Characterizing 2-crossing-critical graphs

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Abstract

It is very well-known that there are precisely two minimal non-planar graphs: K_5 and $K_{3,3}$ (degree 2 vertices being irrelevant in this context). In the language of crossing numbers, these are the only 1-crossing-critical graphs: they each have crossing number at least one, and every proper subgraph has crossing number less than one. In 1987, Kochol exhibited an infinite family of 3-connected, simple 2-crossing-critical graphs. In this work, we: (i) determine all the 3-connected 2-crossing-critical graphs that contain a subdivision of the Möbius Ladder V_{10}; (ii) show how to obtain all the not 3-connected 2-crossing-critical graphs from the 3-connected ones; (iii) show that there are only finitely many 3-connected 2-crossing-critical graphs not containing a subdivision of V_{10}; and (iv) determine all the 3-connected 2-crossing-critical graphs that do not contain a subdivision of V_8.

Received by the editor December 11, 2013.

2010 *Mathematics Subject Classification.* Primary 05C10.

Key words and phrases. crossing number, crossing-critical graphs.

Bokal acknowledges the support of NSERC and U. Waterloo for 2006-2007, Slovenian Research Agency basic research projects L7-5459, J6-3600, J1-2043, L1-9338, J1-6150, research programme P1-0297, and an international research grant GReGAS.

Richter acknowledges the support of NSERC.

Salazar acknowledges the support of CONACYT Grant 106432.

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CHAPTER 1

Introduction

For a positive integer k, a graph G is *k-crossing-critical* if the crossing number $\mathrm{cr}(G)$ is at least k, but every proper subgraph H of G has $\mathrm{cr}(H) < k$. In general, it is not true that a k-crossing-critical graph has crossing number exactly k. For example, any edge-transitive non-planar graph G satisfies $\mathrm{cr}(G-e) < \mathrm{cr}(G)$, for any edge e of G, so every such graph is k-crossing-critical for any k satisfying $\mathrm{cr}(G-e) < k \le \mathrm{cr}(G)$. If G is the complete graph K_n, then $\mathrm{cr}(K_n) - \mathrm{cr}(K_n - e)$ is of order n^2, so K_n is k-crossing-critical for many different values of k.

Insertion and suppression of vertices of degree 2 do not affect the crossing number of a graph, and a k-crossing-critical graph has no vertices of degree 1 and no component that is a cycle. Thus, if G is a k-crossing-critical graph, the graph G' whose vertex set consists of the *nodes* of G (i.e., the vertices of degree different from 2) and whose edges are the *branches* of G (i.e., the maximal paths all of whose internal vertices have degree 2 in G) is also k-crossing-critical. Our interest is, therefore, in k-crossing-critical graphs with minimum degree at least 3.

By Kuratowski's Theorem, the only 1-crossing-critical graphs are $K_{3,3}$ and K_5. The classification of 2-crossing-critical graphs is currently not known. The earliest published remarks on this classification of which we are aware is by Bloom, Kennedy, and Quintas [**7**], where they exhibit 21 such graphs. Kochol [**20**] gives an infinite family of 3-connected, simple 2-crossing-critical graphs, answering a question of Širáň [**33**] who gave, for each $n \ge 3$, an infinite family of 3-connected n-crossing-critical graphs. Richter [**29**] shows there are just eight cubic 2-crossing-critical graphs.

About 15 years ago, Oporowski gave several conference talks about showing that every large peripherally-4-connected, 2-crossing-critical graph has a very particular structure which was later denoted as 'being composed of tiles'. The method suggested was to show that if a peripherally-4-connected, 2-crossing-critical graph has a subdivision of a particular V_{2k} (that is, k is fixed), then it has the desired structure and that only finitely many peripherally-4-connected, 2-crossing-critical graphs do not have a subdivision of V_{2k}. (The graph V_{2n} is obtained from a $2n$-cycle by adding the n diagonals. Note that V_4 is K_4 and V_6 is $K_{3,3}$.)

Approximately 10 years ago, it was proved by Ding, Oporowski, Thomas, and Vertigan [**13**] that, for any k, a large (as a function of k) 3-connected, 2-crossing-critical graph necessarily has a subdivision of V_{2k}. It remains to show that having the V_{2k}-subdivision implies having the desired global structure. Their proof involves first showing a statement about non-planar graphs that is of significant independent interest: for every k, any large (as a function of k) "almost 4-connected" non-planar graph contains a subdivision of one of four non-planar graphs whose sizes grow with k. One of the four graphs is V_{2k}. This theorem is then used for the crossing-critical application mentioned above.

Tiles have come to be a very fruitful tool in the study of crossing-critical graphs. Their fundamentals were laid out by Pinontoan and Richter [**27**], and later they turned out to be a key in Bokal's solution of Salazar's question regarding average degrees in crossing-critical graphs [**8, 28, 31**]. These results all rely on the ease of establishing the crossing number of a sufficiently large tiled graph, and they generated considerable interest in the reverse question: what is the true structure of crossing-critical graphs? How far from a tiled graph can a large crossing-critical graph be? Hliněný's result about bounded path-width of k-crossing-critical graphs [**18**] establishes a rough structure, but is it possible that, for small values of k, tiles would describe the structure completely? It turns out that, for $k = 2$, the answer is positive. A more detailed discussion of these and other matters relating to crossing numbers can be found in the survey by Richter and Salazar [**30**].

Our goal in this work, not quite achieved, is to classify all 2-crossing-critical graphs. The bulk of our effort is devoted to showing that if G is a 3-connected 2-crossing-critical graph that contains a subdivision of V_{10}, then G is one of a completely described infinite family of 3-connected 2-crossing-critical graphs. These graphs are all composed from 42 tiles. This takes up Chapters $3-13$. This combines with [**13**] to prove that a "large" 3-connected 2-crossing-critical graph is a member of this infinite family.

The remainder of the classification would involve determining all 2-crossing-critical graphs that either are not 3-connected or are 3-connected and do not have a subdivision of V_{10}. In Chapter 14, we deal with the 2-crossing-critical graphs that are not 3-connected: they are either one of a small number of known particular examples, or they are 2-connected and easily obtained from 3-connected examples.

There remains the problem of determining the 3-connected 2-crossing-critical graphs that do not contain a subdivision of V_{10}. In the first five sections of Chapter 15, we explain how to completely determine all the 3-connected 2-crossing-critical graphs from peripherally-4-connected graphs that either have crossing number 1 or are themselves 2-crossing-critical. In the sixth and final subsection, we determine which peripherally-4-connected graphs do not contain a subdivision of V_8 and either have crossing number 1 or are themselves 2-crossing-critical. Combining the two parts yields a definite (and practical) procedure for finding all the 3-connected 2-crossing-critical graphs that do not contain a subdivision of V_8. This leaves open the problem of classifying those that contain a subdivision of V_8 but do not have a subdivision of V_{10}. In Sections 16.1 and 16.2, we show that there are only finitely many. (Although this follows from [11], the approach is different and it keeps our work self-contained.)

There is hope for a complete description. In her master's essay, Urrutia-Schroeder [**36**] begins the determination of precisely these graphs and finds 326 of them. Oporowski (personal communication) had previously determined 531 3-connected 2-crossing-critical graphs, of which 201 contain a subdivision of V_8 but not of V_{10}. Austin [**3**] improves on Urrutia-Schroeder's work, correcting a minor error (only 214 of Urrutia-Schroeder's graphs are actually 2-crossing-critical) and finding several others, for a total of 312 examples. Only 8 of Oporowski's examples are not among the 312. A few have been determined by us as stepping stones in our classification of those that have a subdivision of V_{10}. We have hopes of completing the classification.

The principal facts that we prove in this work are summarized in the following statement.

THEOREM 1.1 (Classification of 2-crossing-critical graphs). *Let G be a 2-crossing-critical graph with minimum degree at least 3. Then either:*

- *if G is 3-connected, then either G has a subdivision of V_{10} and a very particular tile structure or has at most 3 million vertices; or*
- *G is not 3-connected and is one of* 49 *particular examples; or*
- *G is 2- but not 3-connected and is obtained from a 3-connected example by replacing digons by digonal paths.*

We remark again that vertices of degree 2 are uninteresting in the context of crossing-criticality, so we assume all graphs have minimum degree at least 3.

Chapters 2–13 of this work contain the proof of the following, which is the main contribution of this work. (The formal definitions required for the statement given below are presented in Chapter 2.)

THEOREM 1.2 (2-crossing-critical graphs with V_{10}). *Let G be a 3-connected, 2-crossing-critical graph containing a subdivision of V_{10}. Then G is a twisted circular sequence $(T_1, T_2, \dots, T_n)$ of tiles, with each T_i coming from a set of 42 possibilities.*

This is part of the first item in the statement of Theorem 1.1.

Chapter 14 is devoted to 2-crossing-critical graphs that are not 3-connected. (We remind the reader of Tutte's theory of cleavage units and introduce digonal paths in Chapter 14.) The results there are summarized in the following.

THEOREM 1.3 (2-crossing-critical graphs with small cutsets). *Let G be a 2-crossing-critical graph with minimum degree at least 3 that is not 3-connected.*

(1) If G is not 2-connected, then G is one of 13 graphs. (See Figure 14.1.)
(2) If G is 2-connected and has two nonplanar cleavage units, then G is one of 36 graphs. (See Figures 14.2 and 14.3.)
(3) If G is 2-connected with at most one nonplanar cleavage unit, then G has precisely one nonplanar cleavage unit and is obtained from a 3-connected, 2-crossing-critical graph by replacing pairs of parallel edges by digonal paths.

Chapter 15 shows how to reduce the determination of 3-connected 2-crossing-critical graphs to "peripherally-4-connected" 2-crossing-critical graphs. A graph G is *peripherally-4-connected* if G is 3-connected and, for every 3-cut X in G, any partition of the components into nonnull subgraphs H and J has one of H and J being a single vertex. The main result here is the following.

THEOREM 1.4. *Every 3-connected, 2-crossing-critical graph is obtained from a peripherally-4-connected, 2-crossing-critical graph by replacing each degree 3 vertex with one of at most 20 different graphs, each having at most 6 vertices.*

We combine this with Robertson's characterization of V_8-free graphs to explain how to determine all the 3-connected 2-crossing-critical graphs that do not have a subdivision of V_8. This requires a further reduction to "internally 4-connected" graphs.

Chapter 16 shows that a 3-connected, 2-crossing-critical graph with a subdivision of V_8 but no subdivision of V_{10} has at most three million vertices. The general result we prove there is the following.

THEOREM 1.5. *Suppose G is a 3-connected, 2-crossing-critical graph. Let $n \geq 3$ be such that G has a subdivision of V_{2n} but not of $V_{2(n+1)}$. Then $|V(G)| = O(n^3)$.*

CHAPTER 2

Description of 2-crossing-critical graphs with V_{10}

In this section, we describe the structure of the 2-crossing-critical graphs that contain V_{10}. As mentioned in the introduction, they are composed of tiles. This concept was first formalized by Pinontoan and Richter [**27, 28**] who studied large sequences of equal tiles. Bokal [**8**] extended their results to sequences of arbitrary tiles, which are required in this section. In those results, "perfect" tiles were introduced to establish the crossing number of the constructed graphs. However, this property required a lower bound on the number of the tiles that is just slightly too restrictive to include all our graphs. As we are able to establish the lower bound on the crossing number of all these graphs in a different way (Theorem 5.5), we summarize the concepts of [**8**] without reference to "perfect" tiles. Where the reader feels we are imprecise, please refer to [**8**] for details.

Definition 2.1. (1) A *tile* is a triple $T = (G, \lambda, \rho)$, consisting of a graph G and two sequences λ and ρ of distinct vertices of G, with no vertex of G appearing in both λ and ρ.

(2) A *tile drawing* is a drawing D of G in the unit square $[0, 1] \times [0, 1]$ for which the intersection of the boundary of the square with $D[G]$ contains precisely the images of the vertices of the *left wall* λ and the *right wall* ρ, and these are drawn in $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$, respectively, such that the y-coordinates of the vertices are increasing with respect to their orders in the sequences λ and ρ.

(3) The *tile crossing number* $\mathrm{tcr}(T)$ of a tile T is the smallest number of crossings in a tile drawing of T.

(4) The tile T is *planar* if $\mathrm{tcr}(T) = 0$.

(5) A *k-drawing* of a graph or a *k-tile-drawing* of a tile is a drawing or tile-drawing, respectively, with at most k crossings.

It is a central point for us that tiles may be "glued together" to form larger tiles. We formalize this as follows.

Definition 2.2. (1) The tiles $T = (G, \lambda, \rho)$ and $T' = (G', \lambda', \rho')$ are *compatible* if $|\rho| = |\lambda'|$.

(2) A sequence $(T_0, T_1, \dots, T_m)$ of tiles is *compatible* if, for each $i = 1, 2, \dots, m$, T_{i-1} is compatible with T_i.

(3) The *join* of compatible tiles (G, λ, ρ) and (G', λ', ρ') is the tile $(G, \lambda, \rho) \otimes (G', \lambda', \rho')$ whose graph is obtained from G and G' by identifying the sequence ρ term by term with the sequence λ'; left wall is λ; and right wall is ρ'.

(4) As $\otimes$ is associative, the join $\otimes\mathcal{T}$ of a compatible sequence $\mathcal{T} = (T_0, T_1, \dots, T_m)$ of tiles is well-defined as $T_0 \otimes T_1 \otimes \cdots \otimes T_m$.

Note that identifying wall vertices in a join may introduce either multiple edges or vertices of degree two. If we are interested in 3-connected graphs, we may suppress vertices of degree two, but we keep the multiple edges.

We have the following simple observation.

OBSERVATION 2.3. *Let $(T_0, T_1, \ldots, T_m)$ be a compatible sequence $\mathcal{T}$ of tiles. Then*

$$\operatorname{tcr}(\otimes\mathcal{T}) \le \sum_{i=0}^{m} \operatorname{tcr}(T_i). \qquad \blacksquare$$

An important operation on tiles that we need converts a tile into a graph.

DEFINITION 2.4. (1) A tile T is *cyclically compatible* if T is compatible with itself.

(2) For a cyclically-compatible tile T, the *cyclization* of T is the graph $\circ T$ obtained by identifying the respective vertices of the left wall with the right wall. A cyclization of a cyclically-compatible sequence of tiles is defined as $\circ\mathcal{T} = \circ(\otimes\mathcal{T})$.

The following useful observation is easy to prove. Typically, we will apply this to the tile $\otimes\mathcal{T}$ obtained from a compatible sequence $\mathcal{T}$ of tiles.

LEMMA 2.5 ([**8, 28**]). *Let T be a cyclically compatible tile. Then* $\operatorname{cr}(\circ T) \le \operatorname{tcr}(T)$. $\blacksquare$

We now describe various operations that turn one tile into another.

DEFINITION 2.6. (1) For a sequence ω, $\bar{\omega}$ denotes the reversed sequence.

(2)
- The *right-inverted* tile of a tile $T = (G, \lambda, \rho)$ is the tile $T^{\updownarrow} = (G, \lambda, \bar{\rho})$;
- the *left-inverted* tile is ${}^{\updownarrow}T = (G, \bar{\lambda}, \rho)$;
- the *inverted* tile is ${}^{\updownarrow}T^{\updownarrow} = (G, \bar{\lambda}, \bar{\rho})$; and
- the *reversed* tile is $T^{\leftrightarrow} = (G, \rho, \lambda)$. ($T^{\leftrightarrow}$ made an item.)

(3) A tile T is *k-degenerate* if T is planar and, for every edge e of T, $\operatorname{tcr}(T^{\updownarrow} - e) < k$.

Note that our k-degenerate tiles are not necessarily perfect, as opposed to the definition in [**8**]. However, the following analogue of [**8**, Cor. 8] is still true.

LEMMA 2.7. *Let $\mathcal{T} = (T_0, \ldots, T_m)$, $m \ge 0$, be a cyclically-compatible sequence of k-degenerate tiles. Then $\otimes(\mathcal{T})$ is a k-degenerate tile.*

PROOF. By Lemma 2.5, $\otimes\mathcal{T}$ is planar. Let e be any edge of $\otimes\mathcal{T}$. Let T_i be the tile of $\mathcal{T}$ containing e. Let $\mathcal{T}' = (T_0, \ldots, T_{i-1}, T_i^{\updownarrow} - e, {}^{\updownarrow}T_{i+1}{}^{\updownarrow}, \ldots, {}^{\updownarrow}T_m{}^{\updownarrow})$, so $\otimes\mathcal{T}' = \otimes\mathcal{T}^{\updownarrow} - e$; in particular, they have the same tile crossing number. As $T_i{}^{\updownarrow}$ is k-degenerate, $\operatorname{tcr}(T_i^{\updownarrow} - e) < k$. Since all other tiles of $\mathcal{T}'$ are planar, Lemma 2.5 implies $\operatorname{tcr}(\otimes\mathcal{T}^{\updownarrow} - e) \le \operatorname{tcr}(T_i^{\updownarrow} - e) < k$. $\square$

The following is an obvious corollary.

COROLLARY 2.8. *Let T be a k-degenerate tile so that $\operatorname{cr}(\circ(T^{\updownarrow})) \ge k$. Then $\circ(T^{\updownarrow})$ is a k-crossing-critical graph.* $\blacksquare$

DEFINITION 2.9. (1) $\mathcal{T}$ is a compatible sequence $(T_0, T_1, \ldots, T_m)$, then:
- the *reversed sequence* $\mathcal{T}^{\leftrightarrow}$ is the sequence $(T_m^{\leftrightarrow}, T_{m-1}^{\leftrightarrow}, \ldots, T_0^{\leftrightarrow})$;

- the *i-flip* $\mathcal{T}^i$ is the sequence $(T_0, \dots, T_i^{\updownarrow}, {}^{\updownarrow}T_{i+1}, T_{i+2} \dots, T_m)$; and
- the *i-shift* $\mathcal{T}_i$ is the sequence $(T_i, \dots, T_m, T_0, \dots, T_{i+1})$.

(2) Two sequences of tiles are *equivalent* if one can be obtained from the other by a series of shifts, flips, and reversals.

Note that the cyclizations of two equivalent sequences of tiles are the same graph.

DEFINITION 2.10. The set $\mathcal{S}$ of tiles consists of those tiles obtained as combinations of two *frames*, illustrated in Figure 2.1, and 13 *pictures*, shown in Figure 2.2, in such a way, that a picture is inserted into a frame by identifying the two squares. A given picture may be inserted into a frame either with the given orientation or with a 180° rotation (some examples are given in Figure 2.3).

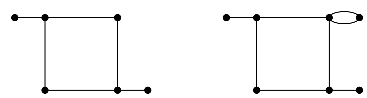

FIGURE 2.1. The two frames.

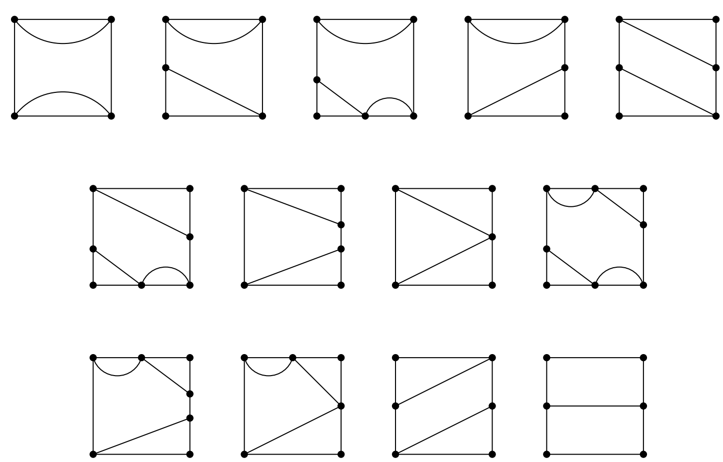

FIGURE 2.2. The thirteen pictures.

We remark that each picture produces either two or four tiles in $\mathcal{S}$; see Figure 2.3

LEMMA 2.11. *Let T be a tile in the set $\mathcal{S}$. Then both T and ${}^{\updownarrow}T_i^{\updownarrow}$ are 2-degenerate.*

PROOF. Figure 2.4 shows that all the tiles are planar. The claim for T implies the result for ${}^{\updownarrow}T_i^{\updownarrow}$, so it is enough to prove the result for an arbitrary $T \in \mathcal{S}$. Let e be an arbitrary edge of T. We consider cases, depending on whether e is either dotted, thin solid, thick solid, thin dashed, or thick dashed in Figure 2.4. Using this classification, we argue that $\mathrm{tcr}(T - e) < 2$.

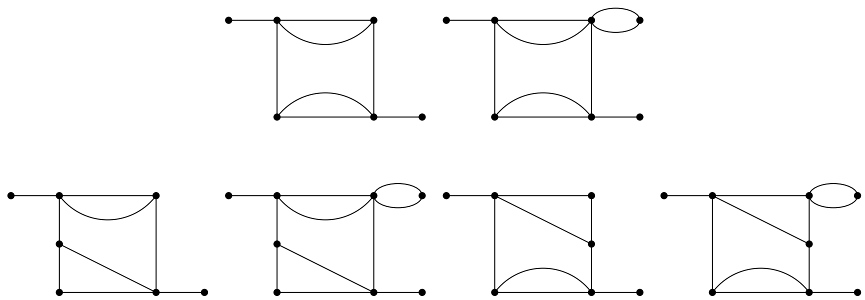

FIGURE 2.3. Each picture produces either two or four tiles.

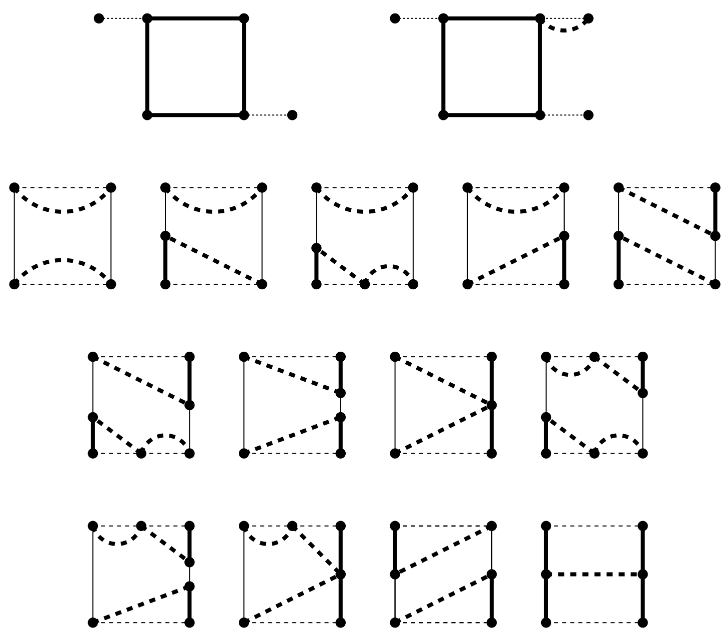

FIGURE 2.4. The different kinds of edges in the pictures.

If e is a dotted edge, then $T - e$ has a wall with a single vertex and $\mathrm{tcr}(T^{\updownarrow} - e) = 0$.

If e is a thin solid edge, then there is a 1-tile-drawing of $T^{\updownarrow}$ with two dotted edges of T crossing each other.

If e is a thick solid edge, then there is a unique thin dashed edge f adjacent to e, and there exists a 1-tile-drawing of $T^{\updownarrow} - e$ with f crossing the dotted edge not on the same horizontal side of T as f.

If e is a thin dashed edge, then there is a unique thick dashed edge e' such that e and e' are in the same face of the exhibited planar drawing of T, as well as a unique dotted edge f, that is not in the same horizontal side of T as e. For such e and e', there exists a 1-tile-drawing of $T^{\updownarrow} - e$ with e' crossing f, as well as a

1-tile-drawing of $T^{\updownarrow} - e'$ with e crossing f. As each thick dashed edge corresponds to at least one thin dashed edge, this concludes the proof. □

We now define the set of graphs that is central to this work.

