The 2-page crossing number of K_n

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Abstract

Around 1958, Hill described how to draw the complete graph K_n with

$$Z(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

crossings, and conjectured that the crossing number $\operatorname{cr}(K_n)$ of K_n is exactly $Z(n)$. This is also known as Guy's conjecture as he later popularized it. Towards the end of the century, substantially different drawings of K_n with $Z(n)$ crossings were found. These drawings are *2-page book drawings*, that is, drawings where all the vertices are on a line ℓ (the spine) and each edge is fully contained in one of the two half-planes (pages) defined by ℓ. The *2-page crossing number* of K_n, denoted by $\nu_2(K_n)$, is the minimum number of crossings determined by a 2-page book drawing of K_n. Since $\operatorname{cr}(K_n) \leq \nu_2(K_n)$ and $\nu_2(K_n) \leq Z(n)$, a natural step towards Hill's Conjecture is the weaker conjecture $\nu_2(K_n) = Z(n)$, popularized by Vrt'o. In this paper we develop a novel and innovative technique to investigate crossings in drawings of K_n, and use it to prove that $\nu_2(K_n) = Z(n)$. To this end, we extend the inherent geometric definition of k-edges for finite sets of points in the plane to topological drawings of K_n. We also introduce the concept of $\leq\leq k$-edges as a useful generalization of $\leq k$-edges and extend a powerful theorem that expresses the number of crossings in a rectilinear drawing of

K_n in terms of its number of ($\leq k$)-edges to the topological setting. Finally, we give a complete characterization of crossing minimal 2-page book drawings of K_n and show that, up to equivalence, they are unique for n even, but that there exist an exponential number of non-homeomorphic such drawings for n odd.

1 Introduction

In a *drawing* of a graph in the plane, each vertex is represented by a point and each edge is represented by a simple open arc, such that if uv is an edge, then the closure (in the plane) of the arc α representing uv consists precisely of α and the points representing u and v. It is further required that no arc representing an edge contains a point representing a vertex.

A crossing in a drawing D of a graph G is a pair $(x, \{\alpha, \beta\})$, where x is a point in the plane, α, β are arcs representing different edges, and $x \in \alpha \cap \beta$. The *crossing number* $\mathrm{cr}(D)$ of D is the number of crossings in D, and the *crossing number* $\mathrm{cr}(G)$ of G is the minimum $\mathrm{cr}(D)$, taken over all drawings D of G.

A drawing is *good* if (i) no three distinct arcs representing edges meet at a common point; (ii) if two edges are adjacent, then the arcs representing them do not intersect each other; and (iii) an intersection point between two arcs representing edges is a crossing rather than tangential. It is well-known (and easy to prove) that every graph has a crossing-minimal drawing which is good (moreover, (ii) and (iii) hold in *every* crossing-minimal drawing). Thus, when our aim (as in this paper) is to estimate the crossing number of a graph, we may assume that all drawings under consideration are good.

As usual, for simplicity we often make no distinction between a vertex and the point representing it, or between an edge and the arc representing it. No confusion should arise from this practice.

Around 1958, Hill conjectured that

$$\mathrm{cr}(K_n) = Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor . \tag{1}$$

This conjecture appeared in print a few years later in papers by Guy [15] and Harary and Hill [16]. Hill described drawings of K_n with $Z(n)$ crossings, which were later corroborated by Blažek and Koman [5]. These drawings show that $\mathrm{cr}(K_n) \leq Z(n)$. The best known general lower bound is $\lim_{n\to\infty} \mathrm{cr}(K_n)/Z(n) \geq 0.8594$, due to de Klerk et al. [11]. For more on the history of this problem we refer the reader to the excellent survey by Beineke and Wilson [3].

One of the major motivations for investigating crossing numbers is their application to VLSI design. With this motivation in mind, Chung, Leighton and Rosenberg [7] analyzed embeddings of graphs in books: the vertices lie on a line (the *spine*) and the edges lie on the *pages* of the book. Book embeddings of graphs have been extensively studied [4, 12]. Now if the book has k pages, and crossings among edges are allowed, the result is a *k-page book drawing.*

Here we concentrate on 2-page book drawings. The *2-page crossing number* $\nu_2(G)$ of a graph G is the minimum of $\mathrm{cr}(D)$ taken over all 2-page book drawings D of G. Alternative terminologies for the 2-page crossing number are *circular crossing number* [17] and *fixed linear crossing number* [8]. We may regard the pages as the closed half-planes defined by the spine, and so every 2-page book drawing can be realized as a plane drawing; it follows that $\mathrm{cr}(G) \le \nu_2(G)$ for every graph G.

In 1964, Blažek and Koman [5] found 2-page book drawings of K_n with $Z(n)$ crossings, thus showing that $\nu_2(K_n) \le Z(n)$ (see also Guy et al. [14], Damiani et al. [9], Harborth [17], and Shahrokhi et al. [20].) Once these constructions were known, the conjecture that $\nu_2(K_n) = Z(n)$ is implicit in the conjecture given by Equation (1) since $\mathrm{cr}(K_n) \le \nu_2(K_n)$. However, the only explicit reference to this weaker conjecture is, as far as we know, from Vrt'o [21].

Buchheim and Zhang [6] reformulated the problem of finding $\nu_2(K_n)$ as a maximum cut problem on associated graphs, and then solved exactly this maximum cut problem for all $n \le 13$, thus confirming Equation (1) for 2-page book drawings for all $n \le 14$ (the case $n = 14$ follows from the case $n = 13$ by an elementary counting argument). Very recently, De Klerk and Pasechnik [10] used this max cut reformulation to find the exact value of $\nu_2(K_n)$ for all $n \le 21$ and $n = 24$, and moreover, by using semidefinite programming techniques, to obtain the asymptotic bound $\lim_{n\to\infty} \nu_2(K_n)/Z(n) \ge 0.9253$. All the results reported in [6] and [10] are computer-aided.

In this paper we prove that $\nu_2(K_n) = Z(n)$. The main technique for the proof is the extension of the concept of *k-edge* of a finite set of points to topological drawings of the complete graph. We do this in a way such that the identities proved by Ábrego and Fernández-Merchant [1] and Lovász et al. [19], that express the crossing number of a rectilinear drawing of K_n in terms of the k-edges or the $(\le k)$-edges of its set of vertices, are also valid in the topological setting.

We recall that a drawing D is *rectilinear* if the edges of D are straight line segments, and the *rectilinear crossing number* $\overline{\mathrm{cr}}(G)$ of a graph G is the minimum of $\mathrm{cr}(D)$ taken over all rectilinear drawings D of G. An edge pq of D is a *k-edge* if the line spanned by pq divides the remaining set of vertices into two subsets of cardinality k and $n-2-k$. Thus a k-edge is also an $(n-2-k)$-edge. Denote by $E_k(D)$ the number of k-edges of D. The following identity [1, 19] has been key to the recent developments on the rectilinear crossing number of K_n.

$$\overline{\mathrm{cr}}\,(D) = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k\,(n-2-k)\,E_k\,(D)\,. \tag{2}$$

In Section 2 we generalize the concept of k-edge to arbitrary (that is, not necessarily rectilinear) drawings of K_n. This allows us to extend Equation (2) to (good) topological drawings of K_n. The key observation to extend the definition of k-edge to the new setting is to observe that, although half-planes are not well defined, we can use the orientation of the triangles defined by three points: the edge pq will be a k-edge of the topological drawing if the set of triangles adjacent to pq is divided, according to its orientation, into two subsets with cardinality k and $n-k-2$. In Section 3 we use this tool to show that $\nu_2(K_n) = Z(n)$. In order to do that, we need to introduce the new concept of $\leq\leq k$-edges, because for topological drawings the lower bound for $\leq k$-edges, $E_{\leq k}(D) \geq 3\binom{k+2}{2}$ does not hold. In Section 4 we analyze crossing optimal 2-page drawings of K_n. We give a complete characterization of their structure, showing that, up to equivalence (see Section 4.1 for a detailed definition), crossing optimal drawings are unique for n even. In contrast, for n odd we provide a family of size $2^{(n-5)/2}$ of non-equivalent crossing optimal drawings. We conclude with some open questions and directions for future research in Section 5.

An extended abstract of this paper [2] has appeared. In it we include some additional observations on the structure of crossing optimal 2-page drawings of K_n. For instance, in these drawings the above mentioned inequality $E_{\leq k}\,(D) \geq 3\binom{k+2}{2}$ does hold.

2 Crossings and k-edges

In this section we generalize the concept of k-edges, which has so far only been used in the geometric setting of finite sets of points in the plane, to topological drawings of K_n. Let D be a good drawing of K_n, let $\overrightarrow{pq}$ be a directed edge of D, and r a vertex of D other than p or q. We say that r *is on the left (respectively, right) side of* $\overrightarrow{pq}$ if the topological triangle pqr traced in that order (its vertices and edges correspond to those in D) is oriented counterclockwise (respectively, clockwise). Note that this is well defined as the three edges pq, qr, and rp in D do not self intersect and do not intersect each other, since D is good. We say that the edge pq is a *k-edge of* D if it has exactly k points of D on the same side (left or right), and thus $n-2-k$ points on the other side. Hence, as in the geometric setting, a k-edge is also an $(n-2-k)$-edge. Note that the direction of the edge pq is no longer relevant and every edge of D is a k-edge for some unique k such that $0 \leq k \leq \lfloor n/2 \rfloor - 1$. Let $E_k(D)$ be the number of k-edges of D.

Theorem 1. *For any good drawing D of K_n in the plane the following identity holds,*

$$\mathrm{cr}\,(D) = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k\,(n-2-k)\,E_k\,(D)\,.$$

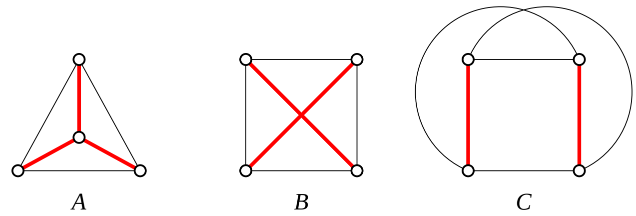

Figure 1: The three different good drawings of K_4, with 3, 2, and 2 separations. The edge of each separation is shown bold.

Proof. In a good drawing of K_n, we say that an edge pq *separates* the vertices r and s if the orientations of the triangles pqr and pqs are opposite. In this case, we say that the set $\{pq, r, s\}$ is a *separation*. It is straightforward to check that, up to ambient isotopy equivalence, there are only three different good drawings A, B, C of K_4; these are shown in Figure 1.

We denote by T_A, T_B, and T_C the number of induced subdrawings of D of type A, B, and C, respectively. Then

$$T_A + T_B + T_C = \binom{n}{4}, \tag{3}$$

and since the subdrawings of types B or C are in one-to-one correspondence with the crossings of D, it follows that

$$\mathrm{cr}\,(D) = T_B + T_C. \tag{4}$$

We count the number of separations in D in two different ways: First, each subdrawing of type A has 3 separations (the edge in each separation is bold in Figure 1), and each subdrawing of types B or C has 2 separations. This gives a total of $3T_A + 2T_B + 2T_C$ separations in D. Second, each k-edge belongs to exactly $k(n-2-k)$ separations. Summing over all k-edges for $0 \leq k \leq \lfloor n/2 \rfloor - 1$ gives a total of $\sum_{k=0}^{\lfloor n/2 \rfloor -1} k(n-2-k)E_k(D)$ separations in D. Therefore

$$3T_A + 2T_B + 2T_C = \sum_{k=0}^{\lfloor n/2 \rfloor -1} k\,(n-2-k)\,E_k(D). \tag{5}$$

Finally, subtracting Equation (5) from three times Equation (3) we get

$$T_B + T_C = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor -1} k\,(n-2-k)\,E_k(D),$$

and thus by Equation (4) we obtain the claimed result. □

For $0 \leq k \leq \lfloor n/2 \rfloor - 1$ and D a good drawing of K_n, we define the set of *$\leq k$-edges* of D as all j-edges in D for $j = 0, \ldots, k$. The number of $\leq k$-edges of D is denoted by

$$E_{\leq k}\,(D) := \sum_{j=0}^{k} E_j\,(D)\,.$$

Similarly, we denote the number of $\leq\leq k$-*edges* of D by

$$E_{\leq\leq k}(D) := \sum_{j=0}^{k} E_{\leq j}(D) = \sum_{j=0}^{k}\sum_{i=0}^{j} E_i(D) = \sum_{i=0}^{k}(k+1-i)\, E_i(D)\,.$$

To avoid special cases we define $E_{\leq\leq -1}(D) = E_{\leq\leq -2}(D) = 0$.

The following result restates Theorem 1 in terms of the number of $\leq\leq k$-edges.

