apply the described process one more time to $D^o$ with $\pi = \pi_0$ and $\pi' = \pi_1$, we obtain a $\{0, 4\}$-free optimal drawing $E$ of $K_{5,n}$. By Proposition 19, $\Phi^1(E)$ contains a 4-cycle $(\pi_0, \pi_2, \pi_3, \pi_4, \pi_0)$. Now if we apply the process to $D^o$ with $\pi = \pi_1$ and $\pi' = \pi_0$, then we obtain another $\{0, 4\}$-free optimal drawing $F$ of $K_{5,n}$. Note that $\pi_2, \pi_3, \pi_4$ are not affected in the process, and so $(\pi_1, \pi_2, \pi_3, \pi_4, \pi_1)$ is a 4-cycle in $\Phi^1(F)$. Thus it follows that $\Phi^1(D^o)$ has two degree-3 vertices $\pi_2$ and $\pi_4$, plus the vertices $\pi_0, \pi_1, \pi_3$, each of which is joined to both $\pi_2$ and $\pi_4$ with an edge labelled 1. This contradicts Claim 13.

□

**Lemma 21.** *Let $D$ be an antipodal-free, optimal drawing of $K_{5,n}$, with $n$ even. Then $\Phi^1(D)$ is isomorphic either to the* 4*-cycle or to $\overline{C}_6$.*

*Proof.* By Proposition 20, $\Phi(D)$ is 4-free. By hypothesis $\Phi(D)$ is also 0-free (since $D$ is antipodal-free), and so $\Phi(D)$ is $\{0, 4\}$-free. The lemma then follows by Proposition 19. □

## 13. Proof of Theorem 1.

We need one final result before moving on to the proof of Theorem 1. In the following statement and its proof, we sometimes use the notation $(i, j, k, \ell, m)$ for cyclic permutations (that is, we separate the elements with commas, as opposed to our usual practice in which for such a cyclic permutation we would have written $(ijk\ell m)$).

**Proposition 22.** *Let $D$ be a drawing of $K_{5,n}$. Suppose that $\Phi(D)$ is $\{0,4\}$-free, and that $\Phi^1(D)$ is a 4-cycle $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$. Suppose that $\pi_0 = (01234)$. Then there exists an $m \in \{0,1,2,3,4\}$ and a relabelling of $\{0,1,2,3,4\}$ that leaves $\pi_0$ invariant, such that (operations are modulo 5):*

- *$\pi_2 = (m, m+1, m+3, m+4,\ m+2)$; and*
- *$\{\pi_1, \pi_3\} = \{(m, m+4, m+2, m+3, m+1), (m, m+4, m+3, m+1, m+2)\}$.*

*Proof.* The reverse permutation $\overline{\pi_0}$ of $\pi_0$ is (43210). Since $\pi_0\pi_1$ and $\pi_0\pi_3$ have label 1 in $\Phi(D)$, it follows that each of $\pi_1$ and $\pi_3$ is obtained from $\overline{\pi_0}$ by performing one transposition. Thus there exist distinct $k, m \in \{0,1,2,3,4\}$ such that $\{\pi_1, \pi_3\} = \{(k, k+4, k+2, k+3, k+1), (m, m+4, m+2, m+3, m+1)\}$.

Suppose that $k = m+3$. Using a relabelling on $\{0,1,2,3,4\}$ that leaves (01234) invariant, we may assume that $m = 2$ and $k = 0$. Then $\{\pi_1, \pi_3\} = \{(04231), (03214)\}$. Now since the edge joining $\pi_2$ to each of $\pi_1$ and $\pi_3$ in $\Phi(D)$ has label 1, it follows that there are antiroutes of size 1 from $\pi_2$ to each of $\pi_1$ and $\pi_3$. It is easy to check that the only such possibility is that $\pi_2 = (04132)$. Using the relabelling $j \mapsto j-2$ on $\{0,1,2,3,4\}$, we get $\{\pi_0, \pi_1, \pi_2, \pi_3\} = \{(01234), (01432), (03241), (04231)\}$. But then $\Phi(D)$ is the labelled graph in Fig. 8, contradicting Proposition 12. An analogous contradiction is obtained under the assumption $k = m+2$. Thus $k = m+1$ or $k = m+4$.