DEFINITION 2.12. The set $\mathcal{T}(\mathcal{S})$ consists of all graphs of the form $\circ((\otimes\mathcal{T})^{\updownarrow})$, where $\mathcal{T}$ is a sequence $(T_0, {}^{\updownarrow}T_1^{\updownarrow}, T_2, \dots, {}^{\updownarrow}T_{2m-1}^{\updownarrow}, T_{2m})$ so that $m \geq 1$ and, for each $i = 0, 1, 2, \dots, 2m$, $T_i \in \mathcal{S}$.

The *rim* of an element of $\mathcal{T}(\mathcal{S})$ is the cycle R that consists of the top and bottom horizontal path in each frame (including the part that sticks out to either side) and, if there is a parallel pair in the frame, one of the two edges of the parallel pair.

The following is an immediate consequence of Lemmas 2.7 and 2.11.

COROLLARY 2.13. *Let $G \in \mathcal{T}(\mathcal{S})$. For every edge e of G,* $\mathrm{cr}(G - e) < 2$. ■

In Theorem 5.5, we complete the proof that each graph G in $\mathcal{T}(\mathcal{S})$ is 2-crossing-critical by proving there that $\mathrm{cr}(G) \geq 2$.

We are now able to state the central result of this work.

THEOREM 2.14. *If G is a 3-connected 2-crossing-critical graph containing a subdivision of V_{10}, then $G \in \mathcal{T}(\mathcal{S})$.*

This theorem is proved in the course of Chapters 3 – 13. We remark that not every graph in $\mathcal{T}(\mathcal{S})$ contains a subdivision of V_{10}.

CHAPTER 3

Moving into the projective plane

It turns out that considering the relation of a 2-crossing-critical graph to its embeddability in the projective plane is useful. This perspective was employed by Richter to determine all eight cubic 2-crossing-critical graphs [**29**]. It is a triviality that, if G has a 1-drawing, then G embeds in the projective plane (put the crosscap on the crossing). Therefore, any graph G that does not embed in the projective plane has crossing number at least 2. Moreover, Archdeacon [**1, 2**] proved that it contains one of the 103 graphs that do not embed in the projective plane but every proper subgraph does. Each obstruction for projective planar embedding has crossing number at least 2. Of these, only the ones in Figure 3.1 are 3-connected and 2-crossing-critical. (The non-projective planar graphs that are not 3-connected are found by different means in Section 14.) These are the ones labelled — left to right, top to bottom — D17, E20, E22, E23, E26, F4, F5, F10, F12, F13, and G1 in Glover, Huneke, and Wang [**15**].

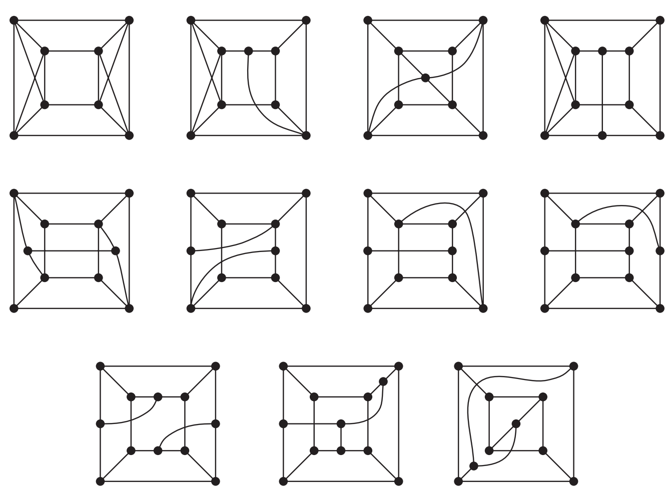

FIGURE 3.1. The 3-connected, 2-crossing-critical graphs that do not embed in $\mathbb{R}P^2$.

DEFINITION 3.1. Let G be a graph embedded in a (compact, connected) surface Σ. Then:

(1) the *representativity* $\operatorname{rep}(G)$ of G is the largest integer n so that every non-contractible, simple, closed curve in Σ intersects G in at least n points (this parameter is undefined when Σ is the sphere);

(2) G is *n-representative* if $n \geq r(G)$;

(3) G is *embedded with representativity* n if $\operatorname{rep}(G) = n$.

Representativity is also known as *face-width* and gained notoriety in the Graph Minors project of Robertson and Seymour. We only require very elementary aspects of this parameter; the reader is invited to consult [**12**] or [**26**] for further information on representativity and Graph Minors.

Barnette [**4**] and Vitray [**37**] independently proved that every 3-representative embedding in the projective plane topologically contains one of the 15 graphs ([**37**, Figure 2.2]). Vitray pointed out in a conference talk [**38**] that each of these 15 graphs has crossing number at least 2. Therefore, any graph that has a 3-representative embedding in the projective plane has crossing number at least 2. One immediate conclusion is that there are only finitely many 2-crossing-critical graphs that embed in $\mathbb{R}P^2$ and do not have a representativity at most 2 embedding in $\mathbb{R}P^2$, and, not only are there only finitely many of these, but they are all known and are shown in Figure 3.2. Vitray went on to show that the only 2-crossing-critical graph whose crossing number is not equal to 2 is $C_3 \square C_3$, whose crossing number is 3.

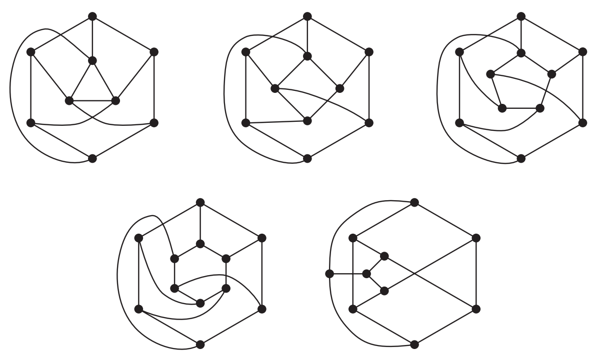

FIGURE 3.2. The 2-crossing-critical 3-representative embeddings in $\mathbb{R}P^2$.

Since every graph that has an embedding in the projective plane with representativity at most 1 is planar, it remains to explore those 2-crossing-critical graphs that have an embedding in $\mathbb{R}P^2$ with representativity precisely 2. To cement some terminology and notation, we have the following.

DEFINITION 3.2. Let $n \geq 3$ be an integer. The graph V_{2n} is the *Möbius ladder* consisting of:

- the *rim* R of V_{2n}, which is a $2n$-cycle $(v_0, v_1, v_2, \dots, v_{2n-1}, v_0)$; and,
- for $i = 0, 1, 2, \dots, n-1$, *the spoke* $v_i v_{n+i}$.

Suppose $V_{2n} \cong H \subseteq G$. (The notation $L \cong H$ means that H is a subdivision of L. Thus, $V_{2n} \cong H \subseteq G$ means H is a subgraph of G and is also a subdivision of V_{2n}.)

- The *H-nodes* are the vertices of H corresponding to v_0, v_1, $\dots$, v_{2n-1} in V_{2n}; the H-nodes are also labelled v_0, v_1, $\dots$, v_{2n-1}.
- For $i = 0, 1, 2, \dots, 2n-1$, the *H-rim branch* r_i is the path in H corresponding to the edge $v_i v_{i+1}$ of V_{2n}.
- For $i = 0, 1, 2, \dots, n-1$, the *H-spoke* is the path s_i in H corresponding to the edge $v_i v_{n+i}$ in V_{2n}.

- We also use H-*rim* and R for the cycle in H corresponding to the rim of V_{2n}.

Whenever we discuss elements of a subdivision H of the Möbius ladder V_{2n}, we presume the indices are read appropriately. For the H-nodes v_k and the H-rim branches r_k, the index k is to be read modulo $2n$. For the H-spokes s_ℓ, the index ℓ is to be read modulo n. Thus, for example, $s_{5+n} = s_5$ and $v_{8+2n} = v_8$, while $r_{8+n} \neq r_8$.

Let G be a 2-crossing-critical graph embedded in $\mathbb{R}P^2$ with representativity 2. Let γ be a simple closed curve in $\mathbb{R}P^2$ meeting G in precisely the two points a and b. We further assume $V_{2n} \cong H \subseteq G$, with $n \geq 3$. Because $G - a$ and $G - b$ have 1-representative embeddings in the projective plane, they are both planar. We note that, for $n \geq 3$, V_{2n} is not planar; therefore, $a, b \in H$.

REMARK 3.3. Throughout this work, we abuse notation slightly. If K is any graph and x is either a vertex or an edge of K, then we write $x \in K$, rather than the technically correct $x \in V(K)$ or $x \in E(K)$. We have taken care so that, in any instance, the reader will never be in doubt about whether x is a vertex or an edge.

If $n \geq 4$, the deletion of a spoke of V_{2n} leaves a non-planar subgraph; thus, when $n \geq 4$, we conclude $a, b \in R$. If γ does not cross R at a, say, then deleting the H-spoke incident with a (if there is one), and shifting γ away from a leaves a subdivision of $K_{3,3}$ in $\mathbb{R}P^2$ that meets the adjusted γ only at b. But then this $K_{3,3}$ has a 1-representative embedding in $\mathbb{R}P^2$, showing $K_{3,3}$ is planar, a contradiction. Therefore, γ must cross R at a and b. As any two non-contractible curves cross an odd number of times, R is contractible and so bounds a closed disc $\mathfrak{D}$ and a closed Möbius strip $\mathfrak{M}$.

Let P and Q be the two ab-subpaths of R, let $\alpha = \gamma \cap \mathfrak{D}$ and $\beta = \gamma \cap \mathfrak{M}$. (We alert the reader that the notations $\mathfrak{D}$, $\mathfrak{M}$, α, β, and γ will be reserved for these objects.) Since each spoke is internally disjoint from γ, the spoke is either contained in $\mathfrak{D}$ or contained in $\mathfrak{M}$. Since the spokes interlace on R, at most one can be embedded in $\mathfrak{D}$.

Moreover, observe that α divides $\mathfrak{D}$ into two regions, one bounded by $P \cup \alpha$ and the other bounded by $Q \cup \alpha$. Thus, if a spoke — label it s_0 — is embedded in $\mathfrak{D}$, then s_0 has both attachments in just one of P and Q, say P. In this case, P contains either all the H-nodes $v_0, v_1, \ldots, v_n$ or all the H-nodes $v_n, v_{n+1}, \ldots, v_{2n-1}, v_0$. It follows that, for $n \geq 4$, there are only two (up to relabelling) representativity 2 embeddings of V_{2n} in the projective plane. See Figure 3.3. We remark that it is possible that one or both of a and b might be an H-node.

We introduce a notation that will be used extensively in this work.

DEFINITION 3.4. The set of 3-connected, 2-crossing-critical graphs is denoted $\mathcal{M}_2^3$.

It is a tedious (and unimportant) exercise to check the observation that none of the graphs in $\mathcal{M}_2^3$ found among the obstructions to having a representativity 2 embedding in $\mathbb{R}P^2$ has a subdivision of V_{10}. We record it in the following assertion.

THEOREM 3.5. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$. Then G has a representativity 2 embedding in $\mathbb{R}P^2$.* ∎

We will also need information about 1-drawings of V_{2n}, for $n \geq 4$. These are similarly straightforward facts that can be proved by considering $K_{3,3}$'s in V_{2n}.

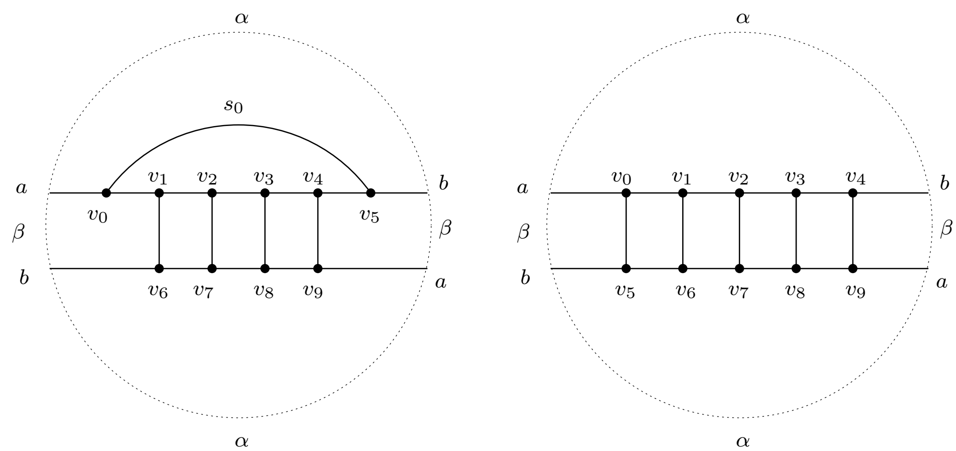

FIGURE 3.3. Standard labellings of the representativity 2 embeddings of V_{10}.

LEMMA 3.6. *Let $n \geq 4$ and let D be a 1-drawing of V_{2n}. Then there is an i so that r_i crosses one of r_{i+n-1}, r_{i+n}, and r_{i+n+1}.* ■

CHAPTER 4

Bridges

The notion of a bridge of a subgraph of a graph is a valuable tool that allows us to organize many aspects of this work. This section is devoted to their definition and an elucidation of their properties that are relevant to us. Bridges are discussed at length in [**35**] and, under the name J-components, in [**34**].

DEFINITION 4.1. Let G be a graph and let H be a subgraph of G.

(1) For a set W of vertices of G, $\|W\|$ consists of the subgraph of G with vertex set W and no edges.
(2) An *H-bridge in G* is a subgraph B of G such that either B is an edge not in H, together with its ends, both of which are in H, or B is obtained from a component K of $G - V(H)$ by adding to K all the edges from vertices in K to vertices in H, along with their ends in H.
(3) For an H-bridge B in G, a vertex u of B is an *attachment of* B if $u \in V(H)$; $\mathrm{att}(B)$ denotes the set of attachments of B.
(4) If B is an H-bridge, then the *nucleus* $\mathrm{Nuc}(B)$ *of* B is $B - \mathrm{att}(B)$.
(5) For $u, v \in V(G)$, a uv-path P in G is *H-avoiding* if $P \cap H \subseteq \|\{u, v\}\|$.
(6) Let A and B be either subsets of $V(G)$ or subgraphs of G. An *AB-path* is a path with an end in each of A and B but otherwise disjoint from $A \cup B$. If, for example, A is the single vertex u, we write uB-path for $\{u\}B$-path.

We will be especially interested in the bridges of a cycle.

DEFINITION 4.2. Let C be a cycle in a graph G and let B and B' be distinct C-bridges.

(1) The *residual arcs* of B in C are the B-bridges in $C \cup B$; if B has at least two attachments, then these are the maximal B-avoiding subpaths of C.
(2) The C-bridges B and B' *do not overlap* if all the attachments of B are in the same residual arc of B'; otherwise, they *overlap*.
(3) The *overlap diagram* $OD(C)$ of C has as its vertices the C-bridges; two C-bridges are adjacent in $OD(C)$ precisely when they overlap.
(4) The cycle C has *bipartite overlap diagram*, denoted BOD, if $OD(C)$ is bipartite; otherwise, C has *non-bipartite overlap diagram*, denoted $NBOD$.

The following is easy to see and well-known.

LEMMA 4.3. *Let C be a cycle in a graph G. The distinct C-bridges B and B' overlap if and only if either:*

(1) there are attachments u, v of B and u', v' of B' so that the vertices u, u', v, v' are distinct and occur in this order in C (in which case B and B' are skew C-bridges*); or*
(2) $\mathrm{att}(B) = \mathrm{att}(B')$ *and* $|\mathrm{att}(B)| = 3$ *(in which case B and B' are* 3-equivalent*).* ■

The following concept plays a central role through the next few sections of this work.

Definition 4.4. Let C be a cycle in a graph G and let B be a C-bridge. Then B is a *planar C-bridge* if $C \cup B$ is planar. Otherwise, B is a *non-planar C-bridge*.

Note that there is a difference between $C \cup B$ being planar and, in some embedding of G in $\mathbb{R}P^2$, $C \cup B$ being *plane*, that is, embedded in some closed disc in $\mathbb{R}P^2$. If $C \cup B$ is plane, then B is planar, but the converse need not hold.

We now present the major embedding and drawing results that we shall use. The theorem is due to Tutte, while the corollary is the form that we shall frequently use.

Theorem 4.5. [**35**, Theorems XI.48 and XI.49] *Let G be a graph.*

(1) G is planar if and only if either G is a forest or there is a cycle C of G having BOD and all C-bridges planar.
(2) G is planar if and only if, for every cycle C of G, C has BOD. ■

For the corollary, we need the following important notion.

Definition 4.6. Let H be a subgraph of a graph G and let D be a drawing of G in the plane. Then H *is clean in* D if no edge of H is crossed in D.

Corollary 4.7. *Let G be a graph and let C be a cycle with BOD. If there is a C-bridge B so that every other C-bridge is planar and there is a 1-drawing of $C \cup B$ in which C is clean, then* $\mathrm{cr}(G) \leq 1$.

Proof. Let $\times$ denote the crossing in a 1-drawing D of $C \cup B$ in which C is clean. As C is not crossed in D, $\times$ is a crossing of two edges of B. Let $G^\times$ denote the graph obtained from G by deleting those two edges and adding a new vertex adjacent to the four ends of the deleted edges. Then C has BOD in $G^\times$ and every C-bridge in $G^\times$ is planar. By Theorem 4.5 (2), $G^\times$ is planar. Any planar embedding of $G^\times$ easily converts to a 1-drawing of G. ■

We will also need the following result.

Lemma 4.8 (Ordering Lemma). *Let G be a graph, C a cycle in G, $\mathcal{B}$ a set of non-overlapping C-bridges. Let P and Q be disjoint paths in C, with $V(C) = V(P \cup Q)$. Suppose that each $B \in \mathcal{B}$ has at least one attachment in each of P and Q. Let P_B and Q_B be the minimal subpaths of P and Q, respectively, containing $P \cap B$ and $Q \cap B$, respectively. Then:*

(1) the $\{P_B\}$ and $\{Q_B\}$ are pairwise internally disjoint and there is an ordering

$$(B_1, \dots, B_k)$$

of $\mathcal{B}$ so that

$$P = P_{B_1} \dots P_{B_2} \dots P_{B_i} \dots P_{B_k} \qquad \textit{and} \qquad Q = Q_{B_1} \dots Q_{B_2} \dots Q_{B_i} \dots Q_{B_k}\,;$$

and

(2) if, for each $B, B' \in \mathcal{B}$, $\mathrm{att}(B) \neq \mathrm{att}(B')$, the order is unique up to inversion.

Proof. Suppose $B, B' \in \mathcal{B}$ are such that P_B and $P_{B'}$ have a common edge e. Then B and B' have attachments x_1, x_2, x'_1, x'_2 in both components of $P - e$ and attachments x, x' in Q. If $|\{x_1, x'_1, x_2, x'_2, x, x'\}| = 3$, then they have 3 common attachments and so overlap, a contradiction. Otherwise, some $y \in \{x'_1, x'_2, x'\}$ is not in $\{x_1, x_2, x\}$. Then y is in one residual arc A of x_1, x_2, x in C and not both of the other two of $\{x'_1, x'_2, x'\}$ are in A. So again B, B' overlap, a contradiction from which we conclude P_B and $P_{B'}$ are internally disjoint.

Let $C = P^{-1}R_1QR_2$. Suppose $B, B' \in \mathcal{B}$ are such that $P = \ldots P_B \ldots P_{B'} \ldots$ and $Q = \ldots Q_{B'} \ldots Q_B \ldots$. We claim that either $P_B = P_{B'}$ or $Q_B = Q_{B'}$. If not, then there is an attachment u_P of one of B and B' in P that is not an attachment of the other and likewise an attachment u_Q of one of B and B' in Q that is not an attachment of the other. Note that u_P and u_Q are not attachments of the same one of B and B', as otherwise the orderings in P and Q imply B and B' overlap.

For the sake of definiteness, we assume $u_P \in \mathrm{att}(B)$, so that $u_Q \in \mathrm{att}(B')$. Let $w_P \in \mathrm{att}(B') \cap P$ and let $w_Q \in \mathrm{att}(B) \cap Q$. The ordering of B and B' in P and Q imply that, in C, these vertices appear in the cyclic order w_P, u_P, u_Q, w_Q. Since u_P, u_Q, w_P, w_Q are all different, we conclude that B and B' overlap on C, a contradiction.

It follows that, by symmetry, we may assume $P_B = P_{B'}$. As P_B and $P_{B'}$ are internally disjoint, they are just a vertex. So if $P = \ldots P_B \ldots P_{B'} \ldots$ and $Q = \ldots Q_{B'} \ldots Q_B \ldots$, we may exchange P_B and P'_B, to see that $P = \ldots P_{B'} \ldots P_B \ldots$ and $Q = \ldots Q_{B'} \ldots Q_B \ldots$. We conclude there is an ordering of $\mathcal{B}$ as claimed.

Let $(B_1, \ldots, B_k)$ and $(B_{\pi(1)}, \ldots, B_{\pi(k)})$ be distinct orderings so that $P = P_{B_1}, \ldots, P_{B_k}$, $P = P_{B_{\pi(1)}}, \ldots, P_{B_{\pi(k)}}$, $Q = Q_{B_1} \ldots Q_{B_k}$ and $Q = Q_{B_{\pi(1)}}, \ldots, Q_{B_{\pi(k)}}$. There exist $i < j$ so that $\pi(i) > \pi(j)$. We may choose the labelling (P versus Q) so that the preceding argument implies that $P_{B_i} = P_{B_j} = u$. If $Q_{B_i} = Q_{B_j}$, then $Q_{B_i} = Q_{B_j} = w$ and $\mathrm{att}(B_i) = \mathrm{att}(B_j)$, which is (2). Therefore, we may assume there is an attachment y of one of B_i and B_j that is not an attachment of the other. Let z be an attachment of the other. Since Q is either (Q_1, y, Q_2, z, Q_3) or $(Q_3^{-1}, z, Q_2^{-1}, y, Q_1^{-1})$, the only possibility is that π is the inversion $(k, k-1, \ldots, 1)$.
■

CHAPTER 5

Quads have BOD

There are two main results in this section. One is to show that each graph in the set $\mathcal{T}(\mathcal{S})$ is 2-crossing-critical and the other, rather more challenging and central to the characterization of 3-connected 2-crossing-critical graphs with a subdivision of V_{10}, is to show that all H-quads and some H-hyperquads have BOD. We start with the definition of quads and hyperquads.

DEFINITION 5.1. Let G be a graph and $V_{10} \cong H \subseteq G$.

(1) For a path P and distinct vertices u and v in P, $[uPv]$ denotes the uv-subpath of P, while $[uPv\rangle$ denotes $[uPv] - v$, $\langle uPv]$ is $[uPv] - u$, and $\langle uPv\rangle$ is $\langle uPv] - v$.
(2) When concatenating a uv-path P with a vw-path Q, we may write either PQ or $[uPvQw]$. If $u = w$ and P and Q are internally disjoint, then both PQ and $[uPvQu]$ are cycles. The reader may have to choose the appropriate direction of traversal of either P or Q in order to make the concatenation meaningful.
(3) If L is a subgraph of G and P is a path in G, then $L - \langle P\rangle$ is obtained from L by deleting all the edges and interior vertices of P. (In particular, this includes the case P has length 1, in which case $L - \langle P\rangle$ is just L less one edge.)
(4) For $i = 0, 1, 2, 3, 4$, the H-quad Q_i is the cycle $r_i\, s_{i+1}\, r_{i+5}\, s_i$.
(5) For $i = 0, 1, 2, 3, 4$, the *H-hyperquad* $\overline{Q}_i$ is the cycle $(Q_{i-1} \cup Q_i) - \langle s_i\rangle$.
(6) The *Möbius bridge of* Q_i is the Q_i-bridge M_{Q_i} in G such that $H \subseteq Q_i \cup M_{Q_i}$.
(7) The *Möbius bridge of* $\overline{Q}_i$ is the $\overline{Q}_i$-bridge $M_{\overline{Q}_i}$ in G for which $(H - \langle s_i\rangle) \subseteq \overline{Q}_i \cup M_{\overline{Q}_i}$.