Proposition 2. *Let D be a good drawing of K_n. Then*

$$\operatorname{cr}(D) = 2\sum_{k=0}^{\lfloor n/2\rfloor-2} E_{\leq\leq k}(D) - \frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor - \frac{1}{2}(1+(-1)^n)E_{\leq\leq\lfloor n/2\rfloor-2}(D).$$

Proof. First note that for $2 \leq k \leq \lfloor n/2\rfloor - 1$ we have that $E_{\leq\leq k}(D) - E_{\leq\leq k-1}(D) = E_{\leq k}(D)$ and $E_{\leq k}(D) - E_{\leq k-1}(D) = E_k(D)$. Thus

$$E_k(D) = E_{\leq\leq k}(D) - 2E_{\leq\leq k-1}(D) + E_{\leq\leq k-2}(D)\,.$$

We rewrite the last term in Theorem 1.

$$\begin{aligned}
&\sum_{k=0}^{\lfloor n/2\rfloor-1} k(n-2-k)E_k(D)\\
&= \sum_{k=2}^{\lfloor n/2\rfloor-1} k(n-2-k)[E_{\leq\leq k}(D) - 2E_{\leq\leq k-1}(D) + E_{\leq\leq k-2}(D)]\\
&= \sum_{k=0}^{\lfloor n/2\rfloor-3} (k(n-2-k) - 2(k+1)(n-3-k) + (k+2)(n-4-k))E_{\leq\leq k}(D)\\
&\quad + \left(\left\lfloor\frac{n}{2}\right\rfloor - 1\right)\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right)E_{\leq\leq\lfloor n/2\rfloor-1}(D) + (-2\left(\left\lfloor\frac{n}{2}\right\rfloor - 1\right)\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right)\\
&\quad + \left(\left\lfloor\frac{n}{2}\right\rfloor - 2\right)\left(n-\left\lfloor\frac{n}{2}\right\rfloor\right))E_{\leq\leq\lfloor n/2\rfloor-2}(D)\\
&= -2\sum_{k=0}^{\lfloor n/2\rfloor-3} E_{\leq\leq k}(D) + \left(\left\lfloor\frac{n}{2}\right\rfloor - 1\right)\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right)E_{\leq\leq\lfloor n/2\rfloor-1}(D)\\
&\quad + (-2\left(\left\lfloor\frac{n}{2}\right\rfloor - 1\right)\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right) + \left(\left\lfloor\frac{n}{2}\right\rfloor - 2\right)\left(n-\left\lfloor\frac{n}{2}\right\rfloor\right))E_{\leq\leq\lfloor n/2\rfloor-2}(D).
\end{aligned}$$

Since $E_{\leq\leq\lfloor n/2\rfloor-1}(D) = E_{\leq\leq\lfloor n/2\rfloor-2}(D) + E_{\leq\lfloor n/2\rfloor-1}(D) = E_{\leq\leq\lfloor n/2\rfloor-2}(D) + \binom{n}{2}$, it follows by

Theorem 1 that

$$\begin{aligned}
\operatorname{cr}(D) &= 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2\rfloor-1} k\,(n-2-k)\,E_k(D) = 3\binom{n}{4} + 2\sum_{k=0}^{\lfloor n/2\rfloor-3} E_{\leq\leq k}(D) \\
&\quad + \left(n+1-2\left\lfloor\frac{n}{2}\right\rfloor\right)E_{\leq\leq\lfloor n/2\rfloor-2}(D) - \left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right)\binom{n}{2} \\
&= 2\sum_{k=0}^{\lfloor n/2\rfloor-3} E_{\leq\leq k}(D) - \frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor + \begin{cases} E_{\leq\leq\lfloor n/2\rfloor-2}(D) & \text{if } n \text{ is even,} \\ 2E_{\leq\leq\lfloor n/2\rfloor-2}(D) & \text{if } n \text{ is odd,}\end{cases}
\end{aligned}$$

which is equivalent to the claimed result. □

3 The 2-page crossing number

We are concerned with 2-page book drawings of K_n. Obviously any line can be chosen as the spine, and for the rest of the paper we will assume that the spine is the x-axis. Moreover, by topological equivalence, we will assume that the vertices are precisely the points with coordinates $(1,0),(2,0),\ldots,(n,0)$.

Consider a 2-page book drawing D of K_n, and label the vertices $1,2,\ldots,n$ from left to right. Our interest lies in crossing optimal drawings, and it is readily seen that in every such drawing, none of the edges $(1,2),(2,3),\ldots,(n-1,n),(n,1)$ is crossed. Thus we may choose to place each of these edges in either the upper closed halfplane (page) or in the lower closed halfplane (page). Moreover, we may choose to place each of the edges $(1,2),(2,3),\ldots,(n-1,n)$ completely on the spine, and this is the convention we shall adopt for the rest of the paper. The edge $(n,1)$ may be placed indistinctly in the upper page or in the lower page, and for the rest of the paper we adopt the convention that it is place in the upper page. Moreover, because we are only concerned with good drawings, we assume without loss of generality that the rest of the edges are semicircles.

Color the edges above or on the spine blue and below the spine red, respectively. We construct an upper triangular matrix which corresponds to the coloring of these edges, see Figure 2. We call this the *2-page matrix* of D and denote it by $M(D)$. Label the columns of the 2-page matrix with $2,\ldots,n$ from left to right and the rows with $1,2,\ldots,n-1$ from top to bottom. For $i<j$ an entry (i,j) (row,column) in the 2-page matrix $M(D)$ is a point with the same color as the edge ij in the drawing D.

Remark. *It follows from the convention laid out above that for every 2-page book drawing D, the entries $(1,2),(2,3),\ldots,(n-1,n)$ and $(1,n)$ in $M(D)$ are all blue.*

We start by proving some basic properties of the 2-page matrix.

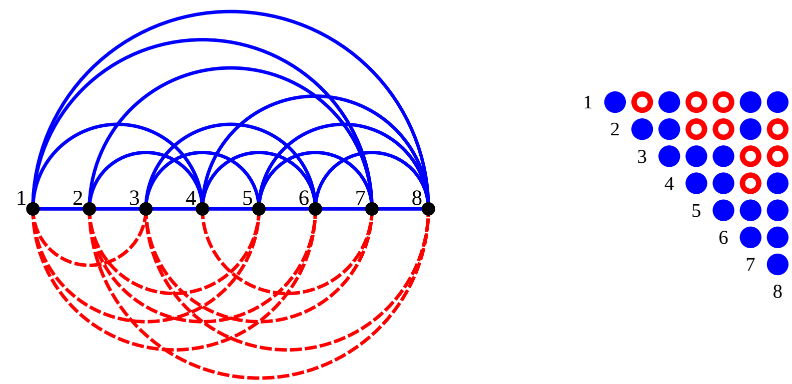

Figure 2: Two-colored diagram for a 2-page book drawing D of K_8 and the corresponding 2-page matrix $M(D)$. Solid dots and lines represent blue edges. Open dots and dashed lines represent red edges. From our convention to place the edges $(1,2),(2,3),\dots,(n-1,n)$ on the spine and the edge $(1,n)$ in the upper page, it follows that all the entries in the main diagonal, as well as the upper right corner entry, are blue.

Lemma 3. *Let D be a 2-page book drawing of K_n. For $1 \le i < j \le n$, let k be the sum of the number of points to the right plus the number of points above the entry (i,j) in the 2-page matrix of D, which have the same color as (i,j). Then the edge ij is a k-edge. (It is possible to have $k > \lfloor n/2 \rfloor - 1$.)*

Proof. Let $1 \le i < j \le n$ and assume that the edge ij is blue (red). We count the number of points l in D to the left (right) of ij. For $l \notin \{i,j\}$ the triangle ijl is oriented counterclockwise (clockwise) if and only if either $l < i$ and the edge lj is blue (red), or $l > j$ and the edge il is blue (red). In the first case these edges correspond to blue (red) points above the entry (i,j), and in the second case to blue (red) points to the right of the entry (i,j), respectively. □

In view of Lemma 3 we say that the point in the entry (i,j) of the 2-page matrix of D *represents* a k-edge if ij is a k-edge (or an $(n-2-k)$-edge) in D.

Lemma 4. *For $k < n/2 - 1$ and for $1 \le j \le k+1$, in the 2-page matrix of a drawing D of K_n there are at least $2\,(k+2-j)$ points in row j representing $\le k$-edges. Similarly, for $n-k \le j \le n$ there are at least $2\,(k+1-n+j)$ points in column j representing $\le k$-edges.*

Proof. For $1 \le j \le k+1$, in row j the rightmost $k+2-j$ points of each color represent $\le k$-edges as they have at most $k+1-j$ points of their color to the right and at most $j-1$ on top. So if each color appears at least $k+2-j$ times in row j, we have guaranteed $2\,(k+2-j)$ $\le k$-edges in row j. If one of the colors appears fewer than $k+2-j$ times, so that there are $k+2-j-e$ blue points in row j for some $1 \le e \le k+2-j$, then there are

$n-j-(k+2-j-e)=n-2-k+e$ red points in this row. In this case we claim that also the leftmost e red points in this row represent $\leq k$-edges. In fact, for $1\leq i\leq e$, the i-th red point (from the left) in row j, has exactly $n-2-k+e-i$ red points to the right and perhaps more red points on top. Since for $n\geq 2$ we have $n-2-k+e-i\geq n/2-k$, this i-th red point also represents a $\leq k$-edge. The equivalent result holds for the rightmost $k+1$ columns. □

Lemma 5. *For $0\leq j<n/2-1$, in the 2-page matrix of a drawing D of K_n there are two points in column n which correspond to j-edges in D. For n even there exists one such point in column n corresponding to an $(n/2-1)$-edge in D.*

Proof. We follow the lines of the proof of Lemma 4. Consider the points in column n in order from top to bottom. By Lemma 3 the i-th vertex of a color corresponds to an $(i-1)$-edge. Thus, if there are at least $j+1$ vertices for each color we are done. Otherwise assume without loss of generality that there are $j+1-e$ blue points in column n for some $1\leq e\leq j+1$. Then there are $n-1-(j+1-e)=n-j+e-2$ red points in this column. For $1\leq i\leq\lfloor n/2\rfloor$ the i-th red point corresponds to an $(i-1)$-edge, and for $\lfloor n/2\rfloor+1\leq i\leq n-j+e-2$ the i-th red point corresponds to an $(i-1)=(n-i-1)$-edge. Thus we get two red points corresponding to j-edges for $i=j+1$ and $i=n-j-1$. Finally, observe that these two points are different for $j<n/2-1$. For n even we get only one such point for $j=n/2-1$. □

The next theorem gives a lower bound on the number of $\leq\leq k$-edges, which will play a central role in deriving our main result. We need the following definitions. Let D be a good drawing of K_n. Let l be a vertex of K_n, and let D' be the (evidently, also good) drawing of K_{n-1} obtained by deleting from D the vertex l and its adjacent edges. Note that a k-edge ij in D' is a k-edge or a $(k+1)$-edge in D. Indeed, if ij has exactly k points to its right in D' (an equivalent argument holds if the k points are on its left), then there are k or $k+1$ points to the right of ij in D depending on whether l is to the left or to the right, respectively, of ij. We say that a k-edge in D is (D,D')-*invariant* if it is also a k-edge in D'. Whenever it is clear what D and D' are, we simply say that an edge is invariant. A (D,D')-*invariant* $\leq k$-*edge* is a (D,D')-invariant j-edge for some $0\leq j\leq k\leq n/2-1$. Denote by $E_{\leq k}(D,D')$ the number of (D,D')-invariant $\leq k$-edges.