Suppose that $k = m+1$. Thus $\{\pi_1, \pi_3\} = \{(m+1, m, m+3, m+4, m+2), (m, m+4, m+2, m+3, m+1)\}$. Using the relabelling $j \mapsto j-1$ on $\{0,1,2,3,4\}$ (which obviously leaves (01234) invariant), we obtain $\{\pi_1, \pi_3\} = \{(m, m+4, m+2, m+3, m+1), (m+4, m+3, m+1, m+2, m)\} = \{(m, m+4, m+2, m+3, m+1), (m, m+4, m+3, m+1, m+2)\}$, as required. Finally, since the edge joining $\pi_2$ to each of $\pi_1$ and $\pi_3$ in $\Phi(D)$ has label 1, it follows that $\pi_2 = (m, m+1, m+3, m+4, m+2)$. The case $k = m+4$ is handled in a totally analogous manner. □

**Proposition 23.** *Suppose that $D$ is a drawing of $K_{5,n}$. Suppose that $\Phi(D)$ is $\{0,4\}$-free, and that $\Phi^1(D)$ is isomorphic to $\overline{C}_6$. Let the vertices of $\Phi^1(D)$ be labeled $\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5$, so that $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_0)$ and $(\pi_0, \pi_4, \pi_5, \pi_3, \pi_0)$ are 4-cycles. Suppose that $\pi_0 = (01234)$. Then there exists an $m \in \{0,1,2,3,4\}$ and a relabelling of $\{0,1,2,3,4\}$ that leaves $\pi_0$ invariant, such that (operations are modulo 5):*

- *$\pi_3 = (m, m+4, m+3,\ m+1, m+2)$;*

- $\{(\pi_1,\pi_2),(\pi_4,\pi_5)\} = \{((m,m+4,m+2,m+3,m+1),(m,m+1,m+3,m+4,m+2)),((m,m+1,m+4,m+3,m+2),(m,m+2,m+3,m+1,m+4))\}$.

*Proof.* By Proposition 22, there exists an $m \in \{0,1,2,3,4\}$ such that $\pi_2 = (m,m+1,m+3,m+4,m+2)$ and $\{\pi_1,\pi_3\} = A := \{(m,m+4,m+2,m+3,m+1),(m,m+4,m+3,m+1,m+2)\}$. By the same proposition, there exists a $k \in \{0,1,2,3,4\}$ such that $\pi_5 = (k,k+1,k+3,k+4,k+2)$ and $\{\pi_3,\pi_4\} = B := \{(k,k+4,k+2,k+3,k+1),(k,k+4,k+3,k+1,k+2)\}$.

Since $\pi_2 \neq \pi_5$, it follows that $m \neq k$. Thus $k$ is either $m+1, m+2, m+3$, or $m+4$. Note that if $k = m+2$ or $k = m+3$ then $A \cap B = \emptyset$, which contradicts that $\{\pi_3\} = A \cap B$. Thus $k$ is either $m+1$ or $m+4$.

We work out the details for the case $k = m+1$; the case $k = m+4$ is handled in a totally analogous manner. Since $\{\pi_3\} = A \cap B$, it follows that $\pi_3 = (m,m+4,m+2,m+3,m+1) = (m+1,m,m+4,m+2,m+3)$. Therefore $\pi_1 = (m,m+4,m+3,m+1,m+2) = (m+1,m+2,m,m+4,m+3)$, $\pi_2 = (m,m+1,m+3,m+4,m+2) = (m+1,m+3,m+4,m+2,m)$, $\pi_4 = (m+1,m,m+3,m+4,m+2)$, and $\pi_5 = (m+1,m+2,m+4,m,m+3)$. Using the relabelling $j \to j-1$ on $\{0,1,2,3,4\}$ (which leaves (01234) invariant), we obtain $\pi_1 = (m,m+1,m+4,m+3,m+2)$, $\pi_2 = (m,m+2,m+3,m+1,m+4)$, $\pi_3 = (m,m+4,m+3,m+1,m+2)$ $\pi_4 = (m,m+4,m+2,m+3,m+1)$, and $\pi_5 = (m,m+1,m+3,m+4,m+2)$. □

*Proof of Theorem 1.* Let $D$ be an antipodal-free drawing of $K_{5,n}$, with $n$ even. In view of Proposition 3 (see Remark 4), we may assume that $D$ is clean, so that $\Phi(D)$ and $\Phi^1(D)$ are well-defined.