The following notions will help our analysis.

DEFINITION 5.2. Let G be a graph, $V_{2n} \cong H \subseteq G$, $n \geq 3$, and let K be a subgraph of G. Then:

(1) a *claw* is a subdivision of $K_{1,3}$ with *centre* the vertex of degree 3 and *talons* the vertices of degree 1;
(2) an $\{x, y, z\}$-*claw* is a claw with talons x, y, and z;
(3) an *open H-claw* is the subgraph of H obtained from a claw in H consisting of the three H-branches incident with an H-node, which is the *centre* of the open H-claw, but with the three talons deleted;
(4) K is *H-close* if $K \cap H$ is contained either in a closed H-branch or in a open H-claw.
(5) A cycle C in K is a *K-prebox* if, for each edge e of C, $K - e$ is not planar.

The following is elementary but not trivial.

LEMMA 5.3. *Let C be an H-close cycle, for some $H \cong V_6$. Then C is a $(C \cup H)$-prebox.*

Proof. For $e \in E(C)$, if $e \notin H$, then evidently $(C \cup H) - e$ contains H, which is a V_6; therefore $(C \cup H) - e$ is not planar. So suppose $e \in H$. Since C is H-close, $C \cap H$ is contained in either a closed H-branch b or an open H-claw Y. There is an H-avoiding path P in $C - e$ having ends in both components of either $b - e$ or $Y - e$. In the former case, $(H - e) \cup P$, and hence $(C \cup H) - e$, contains a V_6. In the latter case, $(Y - e) \cup P$ contains a different claw that has the same talons as Y, so again $(H - e) \cup P$, and $(C \cup H) - e$, contains a V_6. ■

LEMMA 5.4. *Let K be a graph and C a cycle of K. If C is a K-prebox, then, in any 1-drawing of K, C is clean.*

Proof. Let D be a 1-drawing of K and let e be any edge of C. Since $K - e$ is not planar, $D(K - e)$ has a crossing. It must be the only crossing of $D(K)$ and, therefore, e is not crossed in $D(K)$. ■

We can now show that any of the tiled graphs described in Section **??** in fact have crossing number 2, thereby completing the proof that they are all 2-crossing-critical.

THEOREM 5.5. *If $G \in \mathcal{T}(\mathcal{S})$, then $G \in \mathcal{M}_2^3$.*

Proof. By Lemmas 2.7 and 2.11 and Corollary 2.8, we know that if K is a proper subgraph of G, then $\mathrm{cr}(K) \leq 1$. Thus, it suffices to prove that $\mathrm{cr}(G) \geq 2$.

There are two edges in a tile that are not in the corresponding picture and are not part of a parallel pair. An edge of G is a Δ-*base* if it is one of these edges. A Δ-*cycle* is a face-bounding cycle in the natural projective planar embedding of G containing precisely one Δ-base. Recall that the rim R of G is described in Definition 2.12.

There are at least three Δ-cycles contained in G and any two are totally disjoint. From each Δ-cycle we choose either of its RR-paths (by definition, these are R-avoiding) as a "spoke", and, with R as the rim, we find 8 different subdivisions of V_6. There are two of these that are edge-disjoint on the spokes, so if D is a 1-drawing of G, the crossing must involve two edges of R.

CLAIM 1. If e is a rim edge in one of the 13 pictures, then e is in an H'-close cycle C_e, for some $H' \cong V_6$ in G.

The point of this is that Lemmas 5.3 and 5.4 imply that C_e is clean in D. This is also obviously true for the other edges of the rim that are in digons. The conclusion is that we know the two crossing edges must be from among the Δ-bases. We shall show below that no two of these can cross in a 1-drawing of G, the desired contradiction.

Proof of Claim 1. Let e be in edge in the rim R of G that is in the picture T, let r be the component of $T \cap R$ containing e, and let r' be the other component of $T \cap R$. There is a unique cycle in $T - r'$ containing e; this is the cycle C_e. Let e' be the one of the two Δ-bases incident with T that has an end in r. Choose the RR-subpath of the e'-containing Δ-cycle that is disjoint from r. For any other two of the Δ-cycles, choose arbitrarily one of the RR-subpaths. These three "spokes", together with R, constitute a subdivision H' of V_6 for which C_e is H'-close, as required. □

The proof is completed by showing that no two Δ-bases can cross in a 1-drawing of G. If there are at least five tiles, then it is easy to find a subdivision of V_8 so that the two Δ-bases are on disjoint H-quads and therefore cannot be crossed in a 1-drawing of G. Thus, we may assume there are precisely three tiles and the crossing Δ-bases e_1 and e_2 are, therefore, in consecutive Δ-cycles.

Let T be the picture incident with both e_1 and e_2. Choose a subdivision H' of V_6 containing R but so that $T \cap H' = T \cap R$. There is a unique 1-drawing D of H' with e_1 and e_2 being the crossing pair. For $i = 1, 2$, let the H'-branch containing e_i be b_i. The end u_i of e_i that is in T is in the interior of b_i.

The vertices u_1 and u_2 are two of the four attachments of T in G. Let w_1 and w_2 be the other two, labelled so that w_1 is in the same component of $T \cap R$ as u_2. It follows that w_2 is in the same component of $T \cap R$ as u_1. In T, there is a unique pair of totally disjoint R-avoiding u_1w_1- and u_2w_2-paths P_1 and P_2, respectively. The crossing in D is of e_1 with e_2, so $[u_1b_1w_2]$ and $[u_2b_2w_1]$ are both not crossed in D. Therefore, $D[P_1]$ and $D[P_2]$ are both in the same face F of D.

Since the two paths P_1 and P_2 are totally disjoint(text deleted), $D[P_1]$ and $D[P_2]$ are disjoint arcs in F; the contradiction arises from the fact that their ends alternate in the boundary of F, showing there must be a second crossing. ■

One important by-product of cleanliness is that it frequently shows a cycle has BOD.

Lemma 5.6. *Let C be a cycle in a graph G. Let D be a 1-drawing of G in which C is clean. If there is a non-planar C-bridge, then C has BOD and exactly one non-planar bridge.*

Proof. Let B be a non-planar C-bridge. Then $D[C \cup B]$ has a crossing, and, since C is clean in D, the crossing does not involve an edge of C. Therefore, it involves two edges of B. This is the only crossing of D, so inserting a vertex at this crossing turns D into a planar embedding of a graph $G^\times$. As C is still a cycle of $G^\times$, C has BOD in $G^\times$ and all C-bridges in $G^\times$ are planar. But $OD_{G^\times}(C)$ is the same as $OD_G(C)$ and all C-bridges other than B are the same in G and $G^\times$. ■

We shall routinely make use of the following notions.

Definition 5.7. Let G be a connected graph and let H be a subgraph of G. Then:

(1) $H^\#$ is the subgraph of G induced by $E(G) \setminus E(H)$; and
(2) if G is embedded in $\mathbb{R}P^2$, then an *H-face* is a face of the induced embedding of H in $\mathbb{R}P^2$.

We will often use this when B is a C-bridge, for some cycle C in a graph G, in which case $B^\#$ is the union of C and all C-bridges other than B. The following two lemmas are useful examples.

Lemma 5.8. *Let G be a graph embedded in $\mathbb{R}P^2$ with representativity 2 and let γ be a non-contractible curve in $\mathbb{R}P^2$ so that $G \cap \gamma = \{a, b\}$. Let C be a contractible cycle in G and let B be a C-bridge so that* $\mathrm{Nuc}(B) \cap \{a, b\} \neq \varnothing$. *Then $B^\#$ is planar.*

Proof. This is straightforward: $B^{\#} = G - \mathrm{Nuc}(B) \subseteq G - (\{a, b\} \cap \mathrm{Nuc}\, B)$ and the latter has a representativity at most 1 embedding in $\mathbb{R}P^2$. Therefore it is planar. ■

The following result, when combined with the (not yet proved) fact that H-quads and some H-hyperquads have BOD, yields the fact, often used in the sections to follow, that deleting some edge results in a 1-drawing in which a particular H-quad or H-hyperquad must be crossed.

LEMMA 5.9. *Let G be a graph with $\mathrm{cr}(G) \geq 2$ and let C be a cycle in G. If C has BOD in G, then, for any planar C-bridge B, C is crossed in any 1-drawing of $B^{\#}$.*

Proof. Suppose there is a 1-drawing D of $B^{\#}$ with C clean. Since C has BOD and G is not planar, there is a non-planar C-bridge B'. Because C is clean, any crossing in $D[C \cup B']$ involves two edges of B'. The only crossing in D involves two edges of B', so every other C-bridge in $B^{\#}$ is planar. Since B is planar, it follows from Corollary 4.7 that $\mathrm{cr}(G) \leq 1$, a contradiction. ■

We remark that M_Q is a non-planar Q-bridge whenever Q is an H-quad or H-hyperquad.

COROLLARY 5.10. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$. If the H-quad Q_i and H-hyperquad $\overline{Q}_j$ are disjoint, $\overline{Q}_j$ has BOD, and there is a planar $\overline{Q}_j$-bridge B, then Q_i has BOD and there is precisely one non-planar Q_i-bridge.*

Proof. Let B be a planar $\overline{Q}_j$-bridge. Because G is 2-crossing-critical, there is a 1-drawing D of $B^{\#}$. By Lemma 5.9, $\overline{Q}_j$ is crossed in D. Note that $H - \langle s_j \rangle \subseteq B^{\#}$. In any 1-drawing of $H - \langle s_j \rangle$ in which $\overline{Q}_j$ is crossed, the crossing is between $r_{j-2} \cup r_{j-1} \cup r_j \cup r_{j+1}$ and $r_{n+j-2} \cup r_{n+j-1} \cup r_{n+j} \cup r_{n+j+1}$. Since Q_i is edge-disjoint from these crossing rim segments, Q_i is clean in D.

The two graphs $OD_G(Q_i)$ and $OD_{B^{\#}}(Q_i)$ are isomorphic: the Q_i-bridges in both G and $B^{\#}$ are the same, except M_{Q_i} in G becomes $M_{Q_i} - \mathrm{Nuc}(B)$ in $B^{\#}$ and they have the same attachments. Since Q_i is clean in D, $OD_{B^{\#}}(Q_i)$ is bipartite. Furthermore, the crossing in D is between two edges of $\overline{Q}_j$, so D shows that every Q_i-bridge other than M_{Q_i} is planar. ■

We next introduce boxes, which are cycles that, it turns out, cannot exist in a 2-crossing-critical graph G. On several occasions in the subsequent sections, we prove a result by showing that otherwise G has a box.

DEFINITION 5.11. Let C be a cycle in a graph G. Then C is a *box* in G if C has BOD in G and there is a planar C-bridge B so that C is a $B^{\#}$-prebox.

LEMMA 5.12. *Let $G \in \mathcal{M}_2^3$. Then G has no box.*

Proof. Suppose C is a box in G. Then C has BOD and there is a planar C-bridge B so that C is a $B^{\#}$-prebox. As $B^{\#}$ is a proper subgraph of G, there is a 1-drawing D of $B^{\#}$. By Lemma 5.4, $D[C]$ is clean. This contradicts Lemma 5.9. ■

We can now determine the complete structure of a 2-connected H-close subgraph.

LEMMA 5.13. *Let $G \in \mathcal{M}_2^3$ and $V_{2n} \cong H \subseteq G$ with $n \geq 4$. If K is a 2-connected H-close subgraph of G, then K is a cycle.*

Proof. If $K \cap H$ consists of at least two vertices, then we include in K the minimal connected subgraph of the H-branch or open H-claw containing $K \cap H$. Since K is H-close, there is a K-bridge M_K in G so that $H \subseteq K \cup M_K$. Let e be an edge of any H-spoke totally disjoint from K. Note that $M_K - e$ is a K-bridge in $G - e$ and that M_K has the same attachments in G as $M_K - e$ has in $G - e$.

Since K is 2-connected, every edge of K is in an H-close cycle contained in K. Thus, for any 1-drawing D of $G - e$, Lemmas 5.3 and 5.4 imply that $D[K]$ is clean. There is a face F of $D[K]$ containing $D[M_K - e]$. As $D[K]$ is clean and K is 2-connected, F is bounded by a cycle C of K.

Lemma 5.3 implies the cycle C is a $(C \cup H)$-prebox. If K is not just C, then there is a C-bridge B contained on the side of $D[C]$ disjoint from M_K. Evidently B is a planar C-bridge.

Lemma 5.6 implies C has BOD. Since C is a $(C \cup H)$-prebox, C is a $B^{\#}$-prebox. We conclude that C is a box, contradicting Lemma 5.12. This shows that $K = C$. ■

The second of the following two corollaries is used several times later in this work. We recall from Definition 4.1 that, for a set W of vertices, $\|W\|$ is the subgraph with vertex set W and no edges.

COROLLARY 5.14. *Let $G \in \mathcal{M}_2^3$, let $V_{2n} \cong H \subseteq G$ with $n \geq 4$, let B be an H-bridge.*

(1) *If $x, y \in \mathrm{att}(B)$ are such that $\|\{x, y\}\|$ is H-close, then there is a unique H-avoiding xy-path in G.*
(2) *There do not exist vertices $x, y, z \in \mathrm{att}(B)$ so that $\|\{x, y, z\}\|$ is H-close.*

Proof. Suppose P_1 and P_2 are distinct H-avoiding xy-paths. There is either a closed H-branch or an open H-claw containing an xy-path; this subgraph of H contains a unique xy-path P. Then $P \cup P_1 \cup P_2$ is a 2-connected H-close subgraph of G and so, by Lemma 5.13, is a cycle. But it contains three distinct xy-paths, a contradiction.

For the second point, suppose by way of contradiction that such x, y, z exist. Let Y be an $\{x, y, z\}$-claw in B. There is a minimal connected subgraph Z of H contained either in a closed H-branch or in an open H-claw and containing x, y, and z. We note that Z is either a path or an $\{x, y, z\}$-claw. Thus, $Y \cup Z$ is 2-connected and is H-close. It is a cycle by Lemma 5.13, but the centre of Y has degree 3 in $Y \cup Z$, a contradiction. ■

COROLLARY 5.15. *Let $G \in \mathcal{M}_2^3$, let $V_{10} \cong H \subseteq G$, and let B be a Q-local H-bridge, for some H-quad Q. If s is an H-spoke and r is an H-rim branch, both contained in Q, then $|\mathrm{att}(B) \cap s| \leq 2$ and $|\mathrm{att}(B) \cap (Q - [r])| \leq 2$.*

Proof. The first claim follows immediately from Corollary 5.14. For the second, suppose there are three such attachments x, y, and z. Corollary 5.14 implies they are not all in the other H-rim branch r' of Q, so at least one of x, y, and z is in the interior of some H-spoke of Q.

Suppose first that some H-spoke s in Q is such that $\langle s \rangle \cap \{x, y, z\} = \varnothing$. Then let $H' = H - \langle s \rangle$, let B' be the H'-bridge containing B, and let r' and s' be the two H-branches in Q other than r and s. Then x, y, and z are all attachments of B' and they are all in the same open H'-claw containing $(r' \cup s') - r$, contradicting Corollary 5.14.

Otherwise, we may suppose both H-spokes s and s' in Q have one of x, y, and z in their interiors. We may suppose s has no other one of x, y and z. Choose the labelling so that $x \in \langle s \rangle$. Let r' be the H-rim branch in Q other than r and again let $H' = H - \langle s \rangle$ and B' be the H'-bridge containing B. Then y and z are attachments of B', as is the H-node in $s \cap r'$. But now these three attachments of B' contradict Corollary 5.14. ∎

We want to find cycles having BOD in our $G \in \mathcal{M}_2^3$ that is embedded with representativity 2 in the projective plane. The following will be helpful.

Lemma 5.16. *Let G be a graph embedded in $\mathbb{R}P^2$ and let C be a contractible cycle in G. Suppose B is a C-bridge so that $C \cup B$ has no non-contractible cycles and let F be the C-face containing B. If B' is another C-bridge embedded in F, then B and B' do not overlap on C.*

Proof. Let x and y be any distinct attachments of B and let P be a C-avoiding xy-path in B. Then $C \cup P$ has three cycles, all contractible by hypothesis. We claim that one bounds a closed disc Δ so that $C \cup P \subseteq \Delta$. If P is contained in the disc Δ bounded by C, then we are done. In the remaining case, let C' be one of these cycles containing P. If the closed disc Δ' bounded by C' contains C, then we are done. Otherwise, $\Delta \cap \Delta'$ is a path in C and then $\Delta \cup \Delta'$ is the desired closed disc.

As no other C-bridge in F can have attachments in the interiors of both the two xy-subpaths of C and, therefore, there is no C-bridge embedded in F that is skew (see Lemma 4.3 (1)) to B.

Likewise, if x, y, z are three distinct attachments of B, then there is a disc Δ' containing the union of C with a C-avoiding $\{x, y, z\}$-claw in B. This disc shows that no other C-bridge embedded in F can have all of x, y, z as attachments and, therefore, no C-bridge embedded in F is 3-equivalent (see Lemma 4.3 (2)) to B. ∎

The following is an immediate consequence of Lemma 5.16 and the fact that C has only two faces.

Corollary 5.17. *Let G be a graph embedded in $\mathbb{R}P^2$ and let C be a cycle of G bounding a closed disc in $\mathbb{R}P^2$. If at most one C-bridge B is such that $C \cup B$ contains a non-contractible cycle, then C has BOD and, for every other C-bridge B', $C \cup B'$ is planar.* ∎

The following result is surprisingly useful in later sections.

Lemma 5.18. *Let $G \in \mathcal{M}_2^3$ and suppose G is embedded with representativity 2 in the projective plane. Let γ be a non-contractible curve in the projective plane so that $|\gamma \cap G| = 2$ and let C be a cycle of G so that $\gamma \cap C = \varnothing$. If there is a non-planar C-bridge B, then $\gamma \cap G \subseteq B$, C has BOD, and, for every other C-bridge B', $C \cup B'$ is planar.*

Proof. Let a and b be the two points in $\gamma \cap G$. We note that $G - a$ and $G - b$ are planar, as they have representativity 1 embeddings in $\mathbb{R}P^2$. Thus, if, for example, $a \notin B$, then $C \cup B \subseteq G - a$ and so $C \cup B$ is planar, a contradiction.

If B' is any other C-bridge, then $a, b \notin C \cup B'$ and, therefore, $C \cup B'$ is disjoint from γ. Since any non-contractible cycle must intersect γ, $C \cup B'$ has no non-contractible cycles. The result is now an immediate consequence of Corollary 5.17. ■

Here is a simple result that we occasionally use.

LEMMA 5.19. *Suppose $G \in \mathcal{M}_2^3$ and $V_{2n} \cong H \subseteq G$, with $n \geq 4$. Let B be an H-bridge.*

(1) Then $|\mathrm{att}(B)| \geq 2$.
(2) If $|\mathrm{att}(B)| = 2$, then B is isomorphic to K_2.
(3) If $|\mathrm{att}(B)| = 3$, then B is isomorphic to $K_{1,3}$.

Proof. Note that $\mathrm{att}(B) = B \cap B^{\#}$ and $G = B \cup B^{\#}$. If $|\mathrm{att}(B)| \leq 1$, then G is not 2-connected. If $|\mathrm{att}(B)| = 2$ and $\mathrm{Nuc}(B)$ has a vertex, then G is not 3-connected.

Now suppose $|\mathrm{att}(B)| = 3$ and B is not isomorphic to $K_{1,3}$. Let Y be an $\mathrm{att}(B)$-claw contained in B. As $B^{\#} \cup Y$ is a proper subgraph of G, it has a 1-drawing D_1; Y is clean in D_1, as H must be self-crossed. On the other hand, if s is an H-spoke disjoint from B, there is a 1-drawing D_2 of $G - \langle s \rangle$. Again, the crossing in D_2 involves two edges of $H - \langle s \rangle$, so B is clean. We can substitute $D_2[B]$ for $D_1[Y]$ to convert D_1 into a 1-drawing of G, a contradiction. ■

The following lemma is the last substantial one we need before proving that every H-quad has BOD.

LEMMA 5.20. *Let G be a graph that is embedded in $\mathbb{R}P^2$ and let C be a cycle of G. Let B be a C-bridge so that $\mathrm{Nuc}(B)$ contains a non-contractible cycle. Then C is contractible, C has BOD, and every C-bridge other than B is planar.*

Proof. Let N be a non-contractible cycle in $\mathrm{Nuc}(B)$ and let B' be a C-bridge different from B. Then $C \cup B'$ is disjoint from N. Since any two non-contractible cycles in $\mathbb{R}P^2$ intersect, $C \cup B'$ does not contain a non-contractible cycle. Clearly this implies C is contractible and the remaining items are an immediate consequence of Corollary 5.17. ■

We prove below that every H-quad has BOD and that at least two hyperquads have BOD. A standard labelling of the embedded V_{10} will help make the details of the statement comprehensible. We have seen that, up to relabelling, there are two representativity 2 embeddings of V_{10} in $\mathbb{R}P^2$. There is a simple non-contractible curve γ in $\mathbb{R}P^2$ meeting G in two points a and b. These are both in the rim R of H and either none or one of the H-spokes is outside the Möbius band $\mathfrak{M}$ bounded by R. Let α and β be the two ab-subarcs of γ, labelled so that $\beta \subseteq \mathfrak{M}$.

DEFINITION 5.21. Let G be a graph and let $V_{10} \cong H \subseteq G$. If G is embedded in $\mathbb{R}P^2$ so that one H-spoke is not in $\mathfrak{M}$, then H has an *exposed spoke* and the exposed spoke is the H-spoke not in $\mathfrak{M}$.

In this case, the *standard labelling* is chosen so that the exposed spoke is s_0 and so that $v_0, v_1, v_2, v_3, v_4, v_5$ are all incident with one of the two faces of $H \cup \gamma$ incident with s_0.

The faces of $H \cup \gamma$ are bounded by the cycles:

(1) $[a, r_9, v_0]\, s_0\, [v_5, r_5, b, \alpha, a]$;

(2) $r_0\, r_1\, r_3\, r_3\, r_4\, s_0$;
(3) $[a, r_9, v_0]\, r_0\, s_1\, [v_6, r_5, b, \beta, a]$;
(4) Q_1, Q_2, Q_3;
(5) $r_4\, [v_5, r_5, b, \beta, a, r_9, v_9]\, s_4$; and
(6) $[b, r_5, v_6]\, r_6\, r_7\, r_8\, [v_9, r_9, a, \alpha, b]$.

This case is illustrated in the diagram to the left in Figure 3.3.

In the case all the H-spokes are in $\mathfrak{M}$, the labelling of H may be chosen so that the faces of $H \cup \gamma$ are bounded by:

(1) $[a, r_9, v_0] r_0\, r_1\, r_2\, r_3 [v_4, r_4, b, \alpha, a]$;
(2) $[a, r_9, v_0, s_0, v_5, r_4, b, \beta, a]$;
(3) Q_0, Q_1, Q_2, Q_3;
(4) $[v_4, r_4, b, \beta, a, r_9, v_9, s_4, v_4]$; and
(5) $[b, r_4, v_5]\, r_5\, r_6\, r_7\, r_8\, [v_9, r_9, a, \alpha, b]$.

This case is illustrated in the diagram to the right in Figure 3.3.

We need one more technical lemma before the main result of this section.