Theorem 6. *Let $n\geq 3$. For every 2-page book drawing D of K_n and $0\leq k<n/2-1$, we have*

$$E_{\leq\leq k}(D)\geq 3\binom{k+3}{3}.$$

Proof. We proceed by induction on n. The induction base $n=3$ holds trivially. For $n\geq 4$, consider a 2-page book drawing D of K_n with horizontal spine and label the vertices from left to right with $1,2,\dots,n$. Remove the point n and all incident edges to obtain a 2-page book drawing D' of K_{n-1}. To bound $E_{\leq\leq k}(D)$, recall that

$$E_{\leq\leq k}(D)=\sum_{j=0}^{k}(k+1-j)\,E_j(D)\,. \tag{6}$$

All edges incident to n are in D but are not in D'. In fact, by Lemma 5, there are two j-edges adjacent to the vertex n for each $0 \leq j \leq k \leq \lfloor n/2 \rfloor - 2$. These edges contribute with $2\sum_{j=0}^{k}(k+1-j) = 2\binom{k+2}{2}$ to Equation (6). We next compare Equation (6) to

$$E_{\leq\leq k-1}(D') = \sum_{j=0}^{k-1}(k-j)\,E_j(D')\,. \tag{7}$$

Any edge contributing to Equation (7) also contributes to Equation (6), but possibly with a different value. As observed before, a j-edge in D' is a j-edge or a $(j+1)$-edge in D. A j-edge in D' contributes to Equation (7) with $k-j$. A j-edge and a $(j+1)$-edge in D contribute to Equation (6) with $k+1-j$ and $k-j$, respectively. This is a gain of $+1$ or 0, respectively, towards $E_{\leq\leq k}(D)$ when compared to $E_{\leq\leq k-1}(D')$. Finally, a k-edge in both D and D' does not contribute to Equation (7) and contributes to Equation (6) with $+1$. Therefore

$$E_{\leq\leq k}(D) = E_{\leq\leq k-1}(D') + 2\binom{k+2}{2} + E_{\leq k}(D,D').$$

By induction hypothesis, $E_{\leq\leq k-1}(D') \geq 3\binom{k+2}{3}$ and thus

$$E_{\leq\leq k}(D) \geq 3\binom{k+2}{3} + 2\binom{k+2}{2} + E_{\leq k}(D,D') = 3\binom{k+3}{3} - \binom{k+2}{2} + E_{\leq k}(D,D').$$

We finally prove that

$$E_{\leq k}(D,D') \geq \binom{k+2}{2}. \tag{8}$$

In fact, we prove that for each $1 \leq j \leq k+1$ there are at least $k+2-j$ points in row j of $M(D)$ that represent (D,D')-invariant $\leq k$-edges. Suppose that the edge jn is blue (the equivalent argument holds when jn is red). Then any red point in row j with $i \leq k$ red points above or to its right in $M(D)$ represents a (D,D')-invariant i-edge; and any blue point in row j with $i \geq n-2-k$ blue points above or to its right represents a (D,D')-invariant $(n-2-i)$-edge. Thus, the first $k+2-j$ red points from the right in row j (if they exist) represent (D,D')-invariant $\leq k$-edges as they have at most $k+2-j-1$ red points to the right and at most $j-1$ red points above in both $M(D)$ and $M(D')$. If there are fewer than $k+2-j$ red points in row j of $M(D)$, say $k+2-j-e$ for some $1 \leq e \leq k+2-j$, then the first e blue points in row j of $M(D)$ from the left represent $\leq k$-edges, because they have at least $n-j-e \geq n-j-k-2+j = n-k-2$ blue points to their right. Hence there are at least $k+2-j-e$ red points and at least e blue points (for a total of at least $k+2-j$ points) that represent (D,D')-invariant $\leq k$-edges in row j of $M(D)$. Summing over all $1 \leq j \leq k+1$, we get that

$$E_{\leq k}(D,D') \geq \sum_{j=1}^{k+1}(k+2-j) = \binom{k+2}{2}. \qquad \square$$

We are now ready to prove our main result, namely that the 2-page crossing number of K_n is $Z(n)$.

Theorem 7. *For every positive integer n, $\nu_2(K_n) = Z(n)$.*

Proof. The cases $n = 1$ and $n = 2$ are trivial. Let $n \geq 3$. As we mentioned above, 2-page book drawings with $Z(n)$ crossings were constructed by Blažek and Koman [5] (see also Guy et al. [14], Damiani et al. [9], Harborth [17], and Shahrokhi et al. [20].) These drawings show that $\nu_2(K_n) \leq Z(n)$. For the lower bound, let D be a 2-page book drawing of K_n. Using Proposition 2 and Theorem 6, we obtain

$$\begin{aligned}
\operatorname{cr}(D) &\geq 2\sum_{k=0}^{\lfloor n/2\rfloor-2} 3\binom{k+3}{3} - \frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor - \frac{3}{2}\left(1+(-1)^n\right)\binom{\lfloor\frac{n}{2}\rfloor+1}{3} \\
&= 6\binom{\lfloor\frac{n}{2}\rfloor+2}{4} - \frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor - \frac{3}{2}\left(1+(-1)^n\right)\binom{\lfloor\frac{n}{2}\rfloor+1}{3} \\
&= \begin{cases} \frac{1}{64}(n-1)^2(n-3)^2 & \text{if } n \text{ is odd,} \\ \frac{1}{64}n(n-2)^2(n-4) & \text{if } n \text{ is even,} \end{cases} = Z(n).
\end{aligned}$$

□

4 Crossing optimal configurations

In all this section D denotes a 2-page book drawing of K_n and $M(D)$ its 2-page matrix. We say that D is *crossing optimal* if $\nu_2(D) = Z(n)$. Theorem 19 in Subsection 4.3 describes the general structure of the crossing optimal 2-page book drawings of K_n. We use it to prove that, up to the equivalence described below, there is a unique crossing optimal 2-page book drawing of K_n when n is even and, in contrast, there exists an exponential number of non-equivalent crossing optimal 2-page book drawings of K_n when n is odd.

4.1 Equivalent drawings

Let D be a 2-page book drawing of K_n. Recall that we are assuming that the vertices of D are the points $\{(i,0) : 1 \leq i \leq n\}$. Consider the following transformation f that results in the 2-page book drawing $f(D)$ of K_n: move the vertex $(1,0)$ to the point $(n,0)$, and for every $2 \leq k \leq n$ move the vertex $(k,0)$ to the vertex $(k-1,0)$. That is, if an edge $1j$ was drawn above (below) the spine in D, then the edge $(j-1)(n)$ is drawn above (below) the spine in $f(D)$; for all other edges ij with $1 < i < j \leq n$, if ij was drawn above (below) the spine in D, then the edge $(i-1)(j-1)$ is drawn above (below) the spine in $f(D)$. Note that D and $f(D)$ have the same number of crossings, and $f^n(D) = D$. There are two other natural transformations of a drawing D: A vertical reflection $g(D)$ about the line with equation $x = n/2$ and a horizontal reflection $h(D)$ about the spine (or x-axis). In $g(D)$ an edge ij is drawn above (below) the spine if the edge $(n+1-j)(n+1-i)$ is drawn above (below) the spine in D. In $h(D)$ an edge ij is drawn above (below) the spine if the

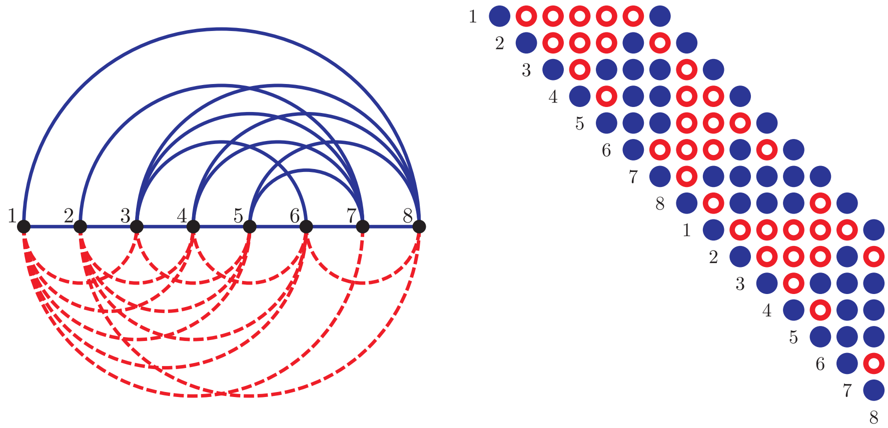

Figure 3: A 2-page drawing of K_8 and its strip diagram.

edge ij is drawn below (above) the spine in D. Note that $g^2(D) = h^2(D) = D$. Given a 2-page drawing D, all drawings obtained by compositions of these transformations from D are said to be *equivalent* to D. All drawings obtained this way are topologically isomorphic (homeomorphic) and thus they all have the same number of crossings as D. The group spanned by these transformations is isomorphic to the direct sum of the dihedral group D_{2n} and the group with 2 elements $\mathbb{Z}_2$. The set $\{f, g, h\}$ is a set of generators such that $g^2 = h^2 = f^n = 1$, $g \circ f = f^{-1} \circ g$, $h \circ f = f \circ h$, and $g \circ h = h \circ g$. Thus the $4n$ transformations in the group can be parametrized by $h^a \circ g^b \circ f^i$ with $i \in \{0, 1, \ldots, n-1\}$ and $a, b \in \{0, 1\}$.

Now we describe these transformations in the 2-page matrix diagram of D: To obtain $M(f(D))$ from $M(D)$, we simply rotate 90 degrees counterclockwise the first row of $M(D)$ and use it as the n^{th} column of $M(f(D))$. The diagram $M(g(D))$ is obtained from $M(D)$ by reflecting with respect to the diagonal $\{(i, n+1-i) : 1 \leq i \leq \lfloor n/2 \rfloor\}$. Finally, $M(h(D))$ is obtained by switching the color of every point except those that join consecutive vertices on the spine or the point $(1, n)$. We can place $M(D)$ and $M(f(D))$ together so that the part they have in common overlaps. Doing this for $M(f^m(D))$ for all integers m we obtain a periodic double infinite strip with period n and with a horizontal section that is $n-1$ units wide. We call this the *strip diagram* of D, or of $f^m(D)$ for any integer m. (See Figure 3.) Any right triangular region with the same dimensions as $M(D)$ obtained from the strip diagram of D by a horizontal and a vertical cut is the matrix diagram of a drawing equivalent to D and thus it has the same number of crossings as D.

4.2 Properties of crossing optimal drawings

We start with a couple of definitions. Consider the entry (i, j) of $M(D)$. We order the entries in row i to the left of (i, j) as follows: first all entries, from right to left, whose color differs to that of (i, j), followed by all other entries (those with the same color as (i, j)) from left to right. This is called the *order associated to* (i, j). Observe that this is the order in which the edges il $(i < l < j)$ appear in the 2-page drawing, ordered bottom to top if the edge ij is blue and top to bottom if the edge ij is red. Let c be an integer such that $0 \leq c \leq n-1$. Denote by D_c the subgraph of D obtained by deleting the c right-most points of D, or equivalently, $M(D_c)$ is obtained by deleting the last c columns of $M(D)$. The following results strongly rely on the proof of Theorem 6.

Lemma 8. *Suppose that $l \geq i+m+1$ for some integers $1 \leq i < l < j \leq n$ and $1 \leq m < j-i$. The entry (i, l) is one of the first m entries in the order associated to (i, j) if and only if (i, l) and (i, j) have different colors.*

Proof. Note that if (i, l) and (i, j) have the same color, then all entries to the left of (i, l) come before (i, l) in the order associated to (i, j). □

Lemma 9. *Let p be an integer such that $0 \leq p \leq \lfloor n/2 \rfloor - 2$. Suppose that $E_{\leq k}(D, D_1) = \binom{k+2}{2}$ for all $0 \leq k \leq p$. Then $M(D)$ satisfies that for $1 \leq i \leq p+1$ in row i there is exactly one (D, D_1)-invariant k-edge for each $i-1 \leq k \leq p$, and there are no (D, D_1)-invariant $(\leq i-2)$-edges. In all other rows there are no (D, D_1)-invariant $\leq p$-edges.*

Proof. In what follows all invariant edges are (D, D_1)-invariant edges. For $k = 0$ the statement implies that there is a unique invariant 0-edge and it appears in row 1. Note that this edge corresponds to the first entry in the order associated to $(1, n)$ in $M(D)$. Following the proof of Theorem 6, $E_{\leq k}(D, D_1) = \binom{k+2}{2}$ implies that for all $1 \leq i \leq k+1$ there are exactly $k+2-i$ invariant $\leq k$-edges in row i of $M(D)$, and for $k+2 \leq i \leq n-1$ there are no invariant $\leq k$-edges in row i of $M(D)$. The second part implies that there are no invariant k-edges in row i for all $k+2 \leq i \leq n-1$ and $0 \leq k \leq p$. Similarly, $E_{\leq k-1}(D, D_1) = \binom{k+1}{2}$ implies that for all $1 \leq i \leq k$ there are exactly $k+1-i$ invariant $(\leq k-1)$-edges in row i of $M(D)$. Therefore for all $1 \leq i \leq k$ there is exactly $(k+2-i)-(k+1-i) = 1$ invariant k-edge, and for $i = k+1$ there is exactly $k+2-(k+1) = 1$ invariant $\leq k$-edge and no invariant $(\leq k-1)$-edge in row i of $M(D)$. Therefore, there is exactly one invariant k-edge in row $k+1$. □

Lemma 10. *Let p be an integer such that $0 \leq p \leq \lfloor n/2 \rfloor - 2$.*

i) Suppose that for some $1 \leq i \leq p+1$ row i of $M(D)$ has exactly one (D, D_1)-invariant k-edge for each $i-1 \leq k \leq p$ and no (D, D_1)-invariant $\leq (i-2)$-edges. If the entry (i, n) in $M(D)$ is blue (red), then the m^{th} entry in row i in the order associated to (i, n) has at least $\min\{p+2-m, i-1\}$ red (blue) entries above for every $1 \leq m \leq \min\{p+1, n-i-1\}$.

ii) Suppose that for some $i \geq p+2$ row i of $M(D)$ does not have (D, D_1)-invariant $\leq p$-edges. If the entry (i, n) in $M(D)$ is blue (red), then the m^{th} entry in row i in the order associated to (i, n) has at least $p+2-m$ red (blue) entries above for every $1 \leq m \leq \min\{p+1, n-i-1\}$.