In view of Lemma 21, $\Phi^1(D)$ is isomorphic either to the 4-cycle or to $\overline{C}_6$.

Case 1. *$\Phi(D)$ is isomorphic to $\overline{C}_6$.*

In this case $\Phi(D)$ has 6 vertices, which we label $\pi_0,\pi_1,\pi_2,\pi_3,\pi_4,\pi_5$, so that $(\pi_0,\pi_1,\pi_2,\pi_3,\pi_0)$ and $(\pi_0,\pi_4,\pi_5,\pi_3,\pi_0)$ are 4-cycles. For $i,j \in \{0,1,2,3,4,5\}$, $i \neq j$, let $\lambda_{ij}$ be the label of the edge $\pi_i\pi_j$. Since $(\pi_0,\pi_1,\pi_2,\pi_3,\pi_0)$ and $(\pi_0,\pi_4,\pi_5,\pi_3,\pi_0)$ are 4-cycles in $\Phi^1(D)$, it follows that all the edges in these 4-cycles have label 1 in $\Phi(D)$; that is, $\lambda_{01} = \lambda_{12} = \lambda_{23} = \lambda_{03} = \lambda_{04} = \lambda_{45} = \lambda_{35} = 1$. By Proposition 8, $\lambda_{02}$ is even. Since $\Phi(D)$ is antipodal-free, and (by Property (2) of a clean drawing) $\lambda_{ij} \leq 4$ for all $i,j$, it follows that $\lambda_{02}$ is either 2 or 4. By Proposition 20 $\Phi(D)$ is 4-free, hence $\lambda_{02} = 2$. The same argument shows that $\lambda_{05} = \lambda_{13} = \lambda_{14} = \lambda_{25} = \lambda_{34} = 2$. Since $\lambda_{35} = 1$ and $\lambda_{13} = 2$, by Proposition 8, $\lambda_{15}$ is odd. If $\lambda_{15} = 1$, then $\{\pi_0,\pi_5\} \cup \{\pi_1,\pi_2,\pi_4\}$ is a $K_{2,3}$ in $\Phi^1(D)$, contradicting Proposition 8; thus $\lambda_{15} = 3$. An analogous argument shows that $\lambda_{24} = 3$.

The linear system $\mathcal{L}(\Phi(D))$ associated to $\Phi(D)$ (see Definition 5) is then:

$$
\begin{array}{lcrcrcrcrcrcrcl}
E_0 & : & 2t_0 & - & t_1 & & & - & t_3 & - & t_4 & & & = & 0.\\
E_1 & : & -t_0 & + & 2t_1 & - & t_2 & & & & & + & t_5 & = & 0.\\
E_2 & : & & - & t_1 & + & 2t_2 & - & t_3 & + & t_4 & & & = & 0.\\
E_3 & : & -t_0 & & & - & t_2 & + & 2t_3 & & & - & t_5 & = & 0.\\
E_4 & : & -t_0 & & & + & t_2 & & & + & 2t_4 & - & t_5 & = & 0.\\
E_5 & : & & + & t_1 & & & - & t_3 & - & t_4 & + & 2t_5 & = & 0.
\end{array}
\tag{7}
$$

It is straightforward to check that if $(t_0, t_1, t_2, t_3, t_4, t_5)$ is a positive solution to this system, then $t_1 = t_2$, $t_4 = t_5$ and $t_0 = t_3 = t_1 + t_4$. By Proposition 6, this implies that $n \equiv 0 \pmod 4$. This proves (1).

We have thus proved that the white vertices of $D$ are partitioned into 6 classes $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5$, such that $|\mathcal{C}_1| = |\mathcal{C}_2|$, $|\mathcal{C}_4| = |\mathcal{C}_5|$, $|\mathcal{C}_0| = |\mathcal{C}_3| = |\mathcal{C}_1| + |\mathcal{C}_4|$, and such that for $i = 0, 1, 2, 3, 4, 5$, each vertex in $\mathcal{C}_i$ has rotation $\pi_i$. Let $r := |\mathcal{C}_1|$ and $s := |\mathcal{C}_4|$, so that $|\mathcal{C}_2| = r$, $|\mathcal{C}_5| = s$, and $|\mathcal{C}_0| = |\mathcal{C}_3| = r + s$. Note that $4(r + s) = n$.