Lemma 5.22. *Let $G \in \mathcal{M}_2^3$, let $V_{10} \cong H \subseteq G$, and let $i, j \in \{0, 1, 2, 3, 4\}$ be such that $\overline{Q}_i$ and $\overline{Q}_j$ have precisely one H-spoke in common. If $\overline{Q}_i$ has BOD and s_i is in a planar $\overline{Q}_i$-bridge, then $(M_{\overline{Q}_j})^{\#}$ is planar.*

Proof. Let e be any edge of s_i and let D be a 1-drawing of $G - e$. By Lemma 5.9, $\overline{Q}_i$ is crossed in D. Thus, the crossing of D involves an edge of $M_{\overline{Q}_j}$, showing that $(M_{\overline{Q}_j})^{\#}$ is planar. ■

The following is the main result of this section.

Theorem 5.23. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$. Let G be embedded with representativity 2 in the projective plane, with the standard labelling. Then:*

(1) each H-quad Q of G has BOD and exactly one non-planar bridge;
(2) $\overline{Q}_2$ has BOD;
(3) for each $i \in \{0, 1, 3, 4\}$, $(M_{\overline{Q}_i})^{\#}$ is planar;
(4) if there is an exposed spoke, then $\overline{Q}_3$ has BOD;
(5) if there is no exposed spoke, then at least one of $\overline{Q}_1$ and $\overline{Q}_3$ has BOD.
(6) if there is no exposed spoke and $\overline{Q}_1$ does not have BOD, then there is a $\overline{Q}_1$-bridge B different from $M_{\overline{Q}_1}$ so that $B \subseteq \mathfrak{D}$ and either:
  (a) $a = v_0$ and B has an attachment at a, an attachment in $r_5\, r_6$, and $\mathrm{att}(B) \subseteq \{a\} \cup r_5\, r_6$; or
  (b) $b = v_5$ and B has an attachment at b, an attachment in $r_0\, r_1$, and $\mathrm{att}(B) \subseteq \{b\} \cup r_0\, r_1$. (The analogous statement holds for $\overline{Q}_3$ in place of $\overline{Q}_1$.)

The following definitions will be useful throughout the remainder of this work.

Definition 5.24. Let G be a graph embedded in $\mathbb{R}P^2$ and let C be a cycle of G bounding a closed disc Δ in $\mathbb{R}P^2$. A C-bridge B is *C-interior* if B is contained in Δ and *C-exterior* otherwise.

Proof of Theorem 5.23. We distinguish two cases.

Case 1: *H has an exposed spoke.*

We adopt the standard labelling, so s_0 is the exposed spoke. We note that Q_2 is disjoint from $G\cap\gamma$ and, therefore, Lemma 5.18 implies Q_2 has BOD and precisely one non-planar bridge, which is part of (1).

The arguments for $Q_1, Q_3, \overline{Q}_2, \overline{Q}_3$ are all analogous and so we do $\overline{Q}_2$. Since s_0 is exposed, the cycle $[a, r_9, v_0]\, s_0\, r_4\, s_4 [v_9, r_9, a]$ is not contractible and is disjoint from $\overline{Q}_2$. Lemma 5.20 shows $\overline{Q}_2$ has BOD and precisely one non-planar bridge, proving (2) and (4). We have also proved (3) for $j = 3$ and (1) for Q_1 and Q_3.

To complete the proof of (1) in Case 1, it remains to deal with Q_0 and Q_4. These two cases are symmetric and so it suffices to prove Q_0 has BOD and only one non-planar bridge. We note that $\overline{Q}_3$ is completely disjoint from Q_0 and we have shown that $\overline{Q}_3$ has BOD. Let B be the $\overline{Q}_3$-bridge containing s_3. As $\overline{Q}_3$ is contractible and B is $\overline{Q}_3$-interior, we conclude that B is planar. Therefore, Corollary 5.10 implies Q_0 has BOD, and each Q_0-bridge except M_{Q_0} is planar, as required for (1).

For (3), it remains to prove that, for $j \in \{0, 1, 4\}$, $(M_{\overline{Q}_j})^{\#}$ is planar. We apply Lemma 5.22: for $j = 0$ or 4, we take $i = 2$; for $j = 1$, we take $i = 3$. In all cases, the result follows.

Case 2: *H has no exposed spoke.*

Lemma 5.18 shows Q_1, Q_2, and $\overline{Q}_2$ all have BOD and just one non-planar bridge. This proves (2) and part of (1). We use this in Corollary 5.10 to see that Q_4 has BOD and just one non-planar bridge, another part of (1). Also, taking $i = 2$ and $j \in \{0, 4\}$ in Lemma 5.22, we see that $(M_{\overline{Q}_j})^{\#}$ is planar, part of (3).

If $\overline{Q}_3$ has BOD, then Corollary 5.10 implies Q_0 has BOD, so in order to show Q_0 has BOD, we may assume $\overline{Q}_3$ has NBOD. There is an analogous situation for Q_3 and $\overline{Q}_1$. We first prove (6) for $\overline{Q}_3$; we will use this to prove both Q_0 has BOD and (5).

If $v_4 \neq b$ and $v_9 \neq a$, then Lemma 5.18 shows that $\overline{Q}_3$ has BOD and exactly one non-planar bridge. So suppose either (or both) $v_4 = b$ or $v_9 = a$. If every $\overline{Q}_3$-bridge other than $M_{\overline{Q}_3}$ has only contractible cycles, then $\overline{Q}_3$ has BOD by Corollary 5.17. Thus, some $\overline{Q}_3$-bridge B other than $M_{\overline{Q}_3}$ is such that $\overline{Q}_3 \cup B$ contains a non-contractible cycle. Evidently, B is $\overline{Q}_3$-exterior. If $B \subseteq \mathfrak{M}$, then again $\overline{Q}_3 \cup B$ has only contractible cycles. Thus, $B \subseteq \mathfrak{D}$.

Any $\overline{Q}_3$-exterior bridge B contained in the face of $H \cup \gamma$ bounded by

$$[a, r_9, v_0]\, r_1\, r_2\, r_3 [v_4, r_4, b, \alpha, a]$$

has all its attachments in $\{a\} \cup r_2\, r_3$. Note that B is planar; moreover, if a is not an attachment, then $\overline{Q}_3 \cup B$ has no non-contractible cycle and, therefore, does not overlap any other $\overline{Q}_3$-exterior bridge. We have the analogous conclusions if B is contained in the face of $H \cup \gamma$ bounded by $[b, r_5, v_6]\, r_6\, r_7\, r_8 [v_9, r_9, a, \alpha, b]$.

We conclude that either B has a as an attachment and also has an attachment in $r_2\, r_3$ or, symmetrically, B has b as an attachment and also has an attachment in $r_7\, r_8$. This proves (6).

We now prove (5). If $\{v_0, v_5\} \cap \{a, b\} = \varnothing$, then $\overline{Q}_1$ has BOD and just one non-planar bridge; likewise if $\{v_4, v_9\} \cap \{a, b\} = \varnothing$, then $\overline{Q}_3$ has BOD and just one non-planar bridge. Up to symmetry, the only other possibility is that $v_0 = a$ and $v_4 = b$.

Now suppose that $\overline{Q}_1$ also has NBOD. Then (6) implies that there must be, up to symmetry, a $\overline{Q}_1$-bridge B_1 different from $M_{\overline{Q}_1}$ having attachments at a and in

$r_5\,r_6$. Likewise, there is an H-bridge B_3 different from $M_{\overline{Q}_3}$ having attachments at b and in $r_7\,r_8$. As B_1 cannot have an attachment at b, $B_1 \neq B_3$. Considering the embedding of G in $\mathbb{R}P^2$, we see that both B_1 and B_3 must be embedded in the face of $H\cup\gamma$ incident with $[b, r_4, v_5]\,r_5\,r_6\,r_7\,r_8[v_9, r_9, a, \alpha, b]$. If B_1, say, has an attachment other than a and v_7, then the H-avoiding path in B_3 from b to any attachment in $r_7\,r_8$ crosses B_1, a contradiction. So $\mathrm{att}(B_1) = \{a, v_7\}$, $\mathrm{att}(B_3) = \{b, v_7\}$, and, by Lemma 5.19, both B_1 and B_2 are just edges.

Now recall that $\overline{Q}_2$ has BOD and, letting B_2 be the $\overline{Q}_2$-bridge containing s_2, Lemma 5.9 implies $\overline{Q}_2$ is crossed in a 1-drawing D of $B_2^{\#}$. The crossing must be between the paths $r_0\,r_1\,r_2\,r_3$ and $r_5\,r_6\,r_7\,r_8$.

There are two maximal uncrossed subpaths of R in D and we know that v_0 and v_9 are on one uncrossed segment, say S_1, of R, while v_4 and v_5 are on S_2. Suppose first that v_7 is on S_1. Then the cycle $[v_0, B_1, v_7]\,r_6\,r_5\,r_4\,s_4\,r_0$ separates v_8 from v_3 in D, yielding the contradiction that s_3 is crossed in D. On the other hand, if v_7 is on S_2, then the same cycle separates v_6 from v_1, yielding the contradiction that s_1 is crossed in D.

We conclude that not both $\overline{Q}_1$ and $\overline{Q}_3$ can have NBOD which is (5). By symmetry, we may assume $\overline{Q}_1$ has BOD. Then Lemma 5.22 shows $(M_{\overline{Q}_3})^{\#}$ is planar. Furthermore, Corollary 5.10 implies Q_3 has BOD and precisely one non-planar bridge.

What remains is to prove that Q_0 has BOD and precisely one non-planar bridge and that there is precisely one non-planar $\overline{Q}_1$-bridge. Recall that symmetry implies this will show the same things for Q_3 and $\overline{Q}_3$, completing the proofs of (1) and (3).

From (6), we may assume that $v_9 = a$ and that there is a $\overline{Q}_3$-bridge B_3 attaching at a and in $r_2\,r_3$. Let w be any attachment of B_3 in $r_2\,r_3$, let P be an H-avoiding v_9w-path in B_3, and let Q be the subpath of $r_2\,r_3$ joining w to v_4. Then the cycle $[v_9, P, w, Q, v_4, s_4, v_9]$ is non-contractible in $\mathbb{R}P^2$ and is disjoint from Q_0. By Lemma 5.20, Q_0 has BOD and has just one non-planar bridge.

As for $\overline{Q}_1$, we consider two cases. If $\overline{Q}_3$ has BOD, then Lemma 5.22 implies $(M_{\overline{Q}_1})^{\#}$ is planar. If $\overline{Q}_3$ has NBOD, then (6) implies either $v_9 = a$ or $v_4 = b$. In both cases, $\mathrm{Nuc}(M_{\overline{Q}_1}) \cap \{a, b\} \neq \varnothing$, so Lemma 5.8 implies $(M_{\overline{Q}_1})^{\#}$ is planar, as required. ■

The following technical corollary of Theorem 5.23 and Lemmas 5.6 and 5.9 will be used in a few different places later.

Corollary 5.25. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$. With indices read modulo 5, suppose, $i \in \{0, 1, 2, 3, 4\}$ is such that $\overline{Q}_i$ has BOD and, where $\{j, k\} = \{i+2, i+3\}$, suppose further that $\overline{Q}_j$ has NBOD. Then s_i is in a planar $\overline{Q}_i$-bridge B_i and $\overline{Q}_k$ has BOD. Moreover, if e_i is any edge of B_i and D_i is a 1-drawing of $G - e_i$, then either $r_{i-1}\,r_i$ crosses whichever of r_{i+3} and r_{i+6} is in $\overline{Q}_j$ or $r_{i+4}\,r_{i+5}$ crosses whichever of r_{i-2} and r_{i+1} is in $\overline{Q}_j$.*

The two possibilities for D_i in the case $j = i + 2$ are illustrated in Figure 5.1.

Proof of Corollary 5.25. By way of contradiction, suppose s_i is not in a planar $\overline{Q}_i$-bridge. We observe that s_0 must be exposed, as otherwise we have the contradiction that, for every $\ell \in \{0, 1, 2, 3, 4\}$, s_ℓ is in a planar $\overline{Q}_\ell$-bridge. It follows that, for $\ell \in \{2, 3\}$, s_ℓ is in a planar $\overline{Q}_\ell$-bridge. Thus, $i \notin \{2, 3\}$.

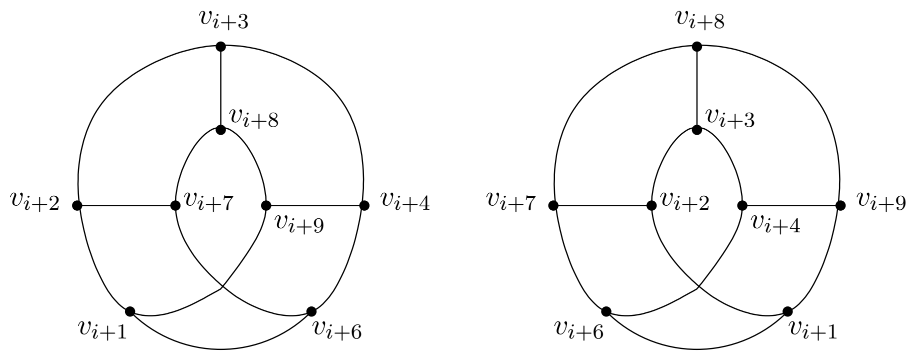

FIGURE 5.1. The two possibilities for D_i when $j = i + 2$.

Let $\ell \in \{2, 3\}$ be such that i and ℓ are not consecutive in the cyclic order $(0, 1, 2, 3, 4)$. Let e_ℓ be the edge of s_ℓ incident with v_ℓ and let D_ℓ be a 1-drawing of $G - e_\ell$. By Lemma 5.9, $\overline{Q}_\ell$ is crossed in D_ℓ.

If $\overline{Q}_\ell$ is self-crossed in D_ℓ, then D_ℓ shows that the $\overline{Q}_i$-bridge containing s_i is planar. Thus, we have that $\overline{Q}_\ell$ is not self-crossed in D_ℓ. One of $s_{\ell-1}$ and $s_{\ell+1}$ is exposed in D_ℓ. If this exposed spoke is not also in $\overline{Q}_i$, then again s_i is in a planar $\overline{Q}_i$-bridge; therefore, we must have that the exposed spoke is in $\overline{Q}_i$. For the sake of definiteness, we assume that $s_{\ell-1}$ is exposed, which implies that $\ell = i + 2$.

As the only non-planar $\overline{Q}_i$-bridge is $M_{\overline{Q}_i}$, we must have an H-avoiding path P from the interior of s_i to the interior of one of $r_{\ell-1}\, r_\ell\, r_{\ell+1}$ and $r_{\ell+4}\, r_{\ell+5}\, r_{\ell+6}$. The drawing D_ℓ restricts the possibility to the interior of one of $r_{\ell-1}\, r_\ell$ and $r_{\ell+4}\, r_{\ell+5}$. But now the embedding in $\mathbb{R}P^2$ implies $i = 0$. This implies $j \in \{2, 3\}$; however, neither $\overline{Q}_2$ nor $\overline{Q}_3$ has NBOD. Therefore, s_i is in a planar $\overline{Q}_i$-bridge.

Because $M_{\overline{Q}_j} - e_i$ and $M_{\overline{Q}_j}$ have the same attachments, $OD_{G-e_i}(\overline{Q}_j)$ and $OD_G(\overline{Q}_j)$ are isomorphic. As the latter is not bipartite, neither is the former. By Lemma 5.6, $\overline{Q}_j$ is not clean in D_i. Thus, either $r_{j-1}\, r_j$ or $r_{j+4}\, r_{j+5}$ is crossed in D_2. These are edge-disjoint from $\overline{Q}_i$.

Lemma 5.9 implies that $\overline{Q}_i$ is also crossed in D_i. Since $\overline{Q}_i$ is crossed and, from the preceding paragraph, something outside of $\overline{Q}_i$ is crossed, either

$$r_{i-1}\, r_i \quad \text{crosses} \quad r_{i+3} \cup r_{i+6}$$

or

$$r_{i+4}\, r_{i+5} \quad \text{crosses} \quad r_{i-2} \cup r_{i+1}\,,$$

as required. ■

Since $\overline{Q}_2$ always has BOD, Corollary 5.25 implies at least one of $\overline{Q}_0$ and $\overline{Q}_4$ has BOD. Together with the fact that, in all cases, at least one of $\overline{Q}_1$ and $\overline{Q}_3$ has BOD, we conclude that at least three of the H-hyperquads have BOD.

The last result in this section will be useful early in the next section.

COROLLARY 5.26. *Let $G \in \mathcal{M}_2^3$ and let $V_{10} \cong H \subseteq G$ and suppose G has a representativity 2 embedding in the projective plane, with the standard labelling. Suppose, for some i, B is an H-bridge having an attachment in both $\langle r_{i-1}\, s_{i-1}\rangle$ and $\langle r_{n+i}\, s_{i+1}\rangle$.*

(1) *if $i \neq 0$, then $B \subseteq \mathfrak{D}$.*
(2) *If $i = 0$, then either $\overline{Q}_3$ has NBOD or B consists only of the edge v_6v_9.*

Proof. For (1), we may assume $B \subseteq \mathfrak{M}$. The two representativity 2 embeddings of V_{10} in $\mathbb{R}P^2$ show that B can only be embedded in a face bounded by either $[a, r_9, v_0]r_1\, s_1[v_6, b, \beta, a]$ or $[b, \beta, a, r_9, v_9]s_4\, r_4[v_5, r_5, b]$ and that s_0 is necessarily exposed in $\mathbb{R}P^2$. Notice that $i = 0$ in both cases, proving (1).

Now assume $i = 0$ and suppose $\overline{Q}_3$ has BOD. From Theorem 5.23, we know that $\overline{Q}_2$ also has BOD. For $j \in \{2, 3\}$, let e_j be the edge of s_j incident with v_j and let D_j be a 1-drawing of $G - e_j$. Because s_j is in a $\overline{Q}_j$-interior bridge, from Lemma 5.9, we know that $\overline{Q}_j$ is crossed in D_j.

If $\overline{Q}_0$ is clean in D_j, then no face of D_j is incident with vertices in both $\langle r_9\, s_4\rangle$ and $\langle s_1\, r_5\rangle$. Therefore, $D_j[B]$ cannot be crossing-free in D_j, a contradiction. Thus, $\overline{Q}_0$ is crossed in D_j. The two possibilities for D_2 are shown in Figure 5.2, while the two possibilities for D_3 are shown in Figure 5.3.

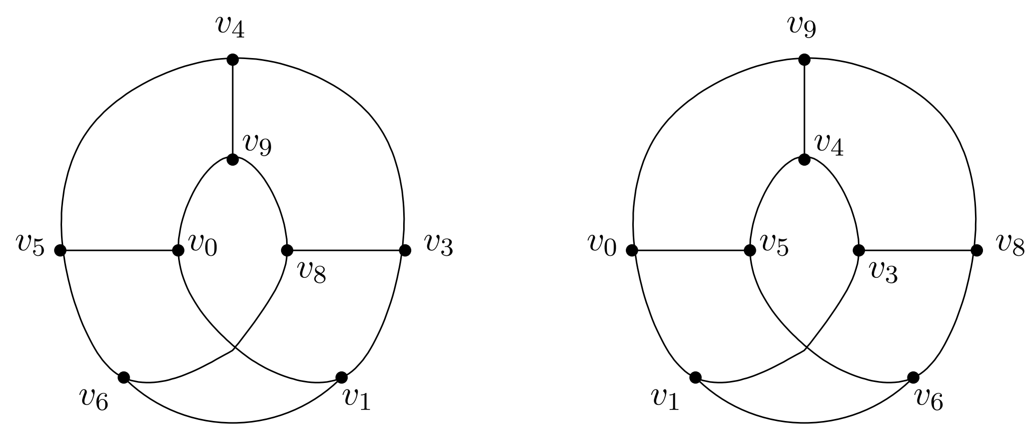

FIGURE 5.2. The two possibilities for D_2.

Let P be an H-avoiding path in G joining a vertex in each of $\langle r_9\, s_4\rangle$ and $\langle s_1\, r_5\rangle$. The left-hand version of D_2 has no face incident with both these paths, and so we must have the right-hand version of D_2. Thus, D_2 implies P has one end in $\langle v_0, r_9, v_9]$ and one end in $\langle v_1, s_1, v_6]$. The right-hand version of D_3 has no face incident with these paths, so it must be the left-hand version of D_3. The only possibility there for the ends of P are v_6 and v_9, as claimed. ∎

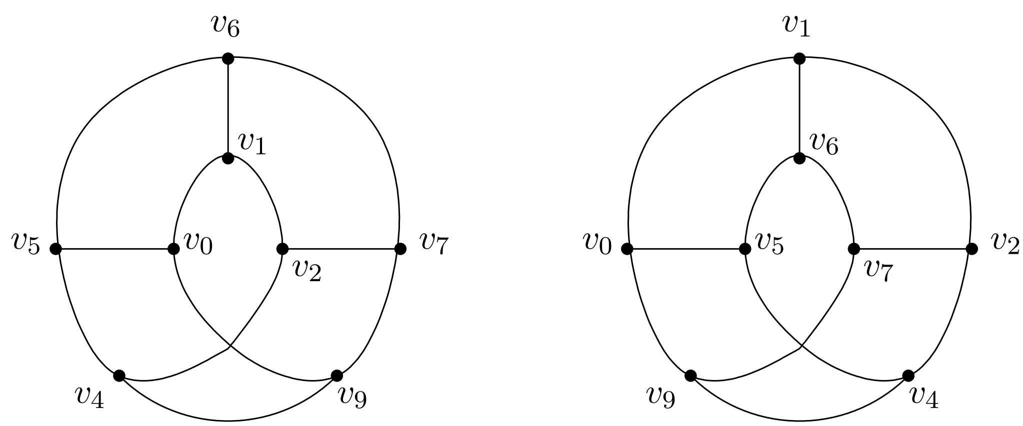

FIGURE 5.3. The two possibilities for D_3.

CHAPTER 6

Green cycles

In this section, we begin our study of the rim edges of H. Ultimately, we will partition them into three types: "green", "yellow", and "red", and it will be the red ones that we focus on to find the desired tile structure. In this section, however, we begin with the study of green edges. We shall show that the cycles C we label green and yellow cannot be crossed in any 1-drawing of $H \cup C$.

Definition 6.1. An edge e of a non-planar graph G is *red* in G if $G - e$ is planar.

We will eventually prove that every edge of R is either in a green cycle, or in a yellow cycle, or red. The main result in this section, one of the three main steps of the entire proof, is that no edge of R is in two green cycles.

Definition 6.2. Suppose G is a graph and $V_{10} \cong H \subseteq G$. Suppose further that G is embedded in $\mathbb{R}P^2$ with representativity 2 and that $\mathfrak{M}$ is the Möbius band bounded by the H-rim R.

(1) A cycle C in G is *H-green* if C is the composition $P_1P_2P_3P_4$ of four paths, such that:
  (a) $P_1 \subseteq R$ and P_1 has length at least 1;
  (b) $P_2P_3P_4$ is R-avoiding;
  (c) $P_2 \cup P_4 \subseteq H$;
  (d) P_3 is H-avoiding (and, therefore, is either trivial or contained in an H-bridge); and
  (e) either
    (i) P_1 contains at most 3 H-nodes or
    (ii) P_1 is *exceptional*, that is, for some $i \in \{0, 1, 2, \dots, 9\}$ and indices read modulo 10,
$$P_1 = r_i\, r_{i+1}\, r_{i+2}\,.$$
(2) An edge of R is *H-green* if it is in an H-green cycle.
(3) A vertex v of R is *H-green* if both edges of R incident with v are in the same H-green cycle.

There is a natural symmetry between P_2 and P_4: if C is an H-green cycle, consisting of the composition $P_1P_2P_3P_4$ as in Definition 6.2, then $P_1^{-1}P_4^{-1}P_3^{-1}P_2^{-1}$ is another H-green cycle. Thus P_4^{-1} and P_2 can both be considered to be P_2. As the orientations of the individual P_i will not be of any importance (except in as much as they are required to make C a cycle), we may say P_2 and P_4 are symmetric.