Proof. In what follows invariant edges refer to (D, D_1)-invariant edges. Denote by (i, e_m) the m^{th} entry in the order associated to (i, n). Note that if (i, e_m) and (i, n) have opposite colors and the number of points above plus the number of points to the right of (i, e_m) with the same color as (i, e_m) is at most p, then (i, e_m) is an invariant $\leq p$-edge. Similarly, if (i, e_m) and (i, n) have the same color and the number of points above plus the number of points to the right of (i, e_m) with the same color as (i, e_m) is more than $n - 2 - p$, then (i, e_m) is an invariant $\leq p$-edge.

Suppose that the entry (i, n) of $M(D)$ is blue (red).

(i) If (i, e_1) is red (blue), then it does not have red entries to its right and it has at most $i - 1$ red (blue) entries above. Since $i - 1 \leq p$, then (i, e_1) is an invariant $(\leq i - 1)$-edge. Because there are no invariant $(\leq i - 2)$-edges in row i, it follows that (i, e_1) is the unique invariant $(i - 1)$-edge in row i and thus all $i - 1$ entries above it are red (blue). Similarly, if the (i, e_1) is blue (red), then all entries in row i are blue (red) and $(i, e_1) = (i, i + 1)$. Hence (i, e_1) has $n - i - 1$ blue (red) entries to its right and perhaps some other blue (red) entries above. Since $n-i-1 \geq n-(p+1)-1 \geq n-2-p$, then (i, e_1) is an invariant $(\leq i-1)$-edge. Because there are no invariant $(\leq i - 2)$-edges in row i, it follows that (i, e_1) is the unique invariant $(i - 1)$-edge in row i and thus all $i - 1$ entries above it are red (blue).

For $2 \leq m \leq p + 2 - i$ assume that the entry $(i, e_{m'})$ is an invariant $(i - 2 + m')$-edge for every $1 \leq m' \leq m - 1$. Note that $i - 1 \leq i - 2 + m' \leq p - 1$.

If (i, e_m) is red (blue), then $(i, e_{m'})$ is red (blue) for every $1 \leq m' \leq m - 1$. So (i, e_m) has exactly $m - 1$ red (blue) entries to its right and at most $i - 1$ red (blue) entries above, that is, (i, e_m) is an invariant $\leq (i-2+m)$-edge. By hypothesis there is a unique invariant k-edge for every $i - 1 \leq k \leq p$ and among the first $m - 1$ entries there is exactly one invariant k-edge for each $i - 1 \leq k \leq i - 2 + (m - 1) = i - 3 + m$. So (i, e_m) is the unique invariant $(i - 2 + m)$-edge (note that $1 \leq i - 2 + m \leq p$) and thus all the entries above it are red (blue).

If (i, e_m) is blue (red), then there are exactly $n - i + m$ blue (red) entries to its right and perhaps some others above it. Since $n - i + m \geq n - i - (p + 2 - i) = n - p + 2$, then (i, e_m) is an invariant $\leq (i - 2 + m)$-edge. As before (i, e_m) must be an invariant $(i - 2 + m)$-edge and thus it must have only red (blue) entries above.

We have already determined the unique invariant k-edge for each $1 \leq k \leq p$. So there are no more invariant $\leq p$-edges in row i. For $p + 3 - i \leq m \leq \min\{p + 1, n - i - 1\}$, we prove that the entry (i, e_m) has at least $p+2-m = \min\{p+2-m, i-1\}$ red (blue) entries above.

If (i, e_m) is red (blue), then it has $m - 1$ red (blue) entries to its right. If (i, e_m) had less than $p + 2 - m$ (note that $p + 2 - m \leq i - 1$) red (blue) entries above, then it would be an invariant $\leq p$-edge (because $(m - 1) + (p + 1 - m) = p$) getting a contradiction.

If (i, e_m) is blue (red), then it has $n - i - m$ blue (red) entries to its right. If (i, e_m) had less than $p + 2 - m$ red (blue) entries above, then it would have a total of at least $n - i - m + (i - 1) - (p + 1 - m) = n - 2 - p$ blue (red) entries above or to its right, and thus it would be an invariant $\leq p$-edge getting a contradiction.

(ii) The proof is the same as for the case $p + 3 - i \leq m \leq \min\{p + 1, n - i - 1\}$ in (i) as we only used that the m^{th} entry in that range was not an invariant $\leq p$-edge. □

Lemma 11. *If D is crossing optimal, then for $0 \leq j \leq \lfloor n/2 \rfloor - 2$ we have*

$$E_{\leq \leq k}(D_j) = 3\binom{k+3}{3} \text{ for all } 0 \leq k \leq \lfloor n/2 \rfloor - 2 - j.$$

Proof. Since D is crossing optimal, then equality must be achieved in the proof of Theorem 7, that is, $E_{\leq\leq k}(D) = 3\binom{k+3}{3}$ for all $0 \leq k \leq \lfloor n/2 \rfloor - 2$. This implies that equality must be achieved throughout the proof of Theorem 6, in particular, $E_{\leq\leq k}(D_1) = 3\binom{k+2}{3}$ for all $0 \leq k \leq \lfloor n/2 \rfloor - 2$, which is equivalent to $E_{\leq\leq k}(D_1) = 3\binom{k+3}{3}$ for all $0 \leq k \leq \lfloor n/2 \rfloor - 3$.

In general, for $0 \leq j \leq \lfloor n/2 \rfloor - 2$, following the proof of Theorem 6, $E_{\leq\leq k}(D_j) = 3\binom{k+3}{3}$ for $1 \leq k \leq \lfloor n/2 \rfloor - 2 - j$ implies that $E_{\leq\leq k-1}(D_{j+1}) = 3\binom{k+2}{3}$ for $1 \leq k \leq \lfloor n/2 \rfloor - 2 - j$, which is equivalent to $E_{\leq\leq k}(D_{j+1}) = 3\binom{k+3}{3}$ for $1 \leq k \leq \lfloor n/2 \rfloor - 2 - j - 1$. □

Lemma 12. *If D is crossing optimal, then in $M(D)$ the m^{th} entry in the order associated to (i, j) has at least $\min\{j - \lceil n/2 \rceil - m, i - 1\}$ entries above with different color than (i, j) for all $1 \leq m \leq \min\{j - \lfloor n/2 \rfloor - 1, j - i - 1\}$.*

Proof. Consider the entry (i, j) of $M(D)$. Because D is crossing optimal, it follows from Lemma 11 that

$$E_{\leq\leq k}(D_{n-j}) = 3\binom{k+3}{3} \text{ for all } 0 \leq k \leq \lfloor n/2 \rfloor - 2 - (n - j) = j - 2 - \lceil n/2 \rceil.$$

Consider row i of D_{n-j}. (Note that D_{n-j} has $j - 1$ rows.) If $1 \leq i \leq j - 1 - \lceil n/2 \rceil$, then by Lemma 9 for $p = j - 2 - \lceil n/2 \rceil$, the 2-page matrix $M(D_{n-j})$ satisfies that in row i there is exactly one (D_{n-j}, D_{n-j+1})-invariant k-edge for each $i - 1 \leq k \leq j - 2 - \lceil n/2 \rceil$ and there are no (D_{n-j}, D_{n-j+1})-invariant $(\leq j - 2 - \lceil n/2 \rceil)$-edges. Then by Lemma 10(i) if the entry (i, j) in $M(D)$ (actually in $M(D_{n-j})$ but we look at it as a submatrix of $M(D)$) is blue (red), then the m^{th} entry in the order associated to (i, j) has at least $\min\{j - \lceil n/2 \rceil - m, i - 1\}$ red (blue) entries above.

If $j - \lceil n/2 \rceil \leq i \leq j - 1$, then by Lemma 9 for $p = j - 2 - \lceil n/2 \rceil$, the 2-page matrix $M(D_{n-j})$ satisfies that in row i there are no (D_{n-j}, D_{n-j+1})-invariant $(\leq j - 2 - \lceil n/2 \rceil)$-edges. Then by Lemma 10(ii) if the entry (i, j) in $M(D)$ is blue (red), then the m^{th} entry in the order associated to (i, j) has at least $j - \lceil n/2 \rceil - m = \min\{j - \lceil n/2 \rceil - m, i - 1\}$ red (blue) entries above. □

Corollary 13. *If D is crossing optimal, then for $2 \leq i \leq \lceil n/2 \rceil$ and $\lceil n/2 \rceil + 2 \leq j \leq n$, each of the first $j - \lceil n/2 \rceil - 1$ entries in the order associated to (i,j) has at least one entry above with different color than (i,j).*

Proof. Let $1 \leq m \leq j - \lceil n/2 \rceil - 1$. Since $\lfloor n/2 \rfloor$ and i are at most $\lceil n/2 \rceil$, then $m \leq \min\{j - \lfloor n/2 \rfloor - 1, j - i - 1\}$. Also $m \leq j - \lceil n/2 \rceil - 1$ and $i \geq 2$ imply that $\max\{j - \lceil n/2 \rceil - m, i-1\} \geq 1$. Thus by Lemma 12, the m^{th} entry in row i in the order associated to (i,j) has at least one entry above with different color than (i,j). □

Corollary 14. *If D is crossing optimal, then for $n \geq 3$, $2 \leq i \leq \lfloor n/2 \rfloor - 1$, and $\lceil n/2 \rceil + i \leq j \leq n$, all entries above the first $j - i + 1 - \lceil n/2 \rceil$ entries in the order associated to (i,j) have different color than (i,j).*

Proof. Let $1 \leq m \leq j - i + 1 - \lceil n/2 \rceil$. Since $i \geq 2$ and $n \geq 3$, then $m \leq \min\{j - \lfloor n/2 \rfloor - 1, j - i - 1\}$. Also $m \leq j - i + 1 - \lceil n/2 \rceil$ implies that $\max\{j - \lceil n/2 \rceil - m, i - 1\} \geq i - 1$. Thus by Lemma 12, the m^{th} entry in row i in the order associated to (i,j) has at least $i-1$ entries above, (i.e., all entries above it) with different color than (i,j) in $M(D)$. □

Lemma 15. *Suppose that D is crossing optimal and $0 \leq k \leq \lfloor n/2 \rfloor - 2$. Then all $\leq k$-edges of D belong to the union of the first $k+1$ rows and the last $k+1$ columns of $M(D)$.*

Proof. Suppose by contradiction that the entry (i,j) of $M(D)$ represents a k-edge and is not in the first $k+1$ rows ($i \geq k+2$) or in the last $k+1$ columns ($j \leq n-k-1$). Since D is crossing optimal, equality must be achieved in Inequality (8) and thus we have that all (D, D_1)-invariant $\leq k$-edges belong to the first $k+1$ columns. So (i,j) is not (D, D_1)-invariant, that is, ij is a $(k-1)$-edge in D_1. Equality in Theorem 6 implies that $E_{\leq \leq k-1}(D_1) = 3\binom{k+2}{2}$ and as before all (D_1, D_2)-invariant $(\leq k-1)$-edges belong to the last k columns of $M(D_1)$, that is, columns $n-k, n-k+1, \ldots, n-1$ of $M(D)$. So ij is not a (D_1, D_2)-invariant edge, that is, ij represents a $(k-2)$-edge in D_2. In general, assuming that ij is a $(k-l)$-edge in D_l and since $E_{\leq \leq k-l}(D_l) = 3\binom{k-l+3}{2}$, then all (D_l, D_{l+1})-invariant $(\leq k-l)$-edges belong to the last $k+1-l$ columns of $M(D_l)$ (i.e., columns $n-k, n-k+1, \ldots, n-l$ of $M(D)$). So (i,j) is not a (D_l, D_{l+1})-invariant edge, that is, (i,j) represents a $(k-l-1)$-edge in D_{l+1}. When $l = k-1$, (i,j) is a 0-edge in $M(D_k)$ that is not in the last column of $M(D_k)$ (column $n-k$ of $M(D)$). Since there are at least three 0-edges in the first column and row of $M(D_k)$ and $i \geq 2$, then $E_{\leq \leq 0}(D_k) \geq 4$, but $E_{\leq \leq 0}(D_k)$ must be 3, getting a contradiction. □

We extend the standard terminology from the geometrical setting, and call a $(\lfloor n/2 \rfloor - 1)$-edge a *halving edge*.

Lemma 16. *If D is crossing optimal, then the entries $(\lfloor n/2 \rfloor, \lceil n/2 \rceil + 1), (\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1)$, and $(\lceil n/2 \rceil, \lceil n/2 \rceil + 1)$ of $M(D)$ are halving edges.*

Proof. This follows from Lemma 15 as all $\leq (\lfloor n/2 \rfloor - 2)$-edges of D belong to the union of the first $\lfloor n/2 \rfloor - 1$ rows (top to bottom) and the last $\lfloor n/2 \rfloor - 1$ columns (left to right) of D. The entries $(\lfloor n/2 \rfloor, \lceil n/2 \rceil + 1), (\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1)$, and $(\lceil n/2 \rceil, \lceil n/2 \rceil + 1)$ are not in the first $\lfloor n/2 \rfloor - 1$ rows or in the last $\lfloor n/2 \rfloor - 1$ columns. □

Lemma 16 guarantees that the entry $(i, i+1)$ in general, and the entry $(i, i+2)$ when n is odd, are halving lines in some drawing equivalent to D. The next result states what this means in D. We state it only for $1 \leq i \leq \lfloor n/2 \rfloor$ (but it can be stated for $\lceil n/2 \rceil \leq i \leq n$ as well) as it is the only case we explicitly use later in the paper.