If necessary, relabel $\{0, 1, 2, 3, 4\}$ so that $\pi_0 = (01234)$. By Proposition 23, perhaps after a further relabelling of $\{0, 1, 2, 3, 4\}$ (that leaves $\pi_0$ invariant), there exists an $m \in \{0, 1, 2, 3,\ 4\}$ such that $\pi_3 = (m, m+4, m+3, m+1, m+2)$, and $\{(\pi_1, \pi_2), (\pi_4, \pi_5)\} = \{((m, m+4, m+2, m+3, m+1), (m, m+1, m+3, m+4, m+2)), ((m, m+1, m+4, m+3, m+2), (m, m+2, m+3, m+1, m+4))\}$. Now perform the further relabelling $j \mapsto j - m$. After this relabelling (which again leaves $\pi_0$ invariant), we have $\pi_3 = (04312)$ and $\{(\pi_1, \pi_2), (\pi_4, \pi_5)\} = \{((04231), (01342)), ((01432), (02314))\}$.

We have thus proved that (perhaps after a relabelling of $\{0, 1, 2, 3, 4\}$) there exist integers $r, s$ such that $D$ has $r + s$ vertices with rotation $\pi_0 = (01234)$, $r$ vertices with rotation $\pi_1 = (04231)$, $r$ vertices with rotation $\pi_2 = (01342)$, $r + s$ vertices with rotation $\pi_3 = (04312)$, $s$ vertices with rotation $\pi_4 = (01432)$, and $s$ vertices with rotation $\pi_5 = (02314)$. That is, $D$ is isomorphic to the drawing $D_{r,s}$ from Section 3.

Case 2. *$\Phi(D)$ is isomorphic to the 4-cycle.*

In this case $\Phi(D)$ has 4 vertices, which we label $\rho_0, \rho_1, \rho_2, \rho_3$, so that $(\rho_0, \rho_1, \rho_2, \rho_3, \rho_0)$ is a cycle. The linear system $\mathcal{L}(\Phi(D))$ associated to $\Phi(D)$ is the one that results by taking $t_4 = t_5 = 0$ in the linear system (7), and omitting the equations $E_4$ and $E_5$.

It is straightforward to check that if $(t_0, t_1, t_2, t_3)$ is a solution to this system, then $t_0 = t_1 = t_2 = t_3$. By Proposition 6, this implies that $n \equiv 0 \pmod 4$. This proves (1).

Thus the white vertices of $D$ are partitioned into 4 classes $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, each of size $n/4$, so that each vertex in class $\mathcal{C}_i$ has rotation $\rho_i$.

Label the vertices $0, 1, 2, 3, 4$ so that $\rho_0 = (01234)$. Then, by Proposition 22, possibly after a relabelling of $\{0, 1, 2, 3, 4\}$ that leaves $\rho_0$ invariant, there is an $m \in \{0, 1, 2, 3, 4\}$ such that $\rho_2 = (m, m+1, m+3, m+4, m+2)$, and $\{\rho_1, \rho_3\} = \{(m, m+4, m+2, m+3, m+1), (m, m+4, m+$

$3, m+1, m+2)\}$. Now we perform the relabelling $j \mapsto j-m$ on $\{0,1,2,3,4\}$ (which obviously leaves $\rho_0$ invariant), we obtain $\rho_2 = (01342)$ and $\{\rho_1, \rho_3\} = \{(04231), (04312)\}$.

We have thus proved that $D$ has $r$ vertices with rotation (01234), $r$ vertices with rotation (01342), $r$ vertices with rotation (04231), and $r$ vertices with rotation (04312). That is, $D$ is isomorphic to the drawing $D_{r,0}$ from Section 3, with $r = n/4$. □

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Instituto de Física, UASLP. San Luis Potosí, México.
*E-mail address*: `cesar@ifisica.uaslp.mx`

Instituto de Física, UASLP. San Luis Potosí, México.
*E-mail address*: `cmedina@ifisica.uaslp.mx`

Instituto de Física, UASLP. San Luis Potosí, México.
*E-mail address*: `gsalazar@ifisica.uaslp.mx`