Note that the exceptional case 1(e)ii is the only one in which P_1 has 4 H-nodes.

Lemma 6.3. *Suppose G is a graph and $V_{10} \cong H \subseteq G$. Let C be any H-green cycle expressed as the composition $P_1P_2P_3P_4$ as in Definition 6.2.*

(1) *If $i \in \{2,4\}$, then P_i has an end in R and is either trivial or contained in an H-spoke.*
(2) *The path P_3 is not trivial.*
(3) *If P_2 and P_4 are both non-trivial, then they are contained in different H-spokes.*

Proof. **(1)** For sake of definiteness, we assume $i = 2$. If P_2 is not trivial, then there is an edge e in P_2. From the definition, e is in H but not in R. Therefore, there is a spoke s containing e. If P_2 has a vertex u not in s, then P_2 is a path contained in H and containing e and u. This implies that one end of s, a vertex of R, is internal to P_2, contradicting the fact that $P_2P_3P_4$ is R-avoiding. So $P_2 \subseteq s$, as required. Since $P_1 \subseteq R$ and P_2 has an end in common with P_1, P_2 has an end in R.

(2) Suppose P_3 is trivial. Then P_2P_4 is an R-avoiding path joining the ends of P_1. Each of P_2 and P_4 is either trivial or in a spoke and, since P_2P_4 is R-avoiding, either both are trivial or P_2P_4 is contained in a single spoke. If both are trivial, then P_1 is the cycle $P_1P_2P_3P_4$, which is impossible, since P_1 is properly contained in the cycle R. Each of P_2 and P_4 has an end in R (or is trivial) and P_2P_4 has both ends in common with P_1, so P_2P_4 is the entire spoke. But then P_1 contains six H-nodes, a contradiction.

(3) For $j = 2, 4$, P_j is non-trivial by hypothesis. Therefore, (1) shows it is contained in an H-spoke s. As it has a vertex in common with P_1, P_j has a vertex in R. This vertex is an H-node incident with s. If P_2 and P_4 are contained in the same spoke s, then, as in the proof of (2), they contain different H-nodes. But then P_1 contains six H-nodes, contradicting Definition 6.2. ■

There is a small technical point that must be dealt with before we can successfully analyze the relation of an H-green cycle to the embedding of G in $\mathbb{R}P^2$.

DEFINITION 6.4. Let Π be a representativity 2 embedding of a graph G in $\mathbb{R}P^2$ and let $V_{10} \cong H \subseteq G$. Then Π is *H-friendly* if, for each H-green cycle C of G and any non-contractible simple closed curve γ in $\mathbb{R}P^2$ meeting $\Pi(G)$ in precisely two points, $\Pi[C]$ is contained in the closure of some face of $\Pi[H] \cup \gamma$.

LEMMA 6.5. *Suppose $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$. Let Π be any representativity 2 embedding of G in $\mathbb{R}P^2$, let γ be a non-contractible simple closed curve in $\mathbb{R}P^2$ meeting $\Pi(G)$ in precisely two points, and let C be an H-green cycle in G. Give H the standard labelling relative to γ.*

(1) *Either $\Pi[C]$ is contained in the closure of some face of $\Pi[H] \cup \gamma$ or v_6v_9 is an edge of G embedded in $\mathfrak{M}$ and $C = r_6\, r_7\, r_8[v_9, v_6v_9, v_6]$. In particular, if $\Pi[H] \subseteq \mathfrak{M}$, then Π is H-friendly.*
(2) *If Π is not H-friendly, then there is an H-friendly embedding of G in $\mathbb{R}P^2$ obtained from Π by reembedding only v_6v_9.*
(3) *In particular, there is an H-friendly embedding of G in $\mathbb{R}P^2$.*

Proof. Suppose $\Pi[C]$ is not contained in the closure of any face of $\Pi[H] \cup \gamma$ and let $P_1P_2P_3P_4$ be the decomposition of C as in Definition 6.2. As P_3 is $(H \cup \gamma)$-avoiding and non-trivial by Lemma 6.3 (2), there is an $(H \cup \gamma)$-face F_3 containing P_3. Note that, if P_2 is not trivial, then Lemma 6.3 (1) asserts it is contained in an H-spoke s and it contains an end of P_3, so P_2 is contained in the boundary of F_3. Likewise for P_4. We assume by way of contradiction that $P_1 \not\subseteq \mathrm{cl}(F_3)$.

CLAIM 1. Then:

(1) $P_1 = r_6\, r_7\, r_8$;
(2) s_0 is exposed;
(3) either $a = v_9$ or $b = v_6$; and
(4) if $F_3 \subseteq \mathfrak{D}$, then both $v_6 = b$ and $v_9 = a$.

PROOF. We first consider the case $F_3 \subseteq \mathfrak{D}$. Both ends of P_1 are contained in one of the ab-subpaths of R. If P_1 is not contained in the boundary of F_3, then it must contain the other complete ab-subpath of R. As each of these has at least 4 H-nodes, the only possibility is that it is precisely 4 H-nodes. In this case, P_1 must be exceptional and s_0 must be exposed. In particular, $P_1 = r_6\, r_7\, r_8$ and P_3 has ends v_6 and v_9. The paths P_2 and P_4 are both trivial. Moreover, as P_1 is not incident with F_3, we must have $v_6 = b$ and $v_9 = a$.

In the other case, $F_3 \subseteq \mathfrak{M}$. If F_3 is contained in the interior of an H-quad, then P_1 joins two vertices in the same quad and is not contained in the quad. In this case, P_1 must have at least 5 H-nodes, which is impossible. Therefore, F_3 is not contained in the interior of an H-quad, and so is bounded by one of $[a, r_9, v_0]\, r_0\, s_1 [v_6, r_5, b, \beta, a]$ and $[a, r_9, v_9]\, s_4\, r_4 [v_5, r_5, b, \beta, a]$. (Recall $\beta = \gamma \cap \mathfrak{M}$.) Notice that s_0 is exposed.

These cases are symmetric; for sake of definiteness, we presume F_3 is bounded by $[a, r_9, v_0]\, r_0\, s_1 [v_6, r_5, \; b, \beta, a]$. The path P_1 has at most 4 H-nodes and joins two vertices on $\overline{Q}_0$. If $P_1 \subseteq \overline{Q}_0$, then $\Pi[C]$ is contained in the closure of one of the two $(\Pi[H] \cup \gamma)$-faces whose boundary is contained in $\Pi[\overline{Q}_0] \cup \gamma$; thus, $P_1 \not\subseteq \overline{Q}_0$. Therefore, P_1 has at least 4 H-nodes; by definition it has at most 4, so P_1 has precisely 4 H-nodes. In particular, P_1 can only be $r_6\, r_7\, r_8$ and $v_9 = a$. □

Because s_0 is exposed, Theorem 5.23 implies that both $\overline{Q}_2$ and $\overline{Q}_3$ have BOD. Let e be any edge in s_2 and let D_2 be a 1-drawing of $G - e$. Since $\overline{Q}_2$ has BOD, Lemma 5.9 shows $\overline{Q}_2$ is crossed in D_2, so $r_0\, r_1\, r_2\, r_3$ crosses $r_5\, r_6\, r_7\, r_8$. This implies that neither s_0 nor s_4 is exposed in D_2 and, therefore, P_3 cannot be in the same $(H - \langle s_0 \rangle)$-bridge as s_0.

Let B_0 and B be the $(H - \langle s_0 \rangle)$-bridges containing s_0 and P_3, respectively. These evidently overlap on $\overline{Q}_0$ and they both overlap $M_{\overline{Q}_0} - e$ (in $G - e$). Therefore, $\overline{Q}_0$ has NBOD. Since $M_{\overline{Q}_0} - e$ is a non-planar $\overline{Q}_0$-bridge in $G - e$, Lemma 5.6 implies that $\overline{Q}_0$ is not clean in D_2.

As $\overline{Q}_0$ and $\overline{Q}_2$ have only s_1 in common and both are crossed in D_2, s_1 must be exposed in D_2. It follows that $D_2[P_3]$ is in the face of $D_2[H - \langle s_2 \rangle]$ bounded by $s_1\, r_6\, r_7\, r_8\, r_9\, r_0$.

The same arguments apply with $\overline{Q}_3$ in place of $\overline{Q}_2$, showing that $D_3[P_3]$ is in the face of $D_3[H - \langle s_3 \rangle]$ bounded by $s_4\, r_4\, r_5\, r_6\, r_7\, r_8$. These two drawings imply that $\mathrm{att}(B) \subseteq r_6\, r_7\, r_8$.

If $F_3 \subseteq \mathfrak{D}$, then F_3 is bounded by $r_9\, s_0\, r_5 [v_6, \alpha, v_9]$ (recall $\alpha = \gamma \cap \mathfrak{D}$). Thus, $\mathrm{att}(B) = \{v_6, v_9\}$ and Lemma 5.19 implies that P_3 is just the edge $v_6 v_9$. In this case, Claim 1 implies P_3 can obviously be embedded in the other face of $H \cup \gamma$ contained in $\mathfrak{D}$ and incident with both v_6 and v_9.

If $F_3 \subseteq \mathfrak{M}$, then F_3 is bounded by either

$$[a, r_9, v_0]\, r_0\, s_1 [v_6, r_5, b, \beta, a] \text{ or } [a, r_9, v_9]\, s_4\, r_4 [v_5, r_5, b, \beta, a] .$$

Again, this implies that $\mathrm{att}(B) \subseteq \{v_6, v_9\}$, so P_3 is just the edge v_6v_9. In this case, Claim 1 implies only that either $v_6 = b$ or $v_9 = a$. Again these cases are symmetric, so we assume $v_9 = a$.

We remark that if $v \in A \cap B$, then (v) is an AB-path and this is the only path containing v that is an AB-path. We now return to the proof.

We wish to reembed v_6v_9 in the $(H \cup \gamma)$-face incident with v_6, v_7, v_8, and v_9. We need only verify that there is no H-avoiding $[b, r_5, v_6\rangle\,\langle v_6, r_6, v_7, r_7, v_8, r_8, v_9]$-path. But such a path would have to appear in D_3, where it can only also be in the face of $D_3[H - \langle s_3\rangle]$ bounded by $s_4\, r_4\, r_5\, r_6\, r_7\, r_8$. But then it crosses v_6v_9 in D_3, a contradiction completing the proof. ■

We are now prepared for our analysis of H-green cycles.

Lemma 6.6. *Let $G \in \mathcal{M}_2^3$, $V_{10} \cong H \subseteq G$, and let Π be an H-friendly embedding of G in $\mathbb{R}P^2$. Let C be an H-green cycle expressed as the composition $P_1P_2P_3P_4$ as in Definition 6.2. Then:*

(1) *P_1 is contained in one of the two ab-subpaths of R;*
(2) *if $C \subseteq \mathfrak{M}$ and s is any H-spoke contained in $\mathfrak{M}$ that is totally disjoint from C, then C is a $(C \cup (H - \langle s\rangle))$-prebox;*
(3) *if C is not contained in $\mathfrak{M}$ and s is any H-spoke contained in $\mathfrak{M}$ having one end in the interior of P_1, then C is a $(C \cup (H - \langle s\rangle))$-prebox;*
(4) *there is a C-bridge M_C so that $H \subseteq C \cup M_C$;*
(5) *C is contractible, C has BOD, and all C-bridges other than M_C are planar;*
(6) *C is a $(C \cup H)$-prebox;*
(7) *M_C is the unique C-bridge (that is, there are no planar C-bridges);*
(8) *C bounds a face of Π;*
(9) *there are at most two H-nodes in the interior of P_1; and*
(10) *in any 1-drawing of $H \cup C$, C is clean.*

Proof. Because Π is H-friendly, there is a face F of $(H \cup \gamma)$ whose closure contains C.

(1) This is an immediate consequence of Definition 6.4, as the boundary ∂ of any face of $H \cup \gamma$ has each component of $\partial \cap R$ contained in one of the ab-subpaths of R.

(2) and (3) Note that $H - \langle s\rangle$ contains a subdivision of V_8. In particular, if e is an edge of C not in R, then $H - \langle s\rangle$ is a non-planar subgraph of $(C \cup (H - \langle s\rangle)) - e$, as required. If $e \in C$ is in R, then we claim the cycle $R' = (R - \langle P_1\rangle) \cup P_2P_3P_4$ is the rim of a V_6. We see this in the two cases.

Case 1: (2) In this case, there are three H-spokes t_1, t_2, t_3 other than s contained in $\mathfrak{M}$. Each t_i has an end v_i in $R - \langle P_1\rangle$ and a maximal R'-avoiding subpath t_i' containing v_i. It is straightforward to verify that $R' \cup t_1' \cup t_2' \cup t_3'$ is a subdivision of V_6, as required.

Case 2: (3) In the exceptional case $P_1 = r_i\, r_{i+1}\, r_{i+2}$, s is different from all of s_i, s_{i+3}, and s_{i+4}, so $R' \cup s_i \cup s_{i+3} \cup s_{i+4}$ is the required V_6. (Note that one of s_i and s_{i+3} can be the exposed spoke and part of that spoke might be in either P_2 or P_4, but whatever part is not in $P_2 \cup P_4$ makes the third spoke.)

In the remaining case, there are two H-spokes s_i and s_{i+1} that are completely disjoint from C. Any other H-spoke s', different from s, s_i, and s_{i+1}, and contained in $\mathfrak{M}$, will connect to R' to make a third spoke, either because both its ends are in

R' or because one end is in R' and the other end is in P_1 and one of the paths in $P_1 - e$ joins the other end of s to a vertex in R'.

(4) Let M_C be the C-bridge containing the ab-subpath Q of R that is P_1-avoiding. We claim $H \subseteq C \cup M_C$. Observe that the maximal P_1-avoiding subpath Q' of R containing Q is contained in M_C and, therefore, $R \subseteq C \cup M_C$. Note that every H-spoke has at least one end in Q' that is not in P_1 and, therefore, that end is in $\mathrm{Nuc}(M_C)$. Thus, if P_3 is not contained in $\mathfrak{M}$, it is obvious that $H \subseteq C \cup M_C$. So suppose P_3 is contained in $\mathfrak{M}$. The H-spokes other than those that contain P_2 and P_4 are obviously in M_C, and the ones containing P_2 and P_4 are in the union of M_C and C.

(5) If either P_1 has at most 3 H-nodes, or s_0 is not exposed, or P_1 is neither $r_1\, r_2\, r_3$ nor $r_6\, r_7\, r_8$, then there is an H-spoke s contained in $\mathfrak{M}$ and totally disjoint from C. The spoke s combines with the one of the two subpaths of R joining the ends of s that is disjoint from P_1 to give a non-contractible cycle disjoint from C. The claim now follows immediately from Lemma 5.20.

We now treat the case s_0 is exposed and P_1 is either $r_1\, r_2\, r_3$ or $r_6\, r_7\, r_8$. In this case, F is a face of $H \cup \gamma$ contained in $\mathfrak{D}$. Let B' be a C-bridge other than M_C. If $B' \subseteq \mathrm{cl}(F)$, then $C \cup B' \subseteq \mathrm{cl}(F)$ and $\mathrm{cl}(F)$ is a closed disc in $\mathbb{R}P^2$. Therefore, $C \cup B'$ has no non-contractible cycles in $\mathbb{R}P^2$. Otherwise, B' is contained in the closure of one of the H-faces bounded by Q_1 or Q_2 or Q_3. For each $i \in \{1, 2, 3\}$, let F_i be the H-face bounded by Q_i. Then $\mathrm{cl}(F_i) \cap \mathrm{cl}(F)$ is a path and, therefore, $\mathrm{cl}(F_i) \cup \mathrm{cl}(F)$ is a closed disc containing $C \cup B'$ and again $C \cup B'$ has no non-contractible cycles. The result now follows from Corollary 5.17.

(6) In the case $P_3 \subseteq \mathfrak{M}$, at most the H-spokes containing P_2 and P_4 meet C. There are at least two others contained in $\mathfrak{M}$ that are disjoint from C; let s be one of these. By (2), for any edge e of C, $(C \cup (H - \langle s \rangle)) - e$ is not planar, so $(C \cup H) - e$ is not planar.

Now suppose $P_3 \subseteq \mathfrak{D}$. If some H-spoke s contained in $\mathfrak{M}$ has an end in the interior of P_1, then (3) implies that, for any edge e of C, $(C \cup (H - \langle s \rangle)) - e$ is not planar, so $(C \cup H) - e$ is not planar.

In the alternative, no H-spoke contained in $\mathfrak{M}$ has an end in the interior of P_1. If e is not in P_1, then $H \cap \mathfrak{M}$, which is a V_8 or V_{10}, is contained in $(C \cup H) - e$, so we may assume $e \in P_1$. But then $(R - \langle P_1 \rangle) \cup P_2 P_3 P_4$ and the H-spokes contained in $\mathfrak{M}$ make a V_8 or V_{10}, showing $(C \cup H) - e$ is not planar.

(7) Observe that (5) shows any other C-bridge is planar and that C has BOD. If B is any other C-bridge, then C is a $B^{\#}$-prebox by (6) and, therefore, is, by definition, a box, contradicting Lemma 5.12.

(8) This is an immediate consequence of the facts that C is contractible (5) and there is only one C-bridge (7).

(9) Suppose by way of contradiction that v_{i-1}, v_i, v_{i+1} are internal to P_1. Notice that P_1 is not exceptional. We claim that $\overline{Q}_i$ is a box, contradicting Lemma 5.12.

For $s \in \{s_{i-1}, s_i, s_{i+1}\}$, s is contained in one of the two faces of R (i.e., the Möbius band $\mathfrak{M}$ and the disc $\mathfrak{D}$). By (8), C is the boundary of some face F of G. Clearly F and s are in different R-faces, so one is in $\mathfrak{M}$ and the other is in $\mathfrak{D}$. Therefore, all of s_{i-1}, s_i, and s_{i+1} are contained in the same one of $\mathfrak{M}$ and $\mathfrak{D}$. Since $\mathfrak{D}$ contains at most one H-spoke, it must be that all three are contained in $\mathfrak{M}$. Clearly, this implies $F \subseteq \mathfrak{D}$ and, therefore, $P_2 P_3 P_4 \subseteq \mathfrak{D}$.

There is another H-spoke s contained in $\mathfrak{M}$ that is totally disjoint from $\overline{Q}_i$. As $P_2P_3P_4 \subseteq \mathfrak{D}$, $R \cup P_2P_3P_4 \cup s$ contains a non-contractible cycle including both $P_2P_3P_4$ and s that is totally disjoint from $\overline{Q}_i$. Thus, Lemma 5.20 implies $\overline{Q}_i$ has BOD and all $\overline{Q}_i$-bridges except $M_{\overline{Q}_i}$ are planar.

We claim $\overline{Q}_i$ is a $(\overline{Q}_i \cup M_{\overline{Q}_i})$-prebox. Note that $\overline{Q}_i \cup M_{\overline{Q}_i}$ contains $H - \langle s_i \rangle$ and so the deletion of any edge in $s_{i-1} \cup s_{i+1}$ leaves a V_6. By (3), C is a $C \cup (H - \langle s_i \rangle)$-prebox, so the deletion of any edge e in $r_{i-1} \cup r_i$ leaves a non-planar subgraph in $(C - e) \cup (H - \langle s_i \rangle)$, which is contained in $(\overline{Q}_i - e) \cup M_{\overline{Q}_i}$. That is, if $e \in r_{i-1} \cup r_i$, then $(\overline{Q}_i - e) \cup M_{\overline{Q}_i}$ is not planar.

We must also consider an edge in $r_{i+4} \cup r_{i+5}$ (these indices are read modulo 10). Let R' be the cycle made up of the following four parts: the two paths in $R - \langle P_1 \rangle - \langle r_{i+4}\, r_{i+5} \rangle$, $P_2P_3P_4$, and $s_{i-1}\, r_{i-1}\, r_i\, s_{i+1}$. To get the V_6, add to R' both H-spokes totally disjoint from P_1 and either of the two R'-avoiding subpaths of P_1 whose ends are in R'. Thus, if $e \in r_{i+4}\, r_{i+5}$, then $(\overline{Q}_i - e) \cup M_{\overline{Q}_i}$ is not planar, completing the proof that $\overline{Q}_i$ is a $(\overline{Q}_i \cup M_{\overline{Q}_i})$-prebox. (See Figure 6.1.)

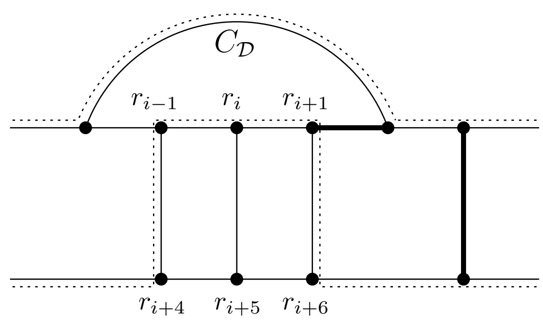

FIGURE 6.1. The case $e \in r_{i+4}\, r_{i+5}$ for $\bar{Q}_i$ being a $(\bar{Q}_i \cup M_{\bar{Q}_i})$-prebox. Only two of the three spokes are shown.

Since the $\overline{Q}_i$-bridge B containing s_i is contained in the closed disc in $\mathbb{R}P^2$ bounded by $\overline{Q}_i$, B is planar and, therefore, $\overline{Q}_i$ is a box, the desired contradiction.

(10) Let D be a 1-drawing of $H \cup C$. Let $P_1P_2P_3P_4$ be the decomposition of C into paths as in Definition 6.2, so $P_1 \subseteq R$ and P_3 is H-avoiding. If C is crossed in D, then it is P_1 that is crossed, while $P_2P_3P_4$, being R-avoiding, is not crossed in D. We claim that there is an H-spoke v_iv_{i+5} disjoint from C that is not exposed in D. The existence of s and the fact that C is crossed in D shows that no face of $R \cup s$ is incident with both ends of P_1 and, therefore, $P_2P_3P_4$ must cross $R \cup s$ in D, the desired contradiction.

To prove the claim, we consider two cases. If P_1 has at most 3 H-nodes, then this is obvious, since only one H-spoke can be exposed. In the alternative, P_1 is exceptional, say $P_1 = r_i\, r_{i+1}\, r_{i+2}$. As the spoke exposed in D is incident with an end of the H-rim branch that is crossed, we see that s_{i+4} is not the exposed spoke and is disjoint from P_1, as required. ■

The next result is the main result of this section and the first of three main steps along the way to obtaining the classification of 3-connected, 2-crossing-critical graphs having a subdivision of V_{10}. The other two major steps are, for $G \in \mathcal{M}_2^3$

containing a subdivision H of V_{10}: (i) G has a representativity 2 embedding in $\mathbb{R}P^2$ so that $H \subseteq \mathfrak{M}$; and (ii) G contains a subdivision of V_{10} with additional properties (that we call "tidiness"). It is this tidy V_{10} for which the partition of the edges of the rim into the red, yellow, and green edges that allows us to find the decomposition into tiles.

THEOREM 6.7. *If $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, then no two H-green cycles have an edge of R in common.*

Proof. Suppose $e_0 \in R$ is in distinct H-green cycles. By Lemma 6.5 (3), there is an H-friendly embedding Π of G in $\mathbb{R}P^2$. By Lemma 6.6 (8), any H-green cycle bounds a face of $\Pi[G]$. As e_0 is in R and R is the boundary of both the (closed) Möbius band $\mathfrak{M}$ and the (closed) disc $\mathfrak{D}$, one of these faces, call it $F_{\mathfrak{M}}$, is contained in $\mathfrak{M}$, while the other, call it $F_{\mathfrak{D}}$, is contained in $\mathfrak{D}$. For $\mathfrak{n} \in \{\mathfrak{M}, \mathfrak{D}\}$, let $C_{\mathfrak{n}}$ be the green cycle bounding $F_{\mathfrak{n}}$ and let $P_1^{\mathfrak{n}} P_2^{\mathfrak{n}} P_3^{\mathfrak{n}} P_4^{\mathfrak{n}}$ be the path decomposition of $C_{\mathfrak{n}}$ as in Definition 6.2; in particular, $P_1^{\mathfrak{n}} \subseteq R$ and $P_3^{\mathfrak{n}}$ is H-avoiding.