Lemma 17. *Let $1 \leq i \leq \lfloor n/2 \rfloor$. If D is crossing optimal, then $M(D)$ satisfies that the number of blue entries in*

$$\begin{aligned} &\{(r, i+1) : 1 \leq r \leq i-1\} \cup \{(i, c) : i+2 \leq c \leq i + \lceil n/2 \rceil\} \\ \cup &\{(i+1, c) : i + \lceil n/2 \rceil + 1 \leq c \leq n\} \end{aligned} \tag{9}$$

is either $\lfloor n/2 \rfloor - 1$ or $\lceil n/2 \rceil - 1$. If n is odd, then the number of entries in

$$\begin{aligned} &\{(r, i+2) : 1 \leq r \leq i-1\} \cup \{(i, c) : i+3 \leq c \leq i + \lceil n/2 \rceil\} \\ \cup &\{(i+2, c) : i + \lceil n/2 \rceil + 1 \leq c \leq n\} \end{aligned} \tag{10}$$

with the same color as the entry $(i, i+2)$ is either $\lfloor n/2 \rfloor - 1$ or $\lfloor n/2 \rfloor$.

Proof. In the strip diagram of D, the entry $(i, i+1)$ of $M(D)$ corresponds to the entry $(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1)$ of $M(f^{i - \lfloor n/2 \rfloor}(D))$, see Figure 4 (left). Applying Lemma 16 to $M(f^{i - \lfloor n/2 \rfloor}(D))$ and noticing that the entries of $M(D)$ in (9) correspond to the entries above plus the entries below the entry $(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1)$ of $M(f^{i - \lfloor n/2 \rfloor}(D))$ gives the result. The proof of the second part is similar, see Figure 4 (right). □

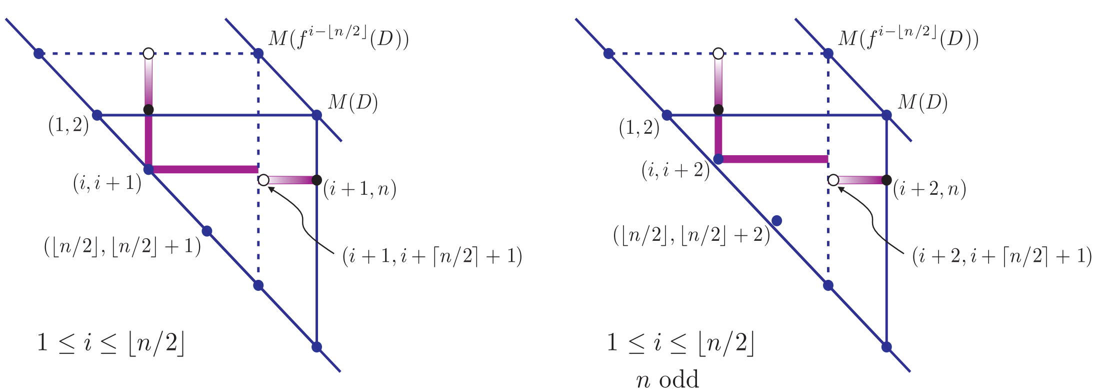

Figure 4: A halving line in a drawing equivalent to D seen in the matrix $M(D)$.

Lemma 18. *If D is crossing optimal, then there exists a drawing D' equivalent to D such that in $M(D')$ the $\lceil n/2 \rceil$ entries $(1, n), (2, n), \ldots,$ and $(\lceil n/2 \rceil, n)$ are blue and the $\lfloor n/2 \rfloor - 1$ entries $(1, \lceil n/2 \rceil + 1)$, $(1, \lceil n/2 \rceil + 2), \ldots$, $(1, n-1)$ are red.*

Proof. For each integer m, let e_m be the largest integer such that the last e_m entries in row $\lfloor n/2 \rfloor$ of $M(f^m(D))$ have the same color. (These entries are $(\lfloor n/2 \rfloor, n - e_m + 1)$, ..., $(\lfloor n/2 \rfloor, n)$.) Similarly, let e'_m be the largest integer such that the first e'_m entries in column $\lceil n/2 \rceil + 1$ of $M(f^m(D))$ have the same color. (These entries are $(1, \lceil n/2 \rceil + 1)$, ..., $(e'_m, \lceil n/2 \rceil + 1)$.) Let $E = \max\{e_m, e'_m : m \in \mathbb{Z}\}$. We claim that $E = \lceil n/2 \rceil$. Indeed, suppose that $E \leq \lceil n/2 \rceil - 1$ and without loss of generality assume that $E = e_{m_0}$ for some integer m_0. (If $E = e'_{m_0}$, start with $g(D)$ instead of D.) Then entry $(\lfloor n/2 \rfloor, n - e_{m_0})$ has a different color than the entries to its right, namely, $(\lfloor n/2 \rfloor, n - e_{m_0} + 1), \ldots, (\lfloor n/2 \rfloor, n)$. By Lemma 12 (for $i = \lfloor n/2 \rfloor$ and $j = n$) the entry $(\lfloor n/2 \rfloor, n - e_{m_0})$ has at least $\min\{n - \lceil n/2 \rceil - 1, \lfloor n/2 \rfloor - 1\} = \lfloor n/2 \rfloor - 1$ entries above with the same color as $(\lfloor n/2 \rfloor, n - e_{m_0})$. But this means that $e'_{m_0 - 1 + \lfloor n/2 \rfloor - e_{m_0}} \geq e_{m_0} + 1 = E + 1$, a contradiction.

Because $E = e_{m_0} = \lceil n/2 \rceil$, all entries in row $\lfloor n/2 \rfloor$ of $M(f^{m_0}(D))$ are blue. By Lemma 16 all entries in column $\lfloor n/2 \rfloor + 1$ of $M(f^{m_0}(D))$ above the entry $(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1)$ are red. This implies that $D' = f^{m_0 + \lfloor n/2 \rfloor}(D)$ satisfies the statement. ☐

4.3 The structure of crossing optimal drawings

We are finally ready to investigate the structure of crossing optimal drawings. The next result is the workhorse behind Theorems 20 and 23, the main results in this section. To help comprehension, we refer the reader to Figure 5.

Theorem 19. *Let $n \geq 6$, $e = 0$ for n even and $e = 1$ for n odd, and let D be a crossing optimal 2-page book drawing of K_n. Then there exists a drawing D' equivalent to D such that $M(D')$ satisfies:*

1. *for $4 + e \leq s \leq \lfloor n/2 \rfloor + 1$ and $n + 2 + e \leq s \leq n + \lfloor n/2 \rfloor + 1$ the entry $(r, s - r)$ is blue for all $\max\{1, s - n\} \leq r \leq (s - 5)/2$;*
2. *for $\lceil n/2 \rceil + 2 + e \leq s \leq n$ and $n + \lceil n/2 \rceil + 2 + e \leq s \leq 2n - 2 - e$ the entry $(r, s - r)$ is red for all $\max\{1, s - n\} \leq r \leq (s - 5)/2$ (except for $(1, n)$, which by convention is blue);*
3. *for n odd, the entries $(1, \lceil n/2 \rceil + 1)$ and $(\lfloor n/2 \rfloor, \lceil n/2 \rceil + 1)$ are red, and the entries $(2, n)$ and $(\lceil n/2 \rceil, \lceil n/2 \rceil + 2)$ are blue.*

Proof. Let

$$\begin{aligned} T_U(D) &= \{(r, c) \in M(D) : 2 \leq c \leq \lceil n/2 \rceil, 1 \leq r \leq c - 1\}, \\ R(D) &= \{(r, c) \in M(D) : \lceil n/2 \rceil + 1 \leq c \leq n, 1 \leq r \leq \lceil n/2 \rceil\}, \text{ and} \\ T_L(D) &= \{(r, c) \in M(D) : \lceil n/2 \rceil + 1 \leq c \leq n, \lceil n/2 \rceil + 1 \leq r \leq c - 1\}. \end{aligned}$$

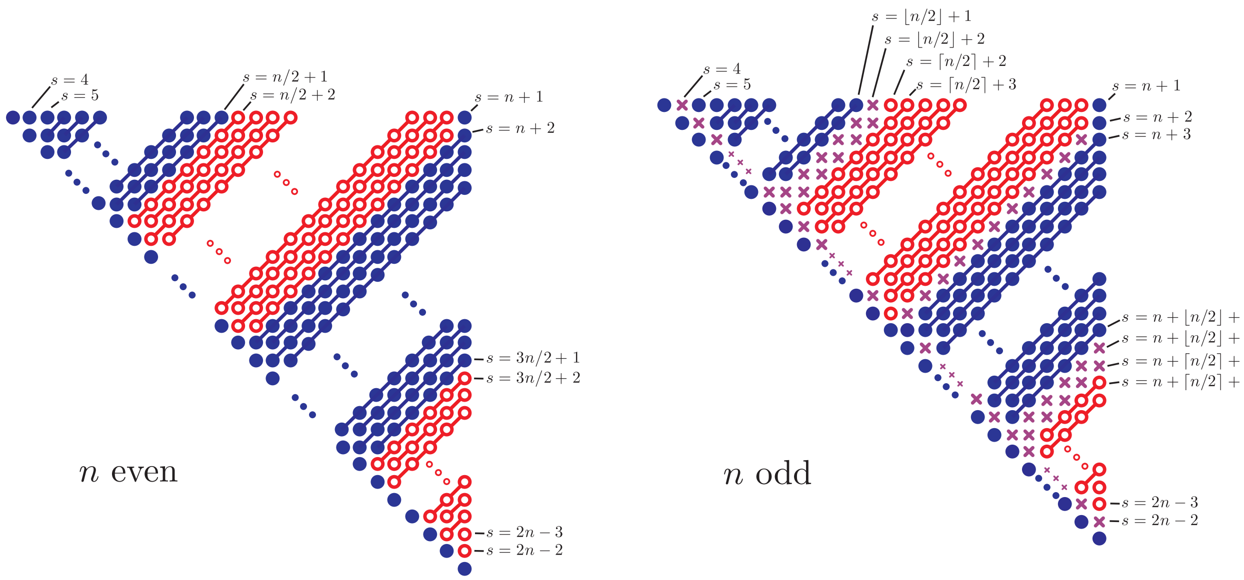

Figure 5: The even and odd cases in Theorem 19. The crosses in the odd case represent points whose color is not fixed.

We shall prove the theorem first for those entries that lie on $R(D)$, then for those that lie on $T_U(D)$, and finally for those that lie on $T_L(D)$.

The entries in $R(D)$

We refer the reader to Figure 6. By Lemma 18, we can assume that in $M(D)$

$$\text{the entries } (1,n),(2,n),\ldots,(\lfloor n/2\rfloor,n) \text{ are blue} \tag{11}$$

(in fact $(\lceil n/2\rceil,n)$ can also be assumed to be blue but we do not use this fact) and

$$\text{the entries } (1,\lceil n/2\rceil+1),\ldots,(1,n-1) \text{ are red.} \tag{12}$$

Moreover, we can assume that

$$\text{the entry } (2,n-1) \text{ is red.} \tag{13}$$

(If it is blue, then $M(h\circ g(D))$ satisfies (11), (12), and (13)).

We now prove that for each r such that $2\le r\le\lfloor n/2\rfloor$,

$$\text{the entries } (r,\lceil n/2\rceil+1),(r,\lceil n/2\rceil+2),\ldots,(r,2\lfloor n/2\rfloor-r+1) \text{ are red} \tag{14}$$

and

$$\text{the entries } \left(r,2\left\lceil n/2\right\rceil-r+2\right),\left(r,2\left\lceil n/2\right\rceil-r+3\right),\ldots,(r,n) \text{ are blue.} \tag{15}$$

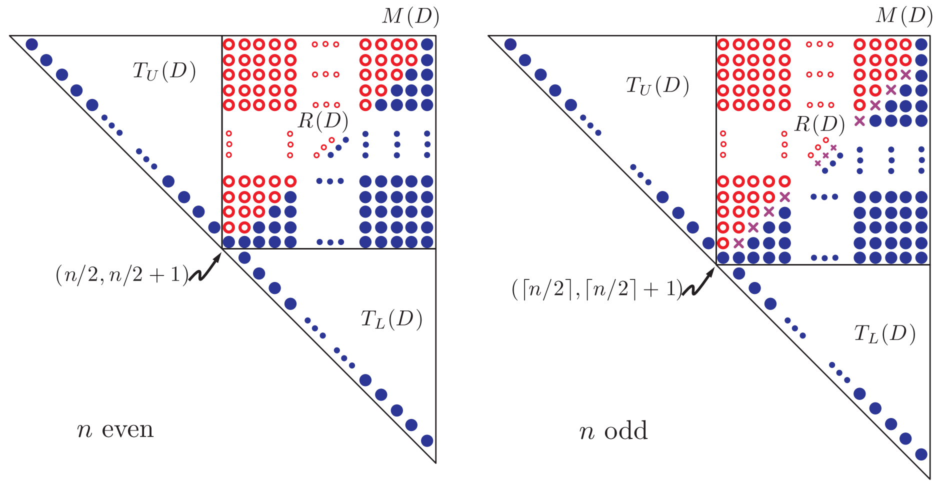

Figure 6: The regions $T_U(D), R(D)$, and $T_L(D)$.

Observe that if $r = 2$ and n is even, then (15) only concerns the entry $(2, n)$, which is blue by (11). (For $r = 2$ and n odd, (15) is an empty claim.) Thus we only need to take care of the base case $r = 2$ for (14). Since (by (13)) the entry $(2, n-1)$ is red, by Corollary 13 the first $\lfloor n/2\rfloor - 2$ entries in the order associated to $(2, n-1)$ have a blue point above. By (12) the only candidates to have blue points above them are the $\lceil n/2\rceil - 2$ entries $(2, 3), (2, 4), \ldots, (2, \lceil n/2\rceil)$. (Note that the order associated to the entry (i, j) only applies to entries in row j to the left of entry (i, j).) Thus the $\lceil n/2\rceil - 2$ entries $(1, 3), (1, 4), \ldots, (1, \lceil n/2\rceil)$ are blue if n is even, and at most one of them, say $(1, c_1)$, is red if n is odd. Moreover, by Lemma 8 the entries $(2, \lceil n/2\rceil + 1), (2, \lceil n/2\rceil + 2), \ldots, (2, n-2)$ are red.

For the inductive step, suppose that for some $3 \leq t \leq \lfloor n/2\rfloor$, each row r with $2 \leq r \leq t-1$ satisfies the result. We now prove (14) and (15) for $r = t$. Suppose that the entry $(t, 2\lceil n/2\rceil - t + 2)$ is red. Then by Corollary 13 each of the first $\lceil n/2\rceil - t + 1$ entries in the order associated to $(t, 2\lceil n/2\rceil - t + 2)$ has at least one blue entry above. Since the entries $(t, \lceil n/2\rceil + 1), \ldots, (t, 2\lfloor n/2\rfloor - t + 2)$ have all red above, the only candidates are the $\lceil n/2\rceil - t$ entries $(t, t+1), (t, t+2), \ldots, (t, \lceil n/2\rceil)$ and the entry $2\lfloor n/2\rfloor - t + 3 = 2\lceil n/2\rceil - t + 1$ for odd n. But, by Lemma 8, to be a candidate this last entry should be blue, which is impossible because it would be the first entry in the order associated to $(t, 2\lceil n/2\rceil - t + 2)$ with at most one blue entry above, contradicting Lemma 12. Since there are not enough candidates, then the entry $(t, 2\lceil n/2\rceil - t + 2)$ is blue.

Now consider the blue entry (t, n). By Corollary 14 the first $\lfloor n/2\rfloor - t + 1$ entries in the order associated to (t, n) have all entries above them red. The only candidates are (t, c_1) if it exists, $(t, \lceil n/2\rceil + 1), \ldots, (t, 2\lceil n/2\rceil - t + 1)$. For n even, there are $\lceil n/2\rceil - t + 1 = \lfloor n/2\rfloor - t + 1$ candidates because (t, c_1) does not exists, and thus all of them are red by Lemma 8. For n odd, there are at most 2 more candidates than we need. By Lemma 8 any blue entry (t, c) with $c \geq \lfloor n/2\rfloor + 2$ is not a candidate. Thus at most two of the last $\lceil n/2\rceil - t + 1$ candidates are blue. Suppose that one of the entries $(t, \lceil n/2 \rceil + 1), (t, \lceil n/2 \rceil + 2), \ldots, (t, 2\lfloor n/2 \rfloor - t + 1)$ is blue. Then there exists $\lceil n/2 \rceil + 1 \leq c \leq 2\lceil n/2 \rceil - t$ such that (t, c) is blue and $(t, c+1)$ is red. Then (t, c) is the first entry in the order associated to $(t, c+1)$ and all entries above it are red, contradicting Corollary 13. Thus (14) holds and, by Lemma 8 for $(i, j) = (t, n)$, the rest of (15) holds too.

Note that (14) is vacuous if $r = \lfloor n/2 \rfloor$ and n is odd. On the other hand, we argue that it is possible to assume that

$$\text{for odd } n, \text{ the entry } (\lfloor n/2 \rfloor, \lceil n/2 \rceil + 1) \text{ is red.} \tag{16}$$

Indeed, suppose that it is blue. Then, by Lemma 16, $(\lfloor n/2 \rfloor, \lceil n/2 \rceil + 1)$ is a blue halving entry with $\lfloor n/2 \rfloor - 1$ red entries above and thus all $\lfloor n/2 \rfloor - 1$ entries to its right are blue. Hence, by Lemma 16, $(\lfloor n/2 \rfloor, \lceil n/2 \rceil)$ is halving with $\lfloor n/2 \rfloor$ blue entries to its right and thus all $\lfloor n/2 \rfloor - 1$ entries above are red. Note that $M(f^{\lfloor n/2 \rfloor}(D))$ satisfies (11), (12), and (13) and its entry $(\lfloor n/2 \rfloor, \lceil n/2 \rceil + 1)$ is red. Then we start with $f^{\lfloor n/2 \rfloor}(D)$ instead of D.

We now prove that the version of (15) for $r = \lceil n/2 \rceil$ also holds:

$$\text{the entries } (\lceil n/2 \rceil, \lceil n/2 \rceil + 2), (\lceil n/2 \rceil, \lceil n/2 \rceil + 3), \ldots, (\lceil n/2 \rceil, n) \text{ are blue.} \tag{17}$$

Note that (17) only needs to be proved for odd n, since for even n this is the case $r = \lfloor n/2 \rfloor$ in (15). Using (11) and (14) it follows that all the entries above $(\lceil n/2 \rceil, \lceil n/2 \rceil + 1)$ are red. By Lemma 16 $(\lceil n/2 \rceil, \lceil n/2 \rceil + 1)$ is a halving entry, and so it follows that all the entries to its right are blue. This proves (17).

We now prove that for $2 \leq r \leq \lfloor n/2 \rfloor - 1$

$$\text{for odd } n, \text{ the entry } (r, n - r + 1) \text{ is red.} \tag{18}$$

Note that (16) is a version of (18) for $r = \lfloor n/2 \rfloor$. Observe that $M(f^{\lceil n/2 \rceil}(D))$ satisfies (11) and (12). If $(2, n-1)$ is red in $M(f^{\lceil n/2 \rceil}(D))$, then the diagonal $(r, n-r)$ with $1 \leq r \leq \lfloor n/2 \rfloor - 1$ in $M(f^{\lceil n/2 \rceil}(D))$ is red by (14). This corresponds to the diagonal $(r, n-r+1)$ with $2 \leq r \leq \lfloor n/2 \rfloor$ in $M(D)$. So now assume that the entry $(2, n-1)$ is blue in $M(f^{\lceil n/2 \rceil}(D))$, which corresponds to $(\lfloor n/2 \rfloor, \lceil n/2 \rceil + 2)$ being blue in $M(D)$. In this case, we can assume that $(1, \lceil n/2 \rceil)$ is blue. (Otherwise start with $M(h \circ g \circ f^{\lceil n/2 \rceil}(D))$ instead of D, which satisfies (11), (12), (13), $(\lfloor n/2 \rfloor, \lceil n/2 \rceil + 1)$ is red, and $(1, \lceil n/2 \rceil)$ is blue.) Now, by Lemma 16, $(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1)$ is a halving entry with $\lfloor n/2 \rfloor$ of the entries in (9) blue, then all others must be red, i.e., $(2, \lceil n/2 \rceil), (3, \lceil n/2 \rceil), \ldots, (\lfloor n/2 \rfloor - 1, \lceil n/2 \rceil)$ are red. Assume by contradiction that $(r, n-r+1)$ is blue for some $2 \leq r \leq \lfloor n/2 \rfloor - 1$. Then $(r, n-r+2)$ is blue, otherwise $(r, n-r+1)$ would be the first entry in the order associated to $(r, n-r+2)$ with no blue entry above, contradicting Corollary 13. But now the red entry $(r, \lceil n/2 \rceil)$ is the $(\lceil n/2 \rceil - r)^{th}$ entry in the order associated to the blue entry (r, n) with a blue entry above, contradicting Corollary 14 and proving (18).

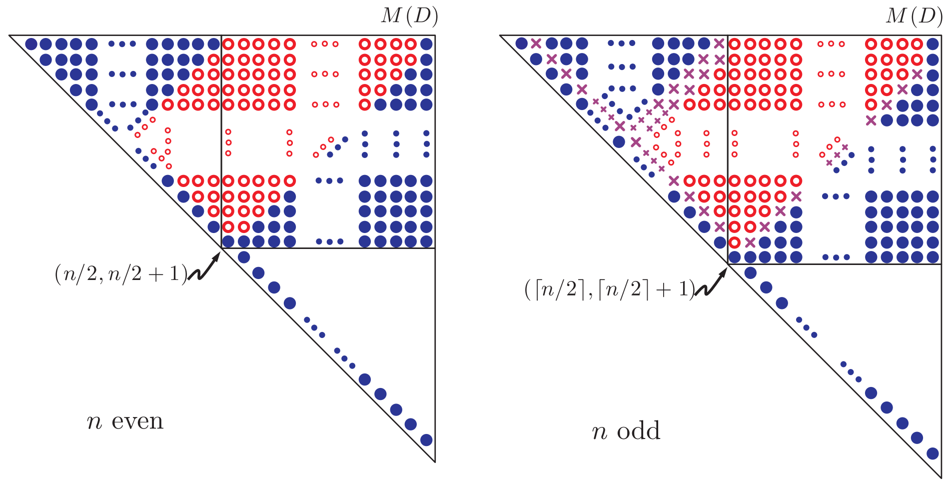

Figure 7: The upper triangle $T_U(D)$ for even and odd n in the proof of Theorem 19.

We finally observe that (11), (12), (13), (14), (15), (16), (17), and (18) prove Theorem 19 for the entries in $R(D)$.

The entries in $T_U(D)$

We refer the reader to Figure 7. We prove by induction on c that for $1 \le c \le \lfloor \frac{1}{2}\lceil n/2 \rceil \rfloor$,

$$\text{the entries } (c+e, \lceil n/2\rceil + 2 - c), \ldots, (\lfloor n/2 \rfloor - c, \lceil n/2 \rceil + 2 - c) \text{ are red,} \tag{19}$$

and

$$\text{the entries } (1, \lceil n/2\rceil + 2 - c), \ldots, (c - 1 - e, \lceil n/2 \rceil + 2 - c) \text{ are blue.} \tag{20}$$

We have proved it for $c = 1$. Suppose that the result holds for all $1 \le c \le d-1$ and we now prove it for $c = d$. By Lemma 17 for $i = \lceil n/2 \rceil + 1 - d$, and since by (15) the $\lfloor n/2 \rfloor - d$ entries $\{(i, b) \mid 2\lceil n/2 \rceil - i + 2 \le b \le i + \lceil n/2 \rceil\} \cup \{(i+1, b) \mid i + \lceil n/2 \rceil + 1 \le b \le n\}$ in (9) are blue, then $(i, i+1)$ has at most $d - 1 + e$ blue entries above. Suppose by contradiction that $(r, i+1)$ is blue for some $d + e \le r \le \lfloor n/2 \rfloor - d$. Then $(r, i+1)$ is the first entry in the order associated to $(r, n-r+1)$ and has at most $\lceil n/2 \rceil - 1 - (\lfloor n/2 \rfloor - d) - 1 = d - 2 + e$ blue entries above. By Lemma 12, $(r, i+1)$ has at least $\min\{\lfloor n/2 \rfloor - r, r - 1\}$ blue entries above and thus $\min\{\lfloor n/2 \rfloor - r, r-1\} \le d - 2 + e$. But $r - 1 > d - 2 + e$ because $r \ge d + e$, and $\lfloor n/2 \rfloor - r \ge d > d - 2 + e$ because $r \le \lfloor n/2 \rfloor - d$. Thus (19) holds for $c = d$.