Note $P_2^{\mathfrak{D}} P_3^{\mathfrak{D}} P_4^{\mathfrak{D}}$ is disjoint from $\mathfrak{M}$ (except for its ends) and $P_2^{\mathfrak{M}} P_3^{\mathfrak{M}} P_4^{\mathfrak{M}}$ is contained in $\mathfrak{M}$. Thus, $C_{\mathfrak{D}} \cap C_{\mathfrak{M}} = P_1^{\mathfrak{D}} \cap P_1^{\mathfrak{M}}$. Lemma 6.6 (9) implies that, for $\mathfrak{n} \in \{\mathfrak{M}, \mathfrak{D}\}$, $P_1^{\mathfrak{n}}$ has at most 4 H-nodes. We conclude that $P_1^{\mathfrak{D}} \cup P_1^{\mathfrak{M}}$ is not all of R, and so $C_{\mathfrak{D}} \cap C_{\mathfrak{M}}$ is a path. Therefore, there is a unique cycle C in $C_{\mathfrak{D}} \cup C_{\mathfrak{M}}$ not containing e_0 and, furthermore, C bounds a closed disc in $\mathbb{R}P^2$ having e_0 in its interior.

On the other hand, Lemma 6.6 (1) shows there is an ab-subpath A_1 of R that contains $P_1^{\mathfrak{D}}$. Since $e_0 \in P_1^{\mathfrak{D}} \cap P_1^{\mathfrak{M}}$, it is also the case that $P_1^{\mathfrak{M}} \subseteq A_1$. Let A be the other ab-subpath of R, so that A is $(C_{\mathfrak{D}} \cup C_{\mathfrak{M}})$-avoiding. In particular, there is a C-bridge M_C containing A. By Lemma 6.6 (7), for $\mathfrak{n} \in \{\mathfrak{M}, \mathfrak{D}\}$, A is in the unique $C_{\mathfrak{n}}$-bridge $M_{C_{\mathfrak{n}}}$. Since $M_{C_{\mathfrak{n}}}$ (and therefore A) is not contained in the face of G bounded by $C_{\mathfrak{n}}$, we conclude that A is not in the disc bounded by C. Therefore, M_C is different from the C-bridge B_C containing e_0.

CLAIM 1. For each H-spoke s, some H-node incident with s is not in $C_{\mathfrak{M}} \cup C_{\mathfrak{D}}$.

PROOF. By Lemma 6.6 (9), there exists an i so that $P_1^{\mathfrak{D}} \subseteq r_i\, r_{i+1}\, r_{i+2}$. In particular, e_0 is in $r_i \cup r_{i+1} \cup r_{i+2}$. Thus, $P_1^{\mathfrak{M}}$ has an edge in at least one of r_i, r_{i+1}, and r_{i+2}.

Lemma 6.6 (8) implies that $C_{\mathfrak{M}}$ bounds a face of G. Therefore, $C_{\mathfrak{M}}$ is contained in the closure $\mathrm{cl}(F)$ of a face F of $\Pi[H]$ and $F \subseteq \mathfrak{M}$. Thus, $P_1^{\mathfrak{M}}$ is contained in one of the two components of $\mathrm{cl}(F) \cap R$. Since such a component is contained in consecutive H-rim branches, if $P_1^{\mathfrak{M}}$ contains an edge in r_j, then $P_1^{\mathfrak{M}}$ is contained in either $r_{j-1}\, r_j$ or $r_j\, r_{j+1}$. From the preceding paragraph, $P_1^{\mathfrak{M}}$ is contained in one of $r_{i-1} r_i$, $r_i r_{i+1}$, $r_{i+1} r_{i+2}$, and $r_{i+2} r_{i+3}$.

We conclude that $P_1^{\mathfrak{D}} \cup P_1^{\mathfrak{M}}$ is contained in either $r_{i-1}\, r_i\, r_{i+1}\, r_{i+2}$ or $r_i\, r_{i+1}\, r_{i+2}\, r_{i+3}$ showing that no H-spoke has both ends in $P_1^{\mathfrak{D}} \cup P_1^{\mathfrak{M}}$. $\square$

CLAIM 2. (1) $H \subseteq C \cup M_C \cup B_C$.

(2) If s is an H-spoke contained in $\mathfrak{M}$ disjoint from $C_{\mathfrak{M}}$, then $(C \cup M_C) - \langle s \rangle$ is not planar.

PROOF. For (1), we note that it is clear that $R \subseteq C \cup M_C \cup B_C$. Now let s be an H-spoke. Suppose first that $s \subseteq \mathfrak{M}$. By Claim 1, there is an H-node v incident with s and not in $C_{\mathfrak{M}} \cup C_{\mathfrak{D}}$. If $s \cap C_{\mathfrak{M}}$ is at most an end of s, then it is evident that $s \subseteq M_C$. If $s \cap C_{\mathfrak{M}}$ is more than just an end of s, then s consists of a $C_{\mathfrak{M}}$-avoiding subpath s' joining v to a vertex in $C_{\mathfrak{M}}$, together with the path $C_{\mathfrak{M}} \cap s$ (which is by Lemma 6.3 (1)) either $P_2^{\mathfrak{M}}$ or $P_4^{\mathfrak{M}}$). But then it is again evident that $s \subseteq C \cup M_C$.

Otherwise, s is exposed, in which case we have the same argument, but replacing $C_{\mathfrak{M}}$ with $C_{\mathfrak{D}}$, completing the proof of (1).

For (2), a V_6 is found whose rim is $(R - \langle P_1^{\mathfrak{M}} \rangle) \cup P_2^{\mathfrak{M}} P_3^{\mathfrak{M}} P_4^{\mathfrak{M}}$. The spokes are contained in the three other spokes in $\mathfrak{M}$, namely they are the parts that are not in $P_2^{\mathfrak{M}} \cup P_4^{\mathfrak{M}}$. □

Claim 3. C has BOD.

Proof. Let S be the set of H-spokes contained in $\mathfrak{M}$ and disjoint from $C_{\mathfrak{M}}$. As $C_{\mathfrak{M}}$ meets at most two H-spokes in $\mathfrak{M}$, $|S| \geq 2$. If some $s \in S$ is also disjoint from $C_{\mathfrak{D}}$, then $R \cup s$ contains a non-contractible cycle disjoint from C, in which case Lemma 5.20 shows C has BOD, as claimed.

So we may assume that no element of S is also disjoint from $C_{\mathfrak{D}}$. Let s be any element of S; then $s \cap C_{\mathfrak{D}}$ is a vertex v of $P_1^{\mathfrak{D}}$. Let e be the edge of s incident with v. In order to show that C has BOD, we will show that: (i) the overlap diagrams $OD_{G-e}(C)$ and $OD_G(C)$ are the same; and (ii) $OD_{G-e}(C)$ is bipartite. For (i), note that $C_{\mathfrak{D}}$ bounds a face in $\mathbb{R}P^2$ and that $\langle s \rangle$ is in the boundary of two $(H \cup \gamma)$-faces. Thus, there can be no C-bridge that overlaps M_C in G because of its attachment at v. That is, $OD_{G-e}(C)$ and $OD_G(C)$ are the same.

For (ii), Lemma 6.6 (2) applied to $C_{\mathfrak{M}}$ and (3) applied to $C_{\mathfrak{D}}$, combined with Lemma 5.4, shows $C_{\mathfrak{D}}$ and $C_{\mathfrak{M}}$ are both clean in D_e. Therefore, C is clean in D_e. By Claim 2 (2), $(C \cup M_C) - e$ is not planar, so Lemma 5.6 shows C has BOD in $G - e$. Therefore, C has BOD in G. □

Claim 4. C is a $C \cup H$-prebox.

Proof. Note that $C_{\mathfrak{D}} \cup C_{\mathfrak{M}} \subseteq C \cup H$. If $e \in C$, then let $i \in \{\mathfrak{M}, \mathfrak{D}\}$ be such that $e \in C_i$. Lemma 6.6 (6) says that C_i is a $(C_i \cup H)$-prebox and, therefore, $(C_i \cup H) - e$ is not planar. Since $(C_i \cup H) - e \subseteq (C \cup H) - e$, we conclude that C is a $(C \cup H)$-prebox. □

Claim 5. $G = C \cup M_C \cup B_C$.

Proof. By way of contradiction, suppose there is another C-bridge B'. Let F be the $(H \cup \gamma)$-face containing B'. Then $C \cup B'$ is contained in the closed disc that is the union of the closure of F and the disc bounded by C, showing B' is planar. By Claim 4 and the fact that $C \cup H \subseteq B'^{\#}$, Lemma 5.4 says that C is clean in a 1-drawing of $B'^{\#}$, of which there is at least one, since G is 2-crossing-critical. This yields a 1-drawing of $C \cup M_C$ with C clean. By Claim 3, C has BOD, B_C is planar because it is contained in the closed disc bounded by C, and above we showed that every other C-bridge is planar; Corollary 4.7 implies $cr(G) \leq 1$, a contradiction. □

We are now on the look-out for a box in G; it is not true that C is necessarily one. Our next claim gives a sufficient condition under which we can find some box and the following two claims show that, in all other cases, C is a box.

Claim 6. Suppose all of the following:

(1) there is an i so that $P_3^{\mathfrak{M}}$ is in a Q_i-local H-bridge;
(2) $P_2^{\mathfrak{M}}$ contains v_i and is a non-trivial subpath of s_i; and

(3) v_{i+2} is in the interior of $P_1^{\mathfrak{D}}$.

Then G has a box.

PROOF. We note that (2) implies $s_i \subseteq \mathfrak{M}$.

SUBCLAIM 1. Both s_{i+1} and s_{i+2} are contained in $\mathfrak{M}$.

PROOF. Suppose first that s_{i+2} is exposed. Then (3) implies $P_2^{\mathfrak{D}}$ and $P_4^{\mathfrak{D}}$ are both trivial. That is, $C_{\mathfrak{D}} = P_1^{\mathfrak{D}} P_3^{\mathfrak{D}}$. But $P_3^{\mathfrak{D}}$ is H-avoiding and overlaps s_{i+2} on R (because $P_1^{\mathfrak{D}}$ has at most four H-nodes, only two of which can be in the interior of $P_1^{\mathfrak{D}}$). Thus, $P_3^{\mathfrak{D}}$ and s_{i+2} cross in $\mathbb{R}P^2$, a contradiction. Therefore, $s_{i+2} \subseteq \mathfrak{M}$.

Next, suppose s_{i+1} is exposed. Then, by symmetry, we may assume $i = 4$ or $i = 9$. In either case, $P_1^{\mathfrak{M}}$ and $P_1^{\mathfrak{D}}$ are in different ab-subpaths of R and so do not have an edge in common, a contradiction. Hence s_{i+1} is also contained in $\mathfrak{M}$. $\square$

Let u be the common end of $P_2^{\mathfrak{M}}$ and $P_3^{\mathfrak{M}}$ and let w be the common end of $P_4^{\mathfrak{M}}$ and $P_1^{\mathfrak{M}}$. By (2), $u \in \langle s_i \rangle$ and, by (1) and (2), $w \in r_i$. Observe that the edge e_0 common to $C_{\mathfrak{M}}$ and $C_{\mathfrak{D}}$ is in $[v_i, r_i, w]$.

Let C' be the cycle $\left[v_{i+5}, s_i, u, P_3^{\mathfrak{M}} P_4^{\mathfrak{M}}, w, r_i, v_{i+1}\right] r_{i+1}\, s_{i+2}\, r_{i+6}\, r_{i+5}$. We note that there are two obvious C'-bridges: the C'-interior bridge $B_{C'}$ containing the edge of s_{i+1} incident with v_{i+6}; and the C'-exterior bridge $M_{C'}$ for which $H - \langle s_{i+1} \rangle \subseteq C' \cup M_{C'}$. To show C' is a box, it suffices to show that C' has BOD and C' is a $(C' \cup M_{C'})$-prebox.

Notice that v_{i+2} is in the interior of $P_1^{\mathfrak{D}}$ by hypothesis and v_{i+1} is in the interior of $P_1^{\mathfrak{D}}$ because $e_0 \in r_i$. Lemma 6.6 (9) implies that the only H-nodes in the interior of $P_1^{\mathfrak{D}}$ are v_{i+1} and v_{i+2}. In particular, v_i and v_{i+3} are in $R - \left\langle P_1^{\mathfrak{D}} \right\rangle$, as are all the ends of s_{i+3} and s_{i+4}.

To see that C' has BOD, we produce a non-contractible cycle in $\mathrm{Nuc}(M_{C'})$. Lemma 5.20 then implies C' has BOD and precisely one non-planar bridge. We start with the two paths $P_2^{\mathfrak{D}} P_3^{\mathfrak{D}} P_4^{\mathfrak{D}}$ and s_{i+4}, and easily complete the required cycle using two paths in R, one containing r_{i+3} and the other containing r_{i+9}.

It remains to show that C' is a $(C' \cup M_{C'})$-prebox. Since $V_8 \cong H - \langle s_{i+1} \rangle \subseteq C' \cup M_{C'}$, it is obvious that, if $e \in C'$ and $e \notin R$, then $(C' \cup M_{C'}) - e$ contains a V_6 and so is not planar. So suppose $e \in C'$ and $e \in R$. There are two cases.

If $e \in r_i\, r_{i+1}$, then take $(R - \left\langle P_1^{\mathfrak{D}} \right\rangle) \cup P_2^{\mathfrak{D}} P_3^{\mathfrak{D}} P_4^{\mathfrak{D}}$ as the rim. We choose as spokes s_i, s_{i+3}, and s_{i+4}.

If $e \in r_{i+5}\, r_{i+6}$, then the rim consists of the two paths $P_2^{\mathfrak{D}} P_3^{\mathfrak{D}} P_4^{\mathfrak{D}}$ and $C' - \langle r_{i+5}\, r_{i+6} \rangle$, together with the two subpaths of R joining them, one containing v_{i+3}, v_{i+4}, and v_{i+5}, and the other containing v_{i+7}, v_{i+8}, v_{i+9}, and v_i. In this case, the spokes are s_{i+3}, s_{i+4}, and $P_2^{\mathfrak{M}}$. $\square$

In the remaining case, we show that C is a box. The following simple observations get us started, the first being the essential ingredient.

CLAIM 7. Either:

(1) there is an i so that
   - $P_3^{\mathfrak{M}}$ is in a Q_i-local H-bridge;
   - s_i contains an edge of $C_{\mathfrak{M}}$; and
   - v_{i+2} is in the interior of $P_1^{\mathfrak{D}}$;

   or (symmetrically)

(2) there is an i so that

- $P_3^{\mathfrak{M}}$ is in a Q_i-local H-bridge;
- s_{i+1} contains an edge of $C_{\mathfrak{M}}$; and
- v_{i-1} is in the interior of $P_1^{\mathfrak{D}}$;

or

(3) there are three H-spokes not having an edge in $C_{\mathfrak{M}}$ and not having an incident vertex in the interior of $P_1^{\mathfrak{D}}$.

Proof. Lemma 6.6 (9) implies there are at most two H-nodes in the interior of $P_1^{\mathfrak{D}}$. Therefore, if no H-spoke contains an edge of $C_{\mathfrak{M}}$, then (3) holds. So we may suppose $C_{\mathfrak{M}}$ has an edge in some H-spoke.

Suppose first that s_0 is exposed, $C_{\mathfrak{M}}$ has an edge in s_1 and e_0 is in either $[a, r_9, v_0, r_0, v_1]$ or $[b, r_5, v_6]$. Therefore, $P_1^{\mathfrak{D}}$ has one end in either $[a, r_9, v_0, r_0, v_1\rangle$ or $[b, r_5, v_6\rangle$. Lemma 6.6 (9) implies at most two H-nodes can be in the interior of $P_1^{\mathfrak{D}}$, so no end of s_3 can be in the interior of $P_1^{\mathfrak{D}}$. We conclude that s_0, s_3 and s_4 are the required three spokes yielding (3).

Symmetry treats the same case on the other side.

In the remaining case, $P_3^{\mathfrak{M}}$ is contained in a Q_i-local H-bridge and both s_i and s_{i+1} are contained in $\mathfrak{M}$. The edge e_0 is in either r_i or r_{i+5}. If the only H-nodes in the interior of $P_1^{\mathfrak{D}}$ are incident with either s_i or s_{i+1}, then the other three H-spokes suffice for (3).

Thus, by symmetry we may assume an end of s_{i+2} is in the interior of $P_1^{\mathfrak{D}}$. This implies that an end of s_{i+1} is also in the interior of $P_1^{\mathfrak{D}}$. Lemma 6.6 (9) shows these are the only H-nodes in the interior of $P_1^{\mathfrak{D}}$. If s_i does not contain an edge of $C_{\mathfrak{M}}$, then the three spokes other than s_{i+1} and s_{i+2} suffice for (3), while if s_i does contain an edge of $C_{\mathfrak{M}}$, then we have (1). □

Claims 6 and 7 show we need only consider the third possibility in Claim 7 to find a box.

Claim 8. If there are three H-spokes not having any edge in $C_{\mathfrak{M}}$ and not having an incident H-node in $\langle P_1^{\mathfrak{D}}\rangle$, then C is a box.

Proof. By Claim 3 and the fact that B_C is a planar C-bridge, it suffices to show C is a $(C \cup M_C)$-prebox. For each $e \in C$, we show that $(C \cup M_C) - e$ contains a V_6.

We note that 3-connection and the fact that $C_{\mathfrak{M}}$ and $C_{\mathfrak{D}}$ both bound faces implies $C_{\mathfrak{M}} \cap C_{\mathfrak{D}}$ is just e_0 and its ends. That is, B_C consists of just e_0 and its ends. Thus, Claim 5 implies that $G - e_0 = C \cup M_C$. In particular, every spoke is in $C \cup M_C$.

Let w be any H-node that is not in C. There are two wC-paths in $R - e_0$; let them be R_x with end $x \in C$ and R_y with end $y \in C$. Thus, R consists of the C-avoiding path $R_x \cup R_y$, a subpath of C, the edge e_0, and another subpath of C. The cycle C consists of two xy-paths; let them be $N^{\mathfrak{D}}$ containing $P_2^{\mathfrak{D}} P_3^{\mathfrak{D}} P_4^{\mathfrak{D}}$ and $N^{\mathfrak{M}}$ containing $P_2^{\mathfrak{M}} P_3^{\mathfrak{M}} P_4^{\mathfrak{M}}$. We note that $N^{\mathfrak{D}} \subseteq \mathfrak{D}$ and $N^{\mathfrak{M}} \subseteq \mathfrak{M}$.

Subclaim 2. Let s be an H-spoke with no edge in $C_{\mathfrak{M}}$ and not having an incident H-node in $\langle P_1^{\mathfrak{D}}\rangle$.

(1) If $s \subseteq \mathfrak{M}$, then $s \cap C$ is either empty, x, or y.
(2) If $s \subseteq \mathfrak{D}$ (that is, $s = s_0$ is exposed), then $s \cap C$ contains at most one of v_0 and v_5.

PROOF. For (1), the alternative is that s contains a vertex u in $\langle N^{\mathfrak{M}}\rangle$. By hypothesis, s has no edge in $C_{\mathfrak{M}}$ and, therefore, s has no edge in C. Being in $N^{\mathfrak{M}}$, the vertex u is either in R or in $P_2^{\mathfrak{M}}P_3^{\mathfrak{M}}P_4^{\mathfrak{M}}$.

Suppose that u is in $P_2^{\mathfrak{M}}P_3^{\mathfrak{M}}P_4^{\mathfrak{M}}$. If u is in $P_3^{\mathfrak{M}}$, then, since $P_3^{\mathfrak{M}}$ is H-avoiding, u is an end of $P_3^{\mathfrak{M}}$, and so is in $P_2^{\mathfrak{M}} \cup P_4^{\mathfrak{M}}$. Thus, if u is in $P_2^{\mathfrak{M}}P_3^{\mathfrak{M}}P_4^{\mathfrak{M}}$, then u is in $P_2^{\mathfrak{M}} \cup P_4^{\mathfrak{M}}$. Since both $P_2^{\mathfrak{M}}$ and $P_4^{\mathfrak{M}}$ are contained in H, are R-avoiding, and neither has an edge of s, the one containing u is trivial and u is in R.

Thus, in every case u is in R and so is an H-node. It follows that one of $[x, N^{\mathfrak{M}}, u]$ and $[u, N^{\mathfrak{M}}, y]$ contains $P_2^{\mathfrak{M}}P_3^{\mathfrak{M}}P_4^{\mathfrak{M}}$ and the other is contained in R. We choose the labelling so that $[x, N^{\mathfrak{M}}, u] \subseteq R$.

As we follow $R - e_0$ from w to x and continue to u along $N^{\mathfrak{M}}$, we see there is an edge of C incident with x and not in R. That it is in $N^{\mathfrak{D}}$ implies it is in $P_2^{\mathfrak{D}}P_3^{\mathfrak{D}}P_4^{\mathfrak{D}}$. All the vertices in $[x, N^{\mathfrak{M}}, u\rangle$ are incident with two rim edges in what we have just traversed. In particular, e_0 is not incident with any of these vertices and, therefore, $[x, N^{\mathfrak{M}}, u]$ is contained in $C^{\mathfrak{D}}$. More precisely, $[x, N^{\mathfrak{M}}, u]$ is contained in $P_1^{\mathfrak{D}}$. As we continue along R past u, we either find e_0 is incident with u or the other edge of C incident with u is in R. In either case, u is in $\langle P_1^{\mathfrak{D}}\rangle$, a contradiction.

For (2), suppose v_0 and v_5 are both in C. Then $P_1^{\mathfrak{M}} \cup P_1^{\mathfrak{D}}$ contains both v_0 and v_5. By Definition 6.2 (1e), v_0 and v_5 are not both in the same one of $P_1^{\mathfrak{M}}$ and $P_1^{\mathfrak{D}}$, so one is in $P_1^{\mathfrak{M}}$ and the other is in $P_1^{\mathfrak{D}}$. By symmetry, we may assume v_0 is in $P_1^{\mathfrak{M}}$. Because Π is H-friendly, $P_1^{\mathfrak{M}}$ is contained in either $[a, r_9, v_0, r_0, v_1]$ or, if $a = v_0$, r_9 (these being the only two faces of $\Pi[H] \cup \gamma$ in $\mathfrak{M}$ that can be incident with v_0).

Recall that e_0 is in both $P_1^{\mathfrak{M}}$ and $P_1^{\mathfrak{D}}$. If $P_1^{\mathfrak{M}} \subseteq [a, r_0, v_0, r_0, v_1]$, then e_0 is in either r_9 or r_0 and $P_1^{\mathfrak{D}}$ is, by Definition 6.2 (1e), contained in either $[a, r_9, v_0]\, r_0\, r_1 [v_2, r_2, v_3\rangle$ or $r_0\, r_1\, r_2\, [v_3, r_4, v_4\rangle$, and v_5 is not in C. If $P_1^{\mathfrak{M}} \subseteq r_9$, then e_0 is in r_9, so $P_1^{\mathfrak{D}}$ is contained in $r_9\, r_8\, r_7\, [v_7, r_6, v_6\rangle$, and again v_5 is not in C. □

The case $e \in N^{\mathfrak{D}}$ is easy: the rim of the V_6 is $(R - \langle P_1^{\mathfrak{M}}\rangle) \cup P_2^{\mathfrak{M}}P_3^{\mathfrak{M}}P_4^{\mathfrak{M}}$ and we choose as spokes any three of the H-spokes that are contained in $\mathfrak{M}$. (If one intersects $C_{\mathfrak{M}}$, then only the part of the spoke that is $C_{\mathfrak{M}}$-avoiding will be the actual spoke of the V_6.)