Look at $(i, i+1)$ again. The $\lfloor n/2 \rfloor - 1 - 3e$ entries $\{(r, i+1) \mid d+e \le r \le i-1-e\} \cup \{(i, b) \mid i+2+e \le b \le n-i+1\}$ in (9) are red and thus, by Lemma 17, at most other $4e$ entries are red. For n even, $4e = 0$ and thus (20) holds. For n odd, suppose by contradiction that $(d-e, i+1)$ has a red entry above. We prove that in this case the entries $(d-e, i+1), (d-e+1, i+1)$, and $(i-1, i+1)$ are red. Since $(d-e, n+1-d+e)$ is red, then by Corollary 14 the first $\lfloor n/2 \rfloor + 2 - 2d + 2e$ entries in the order associated to $(d-e, n+1-d+e)$ have only blue entries above. If $(d-e, i+1)$ were blue, then it would be one of the first two entries in the order associated to $(d-e, n+1-d+e)$ with at least one red point above. This means that $1 \geq \lfloor n/2 \rfloor + 2 - 2d + 2e$ contradicting that $d \leq \lfloor \frac{1}{2} \lceil n/2 \rceil \rfloor$. Thus $(d-e, i+1)$ is red. Similarly, $(d-e+1, i+1)$ cannot be blue as it would be the first entry in the order associated to $(d-e+1, n-d+e)$, which by Lemma 12 should have at most one red entry above, but $(d-e+1, i+1)$ has now at least 2 red entries above. Now $(i-1, i+1)$ is the first entry for $(i-1, n+2-i)$ and, by (19), it has at least $\lceil n/2 \rceil + 1 - 2d + e$ red entries above, i.e., at most $d-2-e$ blue entries above. But by Lemma 12, the first entry in the order associated to the red entry $(i-1, n+2-i)$ has at least $\min\{d-1, i-2\}$ blue entries above. Thus $\min\{d-1, i-2\} \leq d-2-e$, but $d-1 > d-2-e$ and $i-2 > d-2-e$ because $d \leq \lfloor \frac{1}{2} \lceil n/2 \rceil \rfloor$, getting a contradiction. Hence $(i-1, i+1)$ is red. By Lemma 17 at most $\lfloor n/2 \rfloor$ of the entries in (10) are red, yet we already have $\lceil n/2 \rceil$ red entries (namely, at least the $\lceil n/2 \rceil + 1 - 2d + e$ above $(i-1, i+1)$ mentioned before and the $2d-2$ entries $\{(i-1, b) \mid i+2 \leq b \leq n-i+2\}$ to its right), getting a contradiction. Thus (20) holds for $c = d$.

Now we prove that for $2 \leq c \leq \lceil \frac{1}{2} \lceil n/2 \rceil \rceil + 1$,

$$\text{the entries } (1, c), (2, c), \ldots, (c-2-e, c) \text{ are blue.} \tag{21}$$

Since $(c-1, c)$ is one of the $\lfloor n/2 \rfloor + 5 - 2c$ entries in the order associated to the red entry $(c-1, n+2-c)$ (we have shown that the $\lfloor n/2 \rfloor - 1 - e$ entries immediately to the left of $(n+2-c)$ are red), then $(c-1, c)$ has at most one red entry above by Lemma 12. Suppose by contradiction that (r, c) is red for some $1 \leq r \leq c-2-e$. Then $(r+1, c)$ is blue. Since $(r+1, n-r)$ is red, then by Corollary 14 the first $\lfloor n/2 \rfloor - 2r$ entries in the order associated to $(r+1, n-r)$ have only blue entries above. But $(r+1, c)$ is one of the first $\lfloor n/2 \rfloor - 2r$ entries and has the red entry (r, c) above, getting a contradiction.

We finally note that (19), (20), and (21) prove Theorem 19 for the entries in $T_U(D)$.

The entries in $T_L(D)$

We refer the reader to Figure 8. Consider $f^{\lceil n/2 \rceil}(D)$. When n is even, see Figure 8 (left), $R(D)$ and $R(f^{\lceil n/2 \rceil}(D))$ are identical and thus our previous arguments show that $T_U(D)$ and $T_U(f^{\lceil n/2 \rceil}(D)) = T_L(D)$ are identical too, concluding the proof in this case. When n is odd, see Figure 8 (right), $R(D)$ and $R(f^{\lceil n/2 \rceil}(D))$ are slightly different: for $2 \leq r \leq \lfloor n/2 \rfloor$ the diagonal entries $(r, n+1-r)$ are red in $R(D)$ and unfixed in $R(f^{\lceil n/2 \rceil}(D))$, and for $3 \leq r \leq \lfloor n/2 \rfloor$ the diagonal entries $(r, n+2-r)$ are unfixed in $R(D)$ and blue in $R(f^{\lceil n/2 \rceil}(D))$. Also the last row of $R(D)$ is blue and the last row of $R(f^{\lceil n/2 \rceil}(D))$ is unfixed. However, the last column of $T_U(f^{\lceil n/2 \rceil}(D))$ is red and this is what allows us to mimic the arguments used for (19), (20), and (21) to show that $T_L(D)$, which corresponds to $T_U(f^{\lceil n/2 \rceil}(D))$ minus its last column, satisfies the statement. More precisely, it can be proved by induction on c that for $1 \leq c \leq \lfloor \frac{1}{2} \lceil n/2 \rceil \rfloor$, in $M(f^{\lceil n/2 \rceil}(D))$

$$\text{the entries } (c+1, \lceil n/2 \rceil + 1 - c), \ldots, (\lfloor n/2 \rfloor - c - 1, \lceil n/2 \rceil + 1 - c) \text{ are red} \tag{22}$$

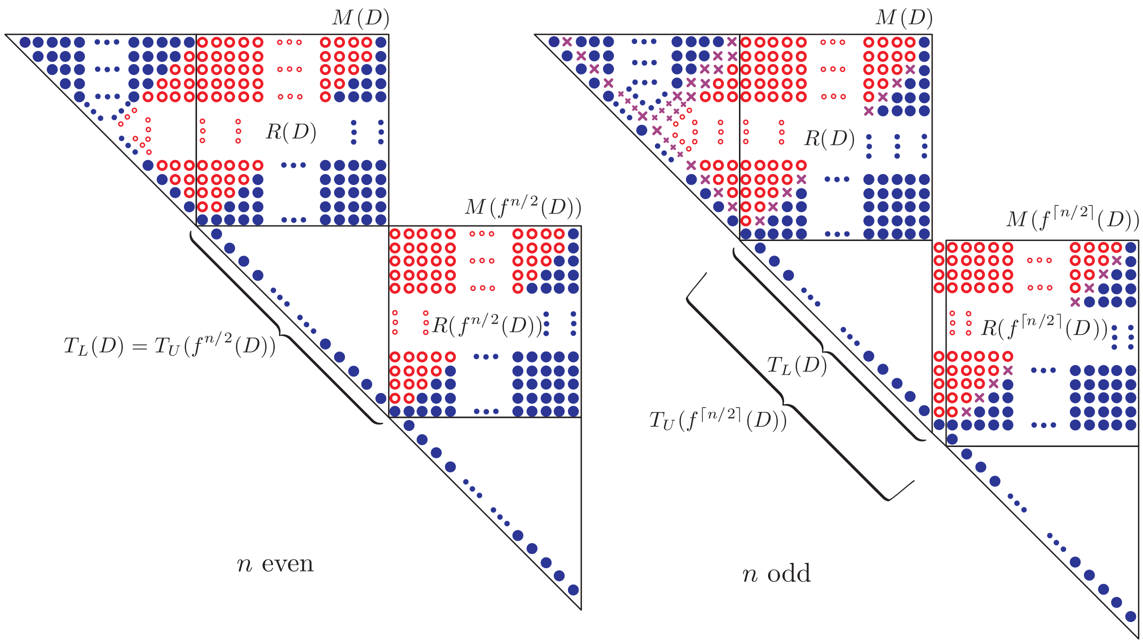

Figure 8: The lower triangle $T_L(D)$ versus the upper triangle $T_U(f^{\lceil n/2 \rceil}(D))$ for even and odd n in the proof of Theorem 19.

and

$$\text{the entries } (1, \lceil n/2 \rceil + 1 - c), \ldots, (c-2, \lceil n/2 \rceil + 1 - c) \text{ are blue.} \tag{23}$$

We omit the proofs of (22) and (23), as they very closely resemble the proofs of (19) and (20).

Similarly, it can be proved by induction that for $2 \le c \le \lceil \frac{1}{2} \lceil n/2 \rceil \rceil$, in $M(f^{\lceil n/2 \rceil}(D))$

$$\text{the entries } (1, c), (2, c), \ldots, (c-3, c) \text{ are blue.} \tag{24}$$

The proof of (24) is also omitted, as it very closely resembles the proof of (21).

We finally note that (22), (23), and (24) prove Theorem 19 for the entries in $T_U(L)$. □

4.4 The number of crossing optimal drawings

Theorem 19 completely determines $M(D')$ when n is even, which means that in this case there is essentially only one crossing optimal drawing.

Theorem 20. *For n even, up to homeomorphism, there is a unique crossing optimal 2-page book drawing of K_n.*

Proof. The result is easily seen to hold for $n = 2$ and $n = 4$. For $n \geq 6$ Theorem 19 completely determines $M(D')$. Note that this matrix corresponds to the drawings by Blažek and Koman [5]. $\square$

In contrast to the even case, for n odd there is an exponential number of non-equivalent crossing optimal 2-page book drawings of K_n. For any odd integer $n \geq 5$, we construct $2^{(n-5)/2}$ non-equivalent crossing optimal drawings of K_n. In fact, these $2^{(n-5)/2}$ drawings are pairwise non-homeomorphic. To prove this, we need the next two results.

Theorem 21. *For every $n \geq 13$ odd, every crossing optimal 2-page book drawing of K_n has exactly one Hamiltonian cycle of non-crossed edges, namely the one obtained from the edges on the spine and the $1n$ edge.*

Proof. Assume $n \geq 13$ is odd. To show that $123\ldots n$ is the only non-crossed Hamiltonian cycle, we show that all other edges are crossed at least once. Assume that D has the form described in Theorem 19. Let (r, c) be an entry of $M(D)$ that does not represent an edge on the spine or the $1n$ edge. Let

$$(r,c)^+ = \begin{cases} (r+1, c+1) & \text{if } c < n, \text{ or} \\ (1, r+1), & \text{if } c = n, \end{cases} \quad \text{and } (r,c)^- = \begin{cases} (r-1, c-1) & \text{if } r > 1, \text{ or} \\ (c-1, n), & \text{if } r = 1. \end{cases}$$

Note that the edges corresponding to $(r, c)^+$ and $(r, c)^-$ cross the edge rc if they have the same color as (r, c).

First assume that $3 \leq c - r \leq n - 3$. Suppose that (r, c) is a blue entry specified by Theorem 19. If $5 \leq r + c \leq \lfloor n/2 \rfloor - 1$ or if $n + 3 \leq r + c \leq n + \lfloor n/2 \rfloor - 1$, then note that the entry $(r, c)^+$ is also blue according to Theorem 19, and thus the edges corresponding to (r, c) and $(r, c)^+$ cross each other.

Because $n \geq 13$, if $\lfloor n/2 \rfloor \leq r + c \leq \lfloor n/2 \rfloor + 1$ or $n + \lfloor n/2 \rfloor \leq r + c \leq n + \lfloor n/2 \rfloor + 1$, then $5 \leq \lfloor n/2 \rfloor - 2 \leq r + c - 2 \leq \lfloor n/2 \rfloor + 1$ or $n + 3 \leq n + \lfloor n/2 \rfloor - 2 \leq r + c - 2 \leq n + \lfloor n/2 \rfloor + 1$, respectively. Thus the entry $(r, c)^-$ is also blue according to Theorem 19, and thus the edges corresponding to (r, c) and $(r, c)^-$ cross each other.

A similar argument shows that for every red entry (r, c) specified by Theorem 19, either $(r, c)^+$ or $(r, c)^-$ is also a red edge.

Second, assume that $c - r = n - 2$, that is $(r, c) \in \{(1, n-1), (2, n)\}$. If $(r, c) = (1, n-1)$, then (r, c) is red and because $(2n - 4) \geq n + \lceil n/2 \rceil + 2$ for $n \geq 13$, it follows that rc crosses the edge corresponding to $(n - 3, n)$, which is red. If $(r, c) = (2, n)$, then (r, c) is blue and because $\lfloor n/2 \rfloor \geq 4$ for $n \geq 13$, it follows that rc crosses the edge corresponding to $(1, 4)$, which is blue.

Suppose now that the color of (r, c) is not determined by Theorem 19. First assume that $r + c \in \{\lfloor n/2 \rfloor + 2, \lceil n/2 \rceil + 2, n + \lfloor n/2 \rfloor + 2, n + \lceil n/2 \rceil + 2\}$. Again, by Theorem 19 note that

$(r,c)^-$ is blue and $(r,c)^+$ is red. Similarly, if $r+c = n+2$, then $(r,c)^-$ is red and $(r,c)^+$ is blue. Thus regardless of its color, the edge rc will cross one of the two edges corresponding to these two entries.

Finally, assume $c - r = 2$. From Theorem 19, the number of red entries of the form $(t, r+1)$ or $(r+1, d)$, with $1 \le t \le r$ and $r+3 \le d \le n$ is at least $\lfloor n/2 \rfloor - 5 \ge 1$. A similar statement holds for the number of blue entries of the same form. Thus there is at least one blue edge (not on the spine) and at least one red edge incident to $r+1$. One of these two edges will necessarily cross the edge rc regardless of its color. □

Note that for $n \le 11$ the above approach cannot guarantee that there are no additional non-crossed edges. For example for $n = 11$ the element $(1, 10)$ cannot be determined. However, these small cases can be handled by exhaustive enumeration, which shows that for crossing optimal drawings there are no such edges for $n = 11$ and no alternative Hamiltonian cycles for $n = 9$. For $n = 5, 7$ there exist alternative Hamiltonian cycles of non-crossed edges, but they do not lead to additional equivalences between the crossing optimal drawings.

Corollary 22. *If D and D' are crossing optimal 2-page book drawings of K_n, then either D and D' are not homeomorphic, or else $M(D)$ and $M(D')$ are equivalent.*

Proof. If n is even the result is trivial by Theorem 20. If n is odd and $n \le 11$, then using Theorem 19 we exhaustively found all equivalence classes of crossing optimal drawings. There are 1, 4, 9, and 25 equivalence classes for $n = 5$, 7, 9, and 11, respectively. We verified that all of these equivalence classes were topologically distinct. If $n \ge 13$ and D and D' are crossing optimal 2-page book drawings, then by the previous theorem both D and D' have only one non-crossed Hamiltonian cycle. Thus if $H : D \to D'$ is a homeomorphism, then H must send the Hamiltonian cycle $123\ldots n$ to itself. It follows that H restricted to this cycle is the composition of a rotation of the cycle with either the identity, or the function that reverses the order of the cycle. Moreover, once the edges on the spine are fixed, the drawing is determined by the colors of the remaining edges. Thus either H is determined by its action on the cycle, or else H switches the blue edges not on the spine with the red edges. In other words, $M(D') = M(H(D)) = M((h^a \circ g^b \circ f^i)(D))$ for some $i \in \{0, 1, 2, \ldots, n-1\}$ and $a, b \in \{0, 1\}$. Thus $M(D)$ and $M(D')$ are equivalent. □

Theorem 23. *For n odd, there are at least $2^{(n-5)/2}$ pairwise non-homeomorphic crossing optimal 2-page book drawings of K_n.*

Proof. As usual let $1, 2, \ldots, n$ be the vertices of K_n. Let rc be an edge of K_n that is not on the Hamiltonian cycle $H = 12\ldots n$, we color rc red or blue according to the following rule: if $r + c \equiv s \pmod{n}$ for some integer $2 \le s \le (n+1)/2$, then we color rc blue, if $r + c \equiv s \pmod{n}$ for some integer $(n+5)/2 \le s \le n+1$, then we color rc red. Finally, if $r + c \equiv (n+3)/2 \pmod{n}$, then we color rc either red or blue. See (Figure 9.)

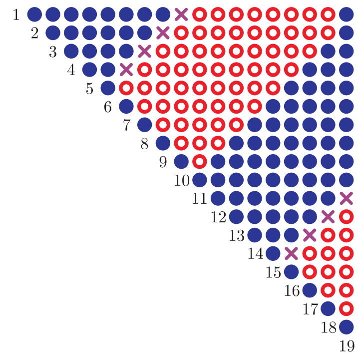

Figure 9: The 2^8 crossing optimal drawings (only 2^7 non-equivalent) for $n = 19$ in Theorem 23. They are obtained by assigning arbitrary colors to the crosses in this matrix.

We first argue that all of these colorings yield crossing optimal drawings of K_n regardless of the color of the $(n-3)/2$ edges rc for which $r + c \equiv (n+3)/2 \pmod{n}$.

For every $1 \leq s \leq n$, let $I_s = \{rc \text{ edge: } rc \notin H \text{ and } r + c \equiv s \pmod{n}\}$. Note that $|I_s| = (n-3)/2$ for all s and $\bigcup_{s=1}^{n} I_s$ is the complete set of edges not in H. Moreover note that each I_s is a matching of pairwise non-crossing edges.

Let rc be an edge such that $r+c \equiv (n+3)/2 \pmod{n}$. Assume without loss of generality that $r < c$. If td is an edge that crosses rc, then t and d are cyclically separated from r and c; that is, we may assume that $r < t < c$ and $d < r$ or $d > c$. To facilitate the case analysis we may assume that the edges that could cross rc are the edges td such that $r < t < c < d < n+r$, with the understanding that d represents the point $d-n$ when $d > n$. Let $C = \{td \text{ edge: } r < t < c < d < n+r\}$ and consider the function $T : C \to C$ defined by $T(td) = t'd'$ where $t' = r + c - t$ and $d' = r + c + n - d$. Note that T is well defined because $r < t' < c < d' < n + r$ and T is one-to-one on C. Moreover, note that

$$\begin{aligned} t' + d' &\equiv r + c + n + r + c - t - d \pmod{n} \\ &\equiv 2(r + c) - (t + d) \pmod{n} \\ &\equiv (n + 3) - (t + d) \equiv 3 - (t + d) \pmod{n}, \end{aligned}$$

so $t + d \equiv s \pmod{n}$ with $2 \leq s \leq (n+1)/2$ if and only if $t' + d' \equiv 3 - (t + d) \equiv n + 3 - s \pmod{n}$ and $(n+5)/2 \leq n + 3 - s \leq n + 1$. Thus td and $T(td)$ have different colors, which means that C contains as many red edges as blue edges. Hence rc crosses the same number of edges independently of its color. This shows that all the drawings we have described have the same number of crossings. Finally, we note that the drawing for which all the arbitrary edges have the same color corresponds to the construction originally found by Blažek and Koman [5] having exactly $Z(n) = \frac{1}{64}(n-1)^2(n-3)^2$ crossings. Hence all the other drawings described are crossing optimal as well.

We now argue that every drawing constructed here is equivalent to exactly one other drawing, and thus we have constructed exactly $2^{(n-5)/2}$ distinct topological drawings. Let D

and D' be two of the crossing optimal drawings we just constructed and suppose that D and D' are homeomorphic. By Corollary 22, $M(D)$ and $M(D')$ are equivalent, thus there exists a transformation $F : D \to D'$ such that $F = h^a \circ g^b \circ f^i$ with $i \in \{0, 1, 2, \ldots, n-1\}$ and $b, a \in \{0, 1\}$. First observe that under f, g, or h, the absolute value difference of the number of red minus blue edges remains invariant. Thus the drawing D in which all of the edges in $I_{(n+3)/2}$ are red can only be homeomorphic to the drawing D' in which all of those edges are blue. These two are indeed homeomorphic under the function $F = h \circ g \circ f^{(n+1)/2}$. Now suppose that the edges $I_{(n+3)/2}$ in D and in D' are not all of the same color. Note that f, g, and h send I_m into another $I_{m'}$, and if I_m is monochromatic (all edges of I_m have the same color) in D, then $I_{m'}$ is monochromatic in $f(D), g(D)$, and $h(D)$. Since I_m is monochromatic in D if and only if $m \neq (n+3)/2$, then F must send $I_{(n+3)/2}$ to itself. If $b = 0$, $rc \in I_{(n+3)/2}$, and $r'c'$ is the image of rc under F, then $r' + c' \equiv r - i + c - i \pmod{n}$. Thus $r' + c' \equiv r + c \pmod{n}$ if and only if $i = 0$. Because the edges I_1 in D are blue and the edges I_1 in $h(D)$ are red, it follows that $a = 0$ and thus F is the identity. Last, if $b = 1$, $rc \in I_{(n+3)/2}$, and $r'c'$ is the image of rc under F, then $r' + c' \equiv (n+1-(c-i)) + (n+1-(r-i)) \equiv 2 + 2i - (r+c) \pmod{n}$. Thus $r' + c' \equiv r + c \pmod{n}$ if and only if $i = (n+1)/2$. Because the edges I_1 in both D and $h(f^{(n+1)/2}(D))$ are blue, it follows that $a = 1$ and thus $F = h \circ g \circ f^{(n+1)/2}$. It can be verified that indeed $F(D)$ is one of the drawings we constructed here, and thus exactly half of the drawings we described are pairwise non-isomorphic. □

n	drawings	n	drawings	n	drawings
5	1	17	324	29	38944
7	4	19	748	31	84064
9	9	21	1672	33	180288
11	25	23	3736	35	385216
13	58	25	8208	37	819328
15	142	27	17968		

Table 1: The number of non-homeomorphic crossing optimal 2-page book drawings of K_n for odd n, $5 \leq n \leq 37$.

The above theorem gives a lower bound of $2^{(n-5)/2}$ for the number of non-equivalent crossing optimal drawings. As in the crossing optimal drawings of Theorem 19 there are $\frac{5}{2}(n-5)$ entries with non-fixed colors, we get an upper bound of $2^{5(n-5)/2}$ non-equivalent crossing optimal drawings. With exhaustive enumeration we have been able to determine the exact numbers of non-equivalent crossing optimal drawings for $n \leq 37$, cf. Table 1. The obtained results suggest an asymptotic growth of roughly $2^{0.54n}$, rather close to our lower bound.

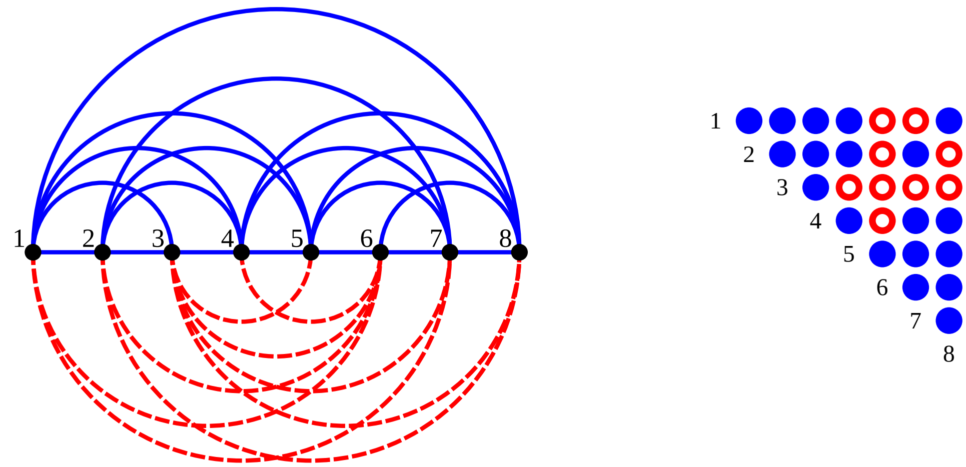

Figure 10: A 2-page book drawing of K_8 with four 0-edges (namely $(1,7)$, $(1,8)$, $(2,7)$, and $(2,8)$) and four 1-edges (namely $(1,5)$ $(1,6)$, $(3,8)$, and $(4,8)$). This shows that the inequality $E_{\leq k}(D) \geq 3\binom{k+2}{2}$, which holds for every geometric drawing D of K_n, does not necessarily hold if D is a topological drawing.

5 Concluding remarks

It was proved by Ábrego and Fernández-Merchant [1] and by Lovász et al. [19] that the inequality $E_{\leq k}(P) \geq 3\binom{k+2}{2}$ holds (in the geometric setting) for every set P of n points in general position in the plane and for every k such that $0 \leq k \leq \lfloor n/2 \rfloor - 2$. This inequality used with the rectilinear version of Theorem 1 gives $Z(n)$ as a lower bound for the rectilinear crossing number of K_n [1]. In contrast to the rectilinear case, the inequality $E_{\leq k}(D) \geq 3\binom{k+2}{2}$ does not hold in general for topological drawings D of K_n, not even for general 2-page drawings (see Figure 10). This shows the relevance of introducing the parameter $E_{\leq\leq k}(D)$ (for which Theorem 6 can be established, leading to the 2-page crossing number of K_n). However, the inequality $E_{\leq k}(D) \geq 3\binom{k+2}{2}$ does hold for crossing optimal 2-page drawings of K_n. For a proof of this, and other interesting observations on crossing optimal drawings of K_n, we refer the reader to Section 4 in the proceedings version of this paper [2].

Our approach to determine k-edges in the topological setting is to define the orientation of three vertices by the orientation of the corresponding triangle in a good drawing of the complete graph. It is natural to ask whether this defines an abstract order type. To this end, the setting would have to satisfy the axiomatic system described by Knuth [18]. But it is easy to construct an example which does not fulfill these axioms, that is, our setting does not constitute an abstract order type as described by Knuth [18]. It is an interesting question for further research how this new concept compares to the classic order type, both in terms of theory (realizability, etc.) and applications.

We believe that the developed techniques of generalized orientation, k-edge for topological drawings, and $\leq\leq k$-edges are of interest in their own. We will investigate their usefulness for related problems in future work. For example, they might also play a central role to approach the crossing number problem for general drawings of complete and complete bipartite graphs.

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