If $e \in N^{\mathfrak{M}}$, then the rim R' of the V_6 is $(R - \langle P_1^{\mathfrak{D}}\rangle) \cup P_2^{\mathfrak{D}}P_3^{\mathfrak{D}}P_4^{\mathfrak{D}}$ and the spokes are the three H-spokes from the hypothesis. If all three hypothesized H-spokes are contained in $\mathfrak{M}$, then it is evident from Subclaim 2 (1) that we have indeed described a V_6 in $(C \cup M_C) - e$.

So suppose that one of the H-spokes in the hypothesis is the exposed spoke s_0. From Subclaim 2 (2), either s_0 is disjoint from C or precisely one H-node incident with s_0 is in C. We may choose the labelling so that v_0 is not in C.

If v_5 is not in C, then s_0 is disjoint from C. Subclaim 2 (1) shows the other two hypothesized H-spokes meet C in at most x or y; it is now obvious that the three hypothesized H-spokes combine with R' to make a V_6.

Finally, suppose v_5 is in C. Because $C_{\mathfrak{D}}$ is H-green, $P_1^{\mathfrak{D}} \subseteq r_2\, r_3\, r_4\, [v_5, r_5, b]$. In particular, s_1 is disjoint from $C_{\mathfrak{M}}$. If s_2 has no edge in $C_{\mathfrak{M}}$, then $R' \cup s_1 \cup s_2$, together with the portion of s_0 from v_0 to $C_{\mathfrak{D}}$ is a V_6 avoiding $N_{\mathfrak{M}}$. If s_2 has an edge in $C_{\mathfrak{M}}$, then $C_{\mathfrak{M}}$ is in the $\Pi[H]$-face bounded by Q_2. In this case, we may replace s_2 with $s_4\, r_4$ to obtained the desired V_6. □

Evidently, Claims 6, 7, and 8 show that G has a box, contradicting Lemma 5.12. ■

CHAPTER 7

Exposed spoke with additional attachment not in $\overline{Q}_0$

The main result of this section is the proof of the following technical theorem, which limits possibilities for the V_{10}-bridges. This will be used in the next section when we get our second major step by showing that there is a representativity 2 embedding of G in $\mathbb{R}P^2$ for which all the H-spokes are contained in the Möbius band.

THEOREM 7.1. *Suppose $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$. Let Π be an H-friendly embedding of G in $\mathbb{R}P^2$, with the standard labelling. Then there is no H-bridge having attachments in both $\langle s_0 \rangle$ and $\langle r_1\, r_2\, r_3 \rangle$.*

At one point in the proof of this theorem, we need the following lemma. Most of it is used again several times.

LEMMA 7.2. *Let G be a graph and let $V_8 \cong H \subseteq G$. Let P be an H-avoiding path in G joining distinct vertices x and y of R and let P' be one of the two xy-subpaths of R. Let D be a 1-drawing of $H \cup P$.*

(1) If P' has at most two H-nodes or, for some i, $P' = r_i\, r_{i+1}$, then P' is not crossed in D.

(2) If there are only the two H-nodes v_i, v_{i+1} in the interior of P' and P' has at most one other H-node, then r_{i+4} is not crossed in D.

(3) Suppose $r_i\, r_{i+1} \subseteq P'$, $P' \not\subseteq r_i\, r_{i+1}$, but $P' \subseteq r_i\, r_{i+1}[v_{i+2}, r_{i+2}, v_{i+3}\rangle$.

(a) Then $r_i\, r_{i+1}$ is not crossed in D.

(b) If P' is crossed in D, then s_{i+3} is exposed in D and $P' \cap r_{i+2}$ crosses r_{i-1}.

Proof. Let x and y be the ends of P and let $R' = (R - \langle P' \rangle) \cup P$. For (1) and (2), we find three spokes to add to R' to find a subdivision of V_6 disjoint from P' — or at least some part of P'. The part of P' disjoint from the V_6 cannot be crossed in any 1-drawing of H.

For (1), if P' contains at most one H-node, then this is easy: any three H-spokes not having an end in P' will suffice. If $P' = r_i\, r_{i+1}$, then the three H-spokes s_i, s_{i+2}, and s_{i+3} suffice.

In the remaining case, P' has precisely two H-nodes. We may express P' in the form

$$P' = [x, r_{j-1}, v_j]\, r_j\, [v_{j+1}, r_{j+1}, y]\,,$$

where either of $[x, r_{j-1}, v_j]$ and $[v_{j+1}, r_{j+1}, y]$ might be a single vertex. In this case, the spokes are s_{j+2}, s_{j+3} and $s_{j+1}[v_{j+1}, r_{j+1}, y]$, showing that $[x, r_{j-1}, v_j]r_j$ is not crossed in D, while replacing $s_{j+1}[v_{j+1}, r_{j+1}, y]$ with $[x, r_{j-1}, v_j]s_j$ shows $[v_{j+1}, r_{j+1}, y]$ is not crossed in D. This completes the proof of (1).

For (2), replace R' with $(R' - \langle r_{i+4}\rangle) \cup (s_i\, r_i\, s_{i+1})$. We now need three spokes. If there is a third H-node in P', then symmetry allows us to assume it is v_{i-1}. In either case, we choose s_{i-1}, $[v_{i+1}, r_{i+1}, y]$, and s_{i+2} as the three spokes for the V_6. This V_6 avoids r_{i+4}, showing it is not crossed in D.

For (3), $x = v_i$ and the hypotheses imply that $y \in \langle r_{i+2}\rangle$. For (3a), we may use the spokes s_i, $s_{i+2}[v_{i+2}, r_{i+2}, y]$, and s_{i+3} to see that $r_i\, r_{i+1}$ is not crossed in D, as required.

For (3b), suppose P' is crossed in D. Part (3a) shows that it must be $P' \cap r_{i+2}$ that is crossed and (2) shows that $r_{i+5} = r_{i-3}$ is not crossed in D. We need only show that r_{i-2} is also not crossed in D. If it were, then $[v_{i+2}, r_{i+2}, y]$ crosses r_{i-2}. But then the cycle $r_{i+3}\, r_{i+4}\, r_{i-3}\, r_{i-2}\, s_{i-1}$ separates $v_i = x$ from y in D, showing that P is also crossed in D, a contradiction. ■

Proof of Theorem 7.1. This is obvious if no spoke is exposed in Π, so we may suppose s_0 is exposed.

Claim 1. There is no H-avoiding $\langle s_0\rangle\, \langle v_1, r_1, v_2]$- or $\langle s_0\rangle\, [v_3, r_3, v_4\rangle$-path.

Proof. By symmetry, it suffices to prove only one. By way of contradiction, we suppose that there is an H-avoiding path P from $x \in \langle s_0\rangle$ to $y \in \langle v_1, r_1, v_2]$.

Let $e \in s_3$ and consider a 1-drawing D of $G - e$. By Lemma 5.9 and Theorem 5.23 (4), we know that $\overline{Q}_3$ is crossed in D. This implies that $r_1\, r_2\, r_3\, r_4$ crosses $r_6\, r_7\, r_8\, r_9$. This already implies neither s_0 nor s_1 is exposed in D. Furthermore, the crossing is of two edges in R and, since P is H-avoiding, we conclude that $D[P]$ is not crossed in D. Therefore, the end of P in $\langle v_1, r_1, v_2]$ must occur in the interval of $r_1\, r_2\, r_3\, r_4$ between the crossing and v_5; that is, the crossing must involve an edge of r_1. In particular, $r_2\, r_3\, r_4\, r_5$ is not crossed in D.

Since $\overline{Q}_3$ is crossed in D and r_1 is crossed in D, the other crossing edge is in $r_7\, r_8$. Thus it is in $r_6\, r_7\, r_8$. It follows that s_2 is exposed in D. Thus, the cycle $r_4\, r_5\, s_1\, r_0\, r_9\, s_4$ separates x from y in D, showing P is crossed in D, a contradiction. □

It follows from Claim 1 that, if there is an H-avoiding path P_0 joining $x \in \langle s_0\rangle$ to $y \in \langle r_1\, r_2\, r_3\rangle$, then $y \in \langle r_2\rangle$. Let $K = H \cup P_0$. See Figure 7.1.

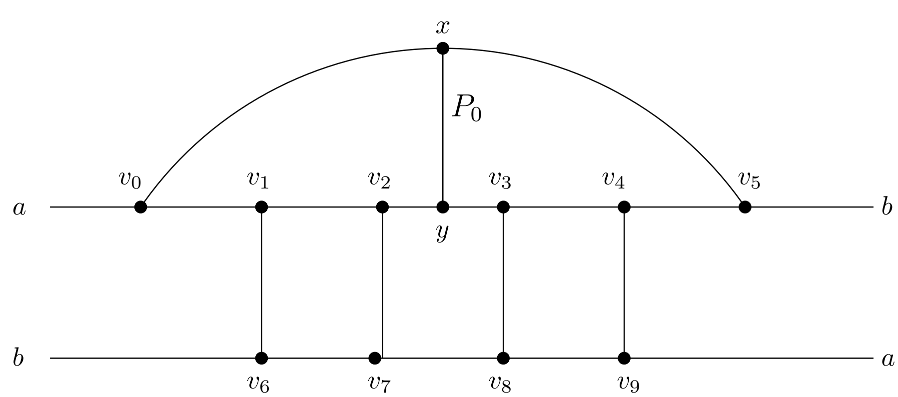

Figure 7.1. The subgraph K of G in $\mathbb{R}P^2$.

Let J_1 and J_2 be the two cycles $r_0\, r_1\, [v_2, r_2, y, P_0, x, s_0, v_0]$ and $r_4\, r_3\, [v_3, r_2, y, P_0, x, s_0, v_5]$, respectively.

Claim 2. The cycles J_1 and J_2 both bound faces of G in $\mathbb{R}P^2$.

Proof. These cycles are both H-green, so this is just Lemma 6.6 (8). □

The following claim completes the determination of the $(H \cap \mathfrak{M})$-bridge containing s_0.

Claim 3. The $(H - \langle s_0 \rangle)$-bridge containing s_0 is $s_0 \cup P_0$.

Proof. Suppose not and let B be the $(H - \langle s_0 \rangle)$-bridge containing s_0. Then Lemma 5.19 implies that B has an attachment z other than v_0, y, and v_5. By Claim 2, $z \in [a, r_9, v_0\rangle \cup \langle v_5, r_5, b]$; by symmetry we may assume the former. Let P be a K-avoiding $z\,\langle s_0 \rangle$-path.

Suppose $z = v_9$. Let e be the edge of s_0 incident with v_0. We show that $\operatorname{cr}((K \cup P) - e) \geq 2$. As this is a proper subgraph of G, we contradict the fact that G is 2-crossing-critical. In $P \cup (s_0 - e) \cup P_0$, there is a claw Y with talons $z = v_9$, y and v_5. We show $\operatorname{cr}((H - \langle s_0 \rangle) \cup Y) \geq 2$.

By way of contradiction, we suppose D is a 1-drawing of $(H - \langle s_0 \rangle) \cup Y$. As $H - \langle s_0 \rangle \cong V_8$, Lemma 7.2 (1) implies that (using the labelling from H) $[y, r_2, v_3]\, r_3\, r_4$ is not crossed in D, while (2) of the same lemma implies neither r_6 nor r_8 is crossed in D. Part (3a) implies $r_9\, r_0\, r_1$ is not crossed, while (3b) implies (since r_9 is not crossed) that $[v_2, r_2, y]$ is not crossed. The only remaining possibilities for crossed $(H - \langle s_0 \rangle)$-rim branches are r_5 and r_7. But no 1-drawing of $H - \langle s_0 \rangle$ has these two rim-branches crossed, the desired contradiction.

So $z \neq v_9$. But then we may replace s_0 with the zv_5-path s_0' in $P \cup s_0$ and replace P_0 with the ys_0'-path in $P_0 \cup s_0$ to get a new subdivision H' of V_{10}. We notice that Lemma 6.5 (1) implies that Π is H'-friendly. However, the analogue J_1' of J_1 does not bound a face, contradicting Claim 2. □

Claim 4. There is a unique 1-drawing of K. In this 1-drawing, s_0 is exposed.

The 1-drawing of K is illustrated in Figure 7.2.

Proof. If D is a 1-drawing of K, then Claim 2 and Lemma 6.6 (10) imply neither J_1 nor J_2 is crossed in D. It follows that none of r_0, r_1, r_2, r_3, and r_4 is crossed in D. Lemma 3.6 implies r_7 cannot be crossed in D, so Q_2 is clean in D. Therefore, s_0 must be in a face of $D[R \cup Q_2]$ incident with r_2. This is only possible if s_0 is exposed, which determines D. □

For $j \in \{2, 3\}$, let D_j be a 1-drawing of $G - \langle s_j \rangle$.

Claim 5. The crossing in $D_2[(H - s_2) \cup P_0]$ is of r_5 with $[y, r_2, v_3]$. Likewise, the crossing in $D_3[(H - s_3) \cup P_0]$ is of r_9 with $[v_2, r_2, y]$.

The 1-drawings of Claim 5 are illustrated in Figure 7.3.

Proof. We treat the case $j = 2$; the case $j = 3$ is very similar. By Theorem 5.23 (2), $\overline{Q}_2$ has BOD, so Lemma 5.9 implies $\overline{Q}_2$ is crossed in D_2. This implies that s_0 is not exposed in D_2. The H-avoiding path P_0 joins $x \in \langle s_0 \rangle$ to $y \in \langle r_2 \rangle$, so y must be on a face incident with s_0. It follows that Q_0 must be crossed in D_2. This

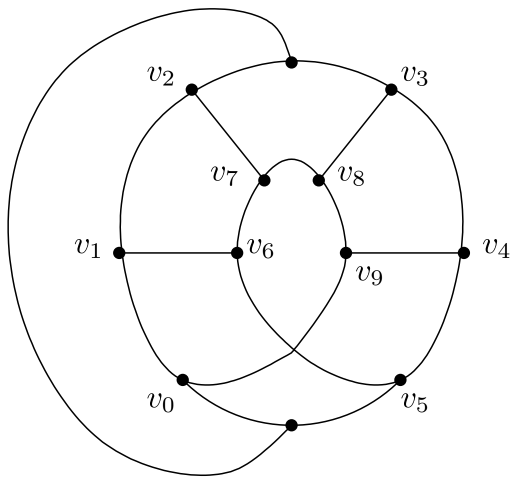

FIGURE 7.2. The 1-drawing of K.

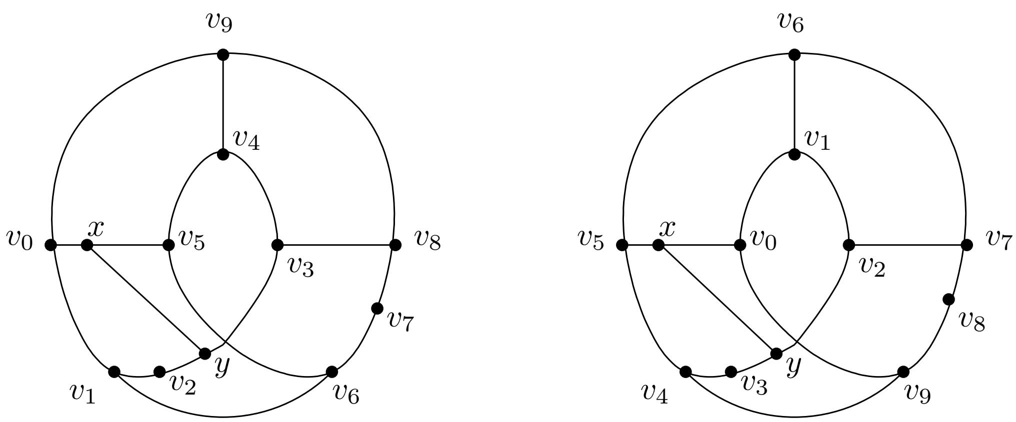

FIGURE 7.3. The 1-drawings $D_2[(K - \langle s_2 \rangle) \cup P_0]$ and $D_3[(K - \langle s_3 \rangle) \cup P_0]$.

implies that s_1 is exposed. We deduce that either r_5 crosses $r_1 \cup r_2$ or r_0 crosses $r_6 \cup r_7$. In the latter case, $D_2[P_0]$ must cross $D_2[H - s_2]$, a contradiction, so it must be the former.

As $D_2[P_0]$ is not crossed, y occurs between v_1 and the crossing in $r_1 \cup r_2$, as required. $\square$

The following claims help us obtain the structure of $(M_{\overline{Q}_0})^{\#}$; we will use this to find a 1-drawing of G, which is the final contradiction.

CLAIM 6. Suppose B is a $\overline{Q}_0$-bridge having an attachment in each of r_9 and r_5. Then B is one of $M_{\overline{Q}_0}$, v_6v_9, v_0v_6, and v_5v_9.

PROOF. We note that $s_0 \cup P_0 \subseteq M_{\overline{Q}_0}$. Either $B = M_{\overline{Q}_0}$, or, in the drawing D_2, B is in a face of $D_2[(H - s_2) \cup P_0]$ incident with both r_9 and r_5. There are only two such faces, namely F, bounded by Q_4, and F', the other face incident with r_9. Whichever face B is in, its attachments are in the intersection of $\overline{Q}_0$ with the boundary of the containing face. Thus, if B is in F, then $\text{att}(B) \subseteq r_4\, s_4\, r_9$. In this case, the only possibility for an attachment in r_5 is v_5, so $v_5 \in \text{att}(B)$. If, on the other hand, B is in F', then $\text{att}(B) \subseteq r_9\, r_0\, s_1$. In this case, $v_6 \in \text{att}(B)$. Similarly, D_3 shows either $B = M_{\overline{Q}_0}$, or $\text{att}(B) \subseteq r_0\, s_1\, r_5$ and $v_0 \in \text{att}(B)$, or $\text{att}(B) \subseteq s_4\, r_4\, r_5$ and $v_9 \in \text{att}(B)$. Comparing these possibilities, we conclude that one of the following four cases holds for $\text{att}(B)$: $\text{att}(B) = \{v_0, v_5\}$; $\text{att}(B) = \{v_6, v_9\}$; $v_5, v_9 \in \text{att}(B)$ and $\text{att}(B) \subseteq r_4 \cup s_4$; and $v_0, v_6 \in \text{att}(B)$ and $\text{att}(B) \subseteq r_0 \cup s_1$.

We claim v_0v_5 is not an H-bridge. For if it were, let D be a 1-drawing of $G - v_0v_5$. Then $s_0 \cup P_0$ is not crossed in D and Claim 3 says the $(H - \langle s_0\rangle)$-bridge containing s_0 is $s_0 \cup P_0$. In particular, s_0 consists of the two edges v_0x and xv_5, and x has degree 3 in G. Thus, we can draw v_0v_5 alongside s_0, yielding a 1-drawing of G, a contradiction.

We must show that, if $v_0, v_6 \in \text{att}(B)$ and $\text{att}(B) \subseteq r_0 \cup s_1$, then $B = v_0v_6$. Likewise, if $v_5, v_9 \in \text{att}(B)$ and $\text{att}(B) \subseteq r_4 \cup s_4$, then $B = v_5v_9$. We consider the former case, the latter being completely analogous. Corollary 5.15 shows that B can have at most one other attachment. Lemma 5.19 shows that either $B = v_0v_6$ or B is a claw with talons v_0, v_6, and $z \in \langle v_0, r_0, v_1, s_1, v_6\rangle$. Since we are trying to show $B = v_0v_6$, we assume the latter. Let e be the edge of B incident with z and let D be a 1-drawing of $G - e$. Since $K \subseteq G - e$, D extends the 1-drawing illustrated in Figure 7.2. We modify D to obtain a 1-drawing of G, which is impossible.

Observe that $B - z$ is an H-avoiding v_0v_6-path P (having length 2); there is only one place $D[P]$ can occur in Figure 7.2. Notice that B is a Q_0-local H-bridge and, furthermore, P overlaps M_{Q_0}.

Theorem 5.23 shows Q_0 has BOD in G; let $(\mathcal{B}, \mathcal{M})$ be the bipartition of $OD(Q_0)$, with $B \in \mathcal{B}$. Then $M_{Q_0} \in \mathcal{M}$. Every Q_0-bridge is drawn in D, with the exception that we have $B - e$ in place of B.

Because we cannot add e back into D to get a 1-drawing of G, there must be an H-avoiding path P' in $G - e$ joining the two components of $[v_0, r_0, v_1, s_1, v_6] - z$ so that $D[P']$ is on the same side — henceforth, the *inside* — of $D[Q_0]$ as P. Let B' be the Q_0-bridge containing P'. If B' has just v_0 and v_6 as attachments, then let D be a 1-drawing of $G - v_0v_6$. As we did above for v_0v_5, we can add v_0v_6 alongside P to recover a 1-drawing of G. Therefore, B' does not have just v_0 and v_6 as attachments.

It follows that B' overlaps B, so it is in $\mathcal{M}$. Therefore, it does not overlap M_{Q_0}; in particular, it cannot have an attachment in both $[v_6, s_1, v_1\rangle$ and $[v_0, r_0, v_1\rangle$. We conclude that, for some $q \in \{r_0, s_1\}$; and (ii) $\text{att}(B') \subseteq q$. Let q' be such that $\{q, q'\} = \{r_0, s_1\}$.

Let $B_1, B_2, \dots, B_k$ be a path in $OD(Q_0) - \{M_{Q_0}, B\}$ so that $B' = B_1$.

Subclaim 1. For $i = 1, 2, \dots, k$, $\text{att}(B_i) \subseteq q$.

Proof. Above, we chose q to contain $\text{att}(B')$, which is the case $i = 1$. Notice that B_1, B_3, $\dots$ are all on the same side of $D[Q_0]$ as B' and P, while B_2, B_4, $\dots$ are all on the other side of $D[Q_0]$. The former are all in $\mathcal{M}$, while the latter are in $\mathcal{B}$. Let i be least so that B_i has an attachment outside q. Then it also has an attachment in $\langle q\rangle$ (in order to overlap B_{i-1}).

If B_i is inside $D[Q_0]$, then B_i does not overlap M_{Q_0}, so it has no attachment in $q' - q$. As B_i cannot cross P in D, $\text{att}(B_i) \subseteq q$, a contradiction.

If B_i is outside $D[Q_0]$, then either $\mathrm{att}(B_i) \subseteq s_1$, so $q = s_1$ and we are done, or $\mathrm{att}(B_i) \subseteq r_0 \cup [v_0, s_0, x]$, so, in particular, $q = r_0$. Furthermore, B_i does not overlap B. Therefore, B_i has no attachment in $\langle v_0, s_0, x]$, so $\mathrm{att}(B_i) \subseteq r_0$. □

Let L be the component of $OD(Q_0) - \{M_{Q_0}, B\}$ containing B'. We can flip the Q_0-bridges in L so that they exchange sides of $D[Q_0]$, yielding a new 1-drawing of $G - e$ with fewer Q_0-bridges in $\mathcal{M}$ on the same side of $D[Q_0]$ as P. Inductively, this shows there is a 1-drawing D' of $G - e$ in which all Q_0-bridges in the face of $D'[K \cup P]$ bounded by $r_0\, s_1\, P$ are in $\mathcal{B}$. As none of these overlaps B, we may add e into D' to obtain a 1-drawing of G, a contradiction. □

Let e_5 be the edge in r_5 that is crossed in D_2 and let e_9 be the edge in r_9 that is crossed in D_3. For $i = 5, 9$, let u_i be the end of e_i nearer to v_i in r_i and let w_i be the other end of e_i. See Figure 7.4. We highlight some relevant "cut" properties of these edges in the next three claims.

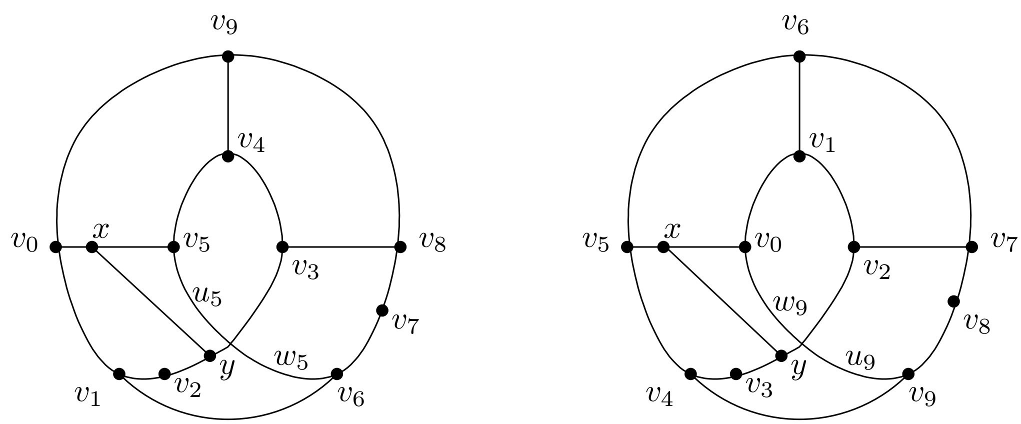

FIGURE 7.4. The 1-drawings $D_2[(K - \langle s_2 \rangle) \cup P_0]$ and $D_3[(K - \langle s_3 \rangle) \cup P_0]$.

CLAIM 7. Any r_9-avoiding $\langle s_4\, r_4]\, \langle r_0\, s_1]$-path in $(M_{\overline{Q}_0})^{\#}$ contains e_5. In particular, there are not two edge-disjoint r_9-avoiding $\langle s_4\, r_4]\, \langle r_0\, s_1]$-paths in $(M_{\overline{Q}_0})^{\#}$.

PROOF. Suppose P is a r_9-avoiding $\langle s_4\, r_4]\, \langle r_0\, s_1]$-path. Let e be any edge of s_2 and let D be any 1-drawing of $G - e$. By Claim 5, $D_2[(H - \langle s_2 \rangle) \cup P_0]$ is illustrated in Figure 7.3. But here we see that the cycle $C = [v_0, s_0, x]\, P_0\, [y, r_2, v_3] s_3\, r_8\, r_9$ separates $\langle s_4\, r_4]$ and $\langle r_0\, s_1]$. Note that C consists of r_9 and a $\overline{Q}_0$-avoiding $v_0 v_9$-path in $M_{\overline{Q}_0}$. Therefore, P is disjoint from C, and so it must cross C in D_2. As this can only happen at the crossing in D_2, it must be that the edge of r_5 crossed in D_2 is in P. □

Analogously, deleting $e \in s_3$ provides a proof of the following claim.

CLAIM 8. Any r_5-avoiding $[s_4\, r_4\rangle\, [r_0\, s_1\rangle$-path in $(M_{\overline{Q}_0})^{\#}$ contains e_9. In particular, there are not two edge-disjoint r_5-avoiding $[s_4\, r_4\rangle\, [r_0\, s_1\rangle$-paths in $(M_{\overline{Q}_0})^{\#}$. □

The final claim is a central point about $M_{\overline{Q}_0}$.

CLAIM 9. Let P_1 and P_2 be the two paths of $\overline{Q}_0 - \{e_5, e_9\}$. Then there is no P_1P_2-path in

$$(M_{\overline{Q}_0})^{\#} - \{e_5, e_9, v_6v_9\}\,.$$

PROOF. Assume that there is a P_1P_2-path P in $(M_{\overline{Q}_0})^{\#} - \{e_5, e_9\}$. For $i = 1, 2$, let z_i be the end of P in P_i.

Suppose first that z_1 is in $\langle s_4\, r_4\rangle$. If z_2 is in $[v_6, r_5, w_5]$, then $P[z_2, r_5, v_6]$ is an r_9-avoiding $\langle s_4\, r_4]\ \langle r_0\, s_1]$-path in $(M_{\overline{Q}_0})^{\#}$ that also avoids e_5, contradicting Claim 7. If z_2 is not in $[v_6, r_5, w_5]$, then there is an r_5-avoiding $[s_4\, r_4\rangle\ [r_0\, s_1\rangle$-path in $(M_{\overline{Q}_0})^{\#}$ that also avoids e_9, contradicting Claim 8. Therefore, z_1 is in $P_1 - \langle s_4\, r_4\rangle$; that is z_1 is in $[v_9, r_9, u_9] \cup [v_5, r_5, u_5]$. Symmetrically, z_2 is in $[w_9, r_9, v_0] \cup [w_5, r_5, v_6]$.

If z_1 is in $[v_5, r_5, u_5]$, then Claim 7 implies z_2 is not in $[w_5, r_5, v_5]$. Therefore, z_2 is in $[w_9, r_9, v_0]$. By Claim 6, P is one of v_6v_9, v_0v_6, and v_5v_9. Clearly, neither z_1 nor z_2 is v_6 and neither is v_9, so none of these outcomes is possible.

Therefore, z_1 is in $[v_9, r_9, u_9]$. Claim 8 implies z_2 is not in $[w_9, r_9, v_0]$. By Claim 6, the only possibility is that $z_1 = v_9$ and $z_2 = v_6$ and P is just the edge v_6v_9, as required. $\square$

We will show that there is an embedding Π' of G in $\mathbb{R}P^2$ and a non-contractible simple closed curve γ' in $\mathbb{R}P^2$ so that $\gamma' \cap G$ consists of one point in each of the interiors of $\Pi'[e_5]$ and $\Pi'[e_9]$. Standard surgery then implies that $\mathrm{cr}(G) \le 1$ (see, for example, [**29**]).

Consider the two faces of $\Pi[K]$ incident with both e_5 and e_9. Let $F_{\overline{Q}_0}$ be the one bounded by $\overline{Q}_0$. Let F' be the other; it is bounded by the cycle $s_0\, r_5\, r_6\, r_7\, r_8\, r_9$, which we call C'. Both $\overline{Q}_0$ and C' contain both e_5 and e_9. What we would like to prove is that, for each such face F with boundary C, there is no K-avoiding path contained in F and having an end in each of the two components of $C - \{e_5, e_9\}$. Although not necessarily true for Π, it is true for an embedding obtained from Π by possibly re-embedding the edges v_0v_6 and v_5v_9.

Let us begin with the possible re-embeddings. We deal with v_0v_6; the argument for v_5v_9 is completely analogous. If v_0v_6 is not embedded in F', then do nothing with it. Otherwise, it is embedded in F' and we claim we can re-embed it in $F_{\overline{Q}_0}$.

The embedding Π shows that v_0v_6 is contained in one of the two faces of $K \cup \gamma$ into which F' is split. Therefore, v_0 and v_6 must be on the same ab-subpath of R. This implies that either $v_0 = a$ or $v_6 = b$, or both. In order not to be able to embed v_0v_6 in $F_{\overline{Q}_0}$, there must be a $\overline{Q}_0$-avoiding path P contained in $F_{\overline{Q}_0}$ joining $\langle r_0\, s_1\rangle$ to $\langle r_5\, r_4\, s_4\, r_9\rangle$.

We first consider where $D_2[P]$ can be. There are only two possibilities: it is either in the face of $D_2[K - \langle s_2\rangle]$ bounded by $[v_2, r_2, \times, r_5, v_6]\, s_1\, r_1$; or in the face incident with both r_0 and s_1. The latter cannot occur, as v_0v_6 is also in that face and they overlap on the boundary of this face. So it must be the former.

However, in this case, both v_0v_6 and P are in the face of $D_3[K - \langle s_3\rangle]$ bounded by Q_0, and they overlap on Q_0, the final contradiction that shows that P does not exist, so we can re-embed v_0v_6 in $F_{\overline{Q}_0}$. Let Π' be the embedding of G obtained by any such re-embeddings of v_0v_6 and v_5v_9.

The faces $F_{\overline{Q}_0}$ and F' of $\Pi[K]$ are also faces of $\Pi'[K]$ with the same boundaries; we will continue to use these names for them, while $\overline{Q}_0$ and C' are still their boundaries.

We now show that there is no K-avoiding path in $F_{\overline{Q}_0}$ joining the two paths P_1 and P_2 of $\overline{Q}_0 - \{e_5, e_9\}$. Such a path is necessarily in $(M_{\overline{Q}_0})^{\#}$. By Claim 9, such a path is necessarily v_6v_9. But Π is H-friendly, so v_6v_9 is not embedded in $\mathfrak{M}$ and so, in particular, is not embedded in $F_{\overline{Q}_0}$. Thus, v_6v_9 is also not in this face of Π', whence there is no P_1P_2-path in $F_{\overline{Q}_0}$, as required.

Now consider the possibility of a K-avoiding path in F' having its ends in each of the two paths in $C' - \{e_5, e_9\}$. Such a path is in a C'-bridge B embedded in F'. By Claim 3, B has no attachment in $\langle s_0 \rangle$. Thus, B has an attachment either in $[v_0, r_9, w_9]$ or in $[v_5, r_5, u_5]$.

We claim it must also have an attachment in $\langle r_6\, r_7\, r_8 \rangle$. If not, then all its attachments are in

$$[v_0, r_9, w_9] \cup [v_5, r_5, u_5] \cup [w_5, r_5, v_6] \cup [v_9, r_9, u_9] \,.$$

But then B is a $\overline{Q}_0$-bridge. If it has an attachment in both r_5 and r_9, then Claim 6 implies B is one of v_0v_6, v_5v_9, and v_6v_9. The first two are not embedded in the Π'-face F' and the last does not have attachments in both components of $C' - \{e_5, e_9\}$. In the alternative, either $\operatorname{att}(B) \subseteq r_5$ or $\operatorname{att}(B) \subseteq r_9$, and then we contradict either Claim 7 or Claim 8.

So B has an attachment in $\langle r_6\, r_7\, r_8 \rangle$. If B has an attachment in $[v_0, r_9, w_9]$, then $D_3[B]$ must have a crossing, which is not possible. If B has an attachment in $[v_5, r_5, u_5]$, then $D_2[B]$ must have a crossing, which is not possible. Therefore, there is no such B, as claimed.

For each of the faces $F_{\overline{Q}_0}$ and F' of Π' and any points x and y in the interiors of $\Pi'[e_5]$ and $\Pi'[e_9]$, the preceding paragraphs show that there is a G-avoiding simple xy-arc in the face. The union of these two arcs is a simple closed curve γ' in G that meets $\Pi'[G]$ in just the two points x and y.

In a neighbourhood of x, there are points of e_5 on both sides of γ'. If γ' were contractible in $\mathbb{R}P^2$, then $\{e_5, e_9\}$ would be an edge-cut of size 2 in the 3-connected graph G, which is impossible. So γ' is non-contractible. But this is also impossible, as it meets G precisely in x and y, showing that G has a 1-drawing, the final contradiction. ■

CHAPTER 8

G embeds with all spokes in $\mathfrak{M}$

In this section, we prove that if $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, then G has a representativity 2 embedding in $\mathbb{R}P^2$ with $H \subseteq \mathfrak{M}$. This is an important step as it provides the embedding structure we need to find the tiles.

It turns out that we need something stronger than $H \subseteq \mathfrak{M}$. We must also show that, in addition to $H \subseteq \mathfrak{M}$, the representativity 2 embedding of G is such that M_{Q_4} is the only Q_4-local H-bridge B for which $Q_4 \cup B$ contains a non-contractible cycle. (We remind the reader that Q_4 is special. Each H-quad bounds a face of $\Pi[H]$. In the standard labelling, the only one of these five faces that contains an arc of γ is the one bounded by Q_4.)

THEOREM 8.1. *Suppose $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$. Then G has a representativity 2 embedding Π in $\mathbb{R}P^2$ so that, with the standard labelling:*

(1) s_0 is not exposed in Π, that is, $\Pi[H] \subseteq \mathfrak{M}$; and,
(2) if B is a Q_4-local H-bridge other than M_{Q_4}, then $\Pi[Q_4 \cup B]$ has no non-contractible cycle.

In principle, these two arguments are consecutive: we first show we can arrange $H \subseteq \mathfrak{M}$, and then deal with the Q_4-bridges. However, the arguments are essentially the same. Therefore, we shall have parallel statements and arguments, one for getting the five H-spokes in $\mathfrak{M}$ and one for getting such an embedding with Q_4 nicely behaved. (If we knew that G had an embedding with H not contained in $\mathfrak{M}$, then we could do both simultaneously.)

DEFINITION 8.2. A *friendly, standard quadruple*, denoted $((G, H, \Pi, \gamma))$, consists of $G \in \mathcal{M}_2^3$, $V_{10} \cong H \subseteq G$, an H-friendly embedding Π of G, and a non-contractible, simple closed curve γ meeting $\Pi[G]$ in precisely two points, used as the reference for giving H the standard labelling relative to Π. We abbreviate friendly, standard quadruple as *fsq*.

Observe that Theorem 3.5 implies G has a representativity 2 embedding in $\mathbb{R}P^2$. Lemma 6.5 (3) implies G has an H-friendly embedding Π. Any non-contractible simple closed curve γ in $\mathbb{R}P^2$ meeting G in precisely two points yields a standard labelling of H relative to Π and γ. Summarizing, we have the following observation.

LEMMA 8.3. *If $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, then there is an fsq $((G, H, \Pi, \gamma))$.*
■

Let Q^* be $\overline{Q}_0$ if s_0 is exposed in Π and let Q^* be Q_4 if s_0 is not exposed in Π, that is, if $\Pi[H] \subseteq \mathfrak{M}$. Our first step is to show that $OD(Q^*)$ is (nearly) bipartite. Theorem 5.23 (1) implies $OD(Q_4)$ is bipartite. For $Q^* = \overline{Q}_0$, this is more involved. In the following statement, v_1v_4 and v_6v_9 are meant to be possible $\overline{Q}_0$-bridges consisting of a single edge joining the two indicated vertices. They need not exist in G.

LEMMA 8.4. *Let* $(\!(G, H, \Pi, \gamma)\!)$ *be an fsq. If* s_0 *is exposed in* Π*, then* $OD(\overline{Q}_0) - \{v_1v_4, v_6v_9\}$ *is bipartite.*

The following observations will be needed throughout the proof of Theorem 8.1 and, in particular, the proof of Lemma 8.4.

DEFINITION 8.5. Let $(\!(G, H, \Pi, \gamma)\!)$ be an fsq and let Q^* be either $\overline{Q}_0$ (if s_0 is exposed) or Q_4 (otherwise). Then $\mathcal{N}$ — a function of $(\!(G, H, \Pi, \gamma)\!)$ — denotes the set of Q^*-bridges B other than M_{Q^*} for which $\Pi[Q^* \cup B]$ has a non-contractible cycle. In the case $Q^* = \overline{Q}_0$, any of v_1v_4 and v_6v_9 that occurs in G is a $\overline{Q}_0$-bridge B for which $\Pi[\overline{Q}_0 \cup B]$ has a non-contractible cycle, and we do not include these in $\mathcal{N}$.

We remark that, if s_0 is exposed in Π, then Theorem 7.1 implies the $(H \cap \mathfrak{M})$-bridge B^0 containing s_0 is distinct from $M_{\overline{Q}_0}$. In this case, $B^0 \in \mathcal{N}$. If s_0 is not exposed in Π, then $Q^* = Q_4$. If $\mathcal{N} = \varnothing$, then Π satisfies the conclusions of Theorem 8.1. Therefore, in this case, we may assume $\mathcal{N} \neq \varnothing$.

Before we can prove Lemma 8.4, we need some results common to both cases.

An easy corollary of the following lemma will be used to deal with the main case in the proof of Lemma 8.4.

LEMMA 8.6. *Let* D *be a 1-drawing of* V_8 *(with the usual labelling) in which* Q_1 *is crossed. Then:*

(1) Q_3 *bounds a face of* D*; and*
(2) if $\overline{Q}_0$ *is crossed in* D*, then either* r_1 *crosses* r_4 *or* r_5 *crosses* r_0*.*

Proof. As Q_1 is crossed in D, either r_1 crosses $r_4\,r_5\,r_6$ in D or r_5 crosses $r_0\,r_1\,r_2$ in D. This already shows that Q_3 bounds a face of D.

As $\overline{Q}_0$ is crossed in D, either $r_7\,r_0$ or $r_3\,r_4$ is crossed in D. Compare each of these with the possible crossing of Q_1. In the former case, r_0 crosses r_5, while in the latter case r_4 crosses r_1. ■

The following is the simple corollary that we will use.

COROLLARY 8.7. *Let* $G \in \mathcal{M}_2^3$ *and* $V_{10} \cong H \subseteq G$*. Let* D_2 *be a 1-drawing of* $G - \langle s_2 \rangle$*. Then:*

(1) Q_4 *bounds a face of* $D_2[H - s_2]$*; and*
(2) if $\overline{Q}_0$ *is crossed in* D_2*, then either* $r_6\,r_7$ *crosses* r_1 *or* $r_1\,r_2$ *crosses* r_5 *(see Figure 8.1 for the possibilities for* $D_2[H - \langle s_2 \rangle]$*).*

Likewise, if D_3 *is a 1-drawing of* $G - \langle s_3 \rangle$ *in which* $\overline{Q}_0$ *is crossed, then the two possibilities for* $D_3[H - \langle s_3 \rangle]$ *are illustrated in Figure 8.2.*

Proof. Theorem 5.23 implies $\overline{Q}_2$ has BOD. Lemma 5.9 implies $\overline{Q}_2$ is crossed in D_2. The results now follow immediately from Lemma 8.6. ■

Let r^* denote $r_9 \cup r_0$ in the case $Q^* = \overline{Q}_0$ and r_9 in the case $Q^* = Q_4$. We also let r^*_{+5} denote the other component of $Q^* \cap R$.

LEMMA 8.8. *Let* $(\!(G, H, \Pi, \gamma)\!)$ *be an fsq. If* $B \in \mathcal{N}$*, then* $\Pi[B] \subseteq \mathfrak{D}$*,* $\mathrm{att}(B) \subseteq r^* \cup r^*_{+5}$*, and* B *has an attachment in each of* r^* *and* r^*_{+5}*.*

Proof. If $\Pi[B] \subseteq \mathfrak{M}$, then $\Pi[Q^* \cup B]$ is contained in a closed disc and, therefore, has only contractible cycles, a contradiction. Thus, $\Pi[B] \subseteq \mathfrak{D}$. It now follows

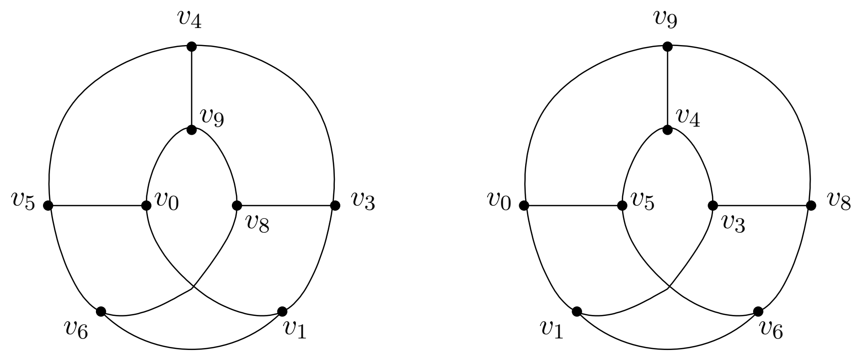

FIGURE 8.1. The two possibilities for D_2.

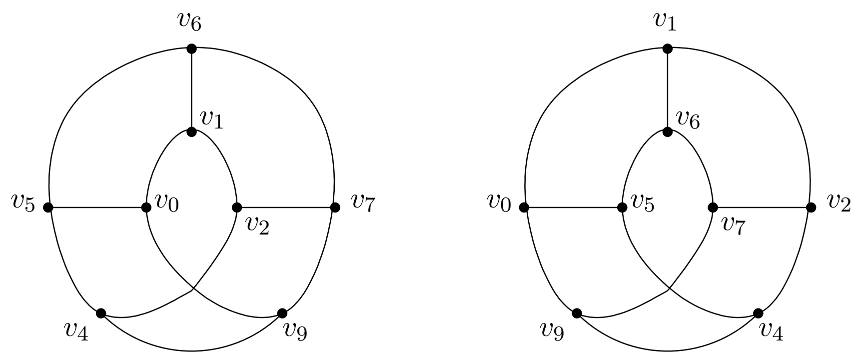

FIGURE 8.2. The two possibilities for D_3.

that $\operatorname{att}(B)$ is contained in the intersection of $\overline{Q}_0$ with the boundary of $\mathfrak{D}$; that is, $\operatorname{att}(B) \subseteq r^* \cup r^*_{+5}$.

Suppose by way of contradiction that $\operatorname{att}(B) \subseteq r^*$. Let $\bar{r}^*$ be a minimal subpath of r^* containing $\operatorname{att}(B)$. Then there is a non-contractible cycle C contained in $B \cup \bar{r}^*$.

Let F be the closed $(\Pi[H] \cup \gamma)$-face containing $\Pi[B]$. Then F contains $\Pi[B \cup \bar{r}^*]$, so the non-contractible cycle $\Pi[C]$ is contained in the closed disc F, a contradiction. So $\operatorname{att}(B)$ is not contained in r^* and, likewise, it is not contained in r^*_{+5}. ■

Let $(\!(G, H, \Pi, \gamma)\!)$ be an fsq, with s_0 exposed in Π. Suppose D_2 is a 1-drawing of $G - \langle s_2 \rangle$ in which $\overline{Q}_0$ is crossed. Corollary 8.7 implies that $D_2[H - \langle s_2 \rangle]$ is one of the two drawings illustrated in Figure 8.1. The *outside of* $D_2[\overline{Q}_0]$ is the face of $D_2[\overline{Q}_0]$ containing $D_2[s_3]$. The *inside* is the other face of $D_2[\overline{Q}_0]$. Likewise, if D_3 is a 1-drawing of $G - \langle s_3 \rangle$ in which $\overline{Q}_0$ is crossed, then the *outside of* $D_3[\overline{Q}_0]$ is the face of $D_3[\overline{Q}_0]$ containing $D_3[s_2]$.

LEMMA 8.9. *Let* $(\!(G, H, \Pi, \gamma)\!)$ *be an fsq, with* s_0 *exposed in* Π. *For* $i = 2, 3$, *let* D_i *be a 1-drawing of* $G - \langle s_i \rangle$ *in which* $\overline{Q}_0$ *is crossed. Suppose* B *is a* $\overline{Q}_0$*-bridge in* $\mathcal{N}$.

(1) If $D_2[B]$ *is outside of* $D_2[\overline{Q}_0]$, *then* $B \in \{v_1v_5, v_0v_6\}$.
(2) If $D_3[B]$ *is outside of* $D_3[\overline{Q}_0]$, *then* $B \in \{v_0v_4, v_5v_9\}$.

Proof. We prove (1); (2) is completely analogous. We remark that $B \neq B^0$ as $D_2[s_0]$ is inside $D_2[\overline{Q}_0]$. Lemma 8.8 shows that either: (i) $\mathrm{att}(B) \subseteq [b, r_5, v_6] \cup [v_9, r_9, a]$ and B has attachments in both $[b, r_5, v_6]$ and $[v_9, r_9, a]$; or (ii) $\mathrm{att}(B) \subseteq [a, r_9, v_0]r_1 \cup r_4[v_5, r_5, b]$ and B has attachments in both $[a, r_9, v_0]r_1$ and $r_4[v_5, r_5, b]$.

Suppose first that D_2 is the left-hand possibility illustrated in Figure 8.1. Considering D_2, we see that v_1 is one attachment of B and the others are in $r_4\, r_5$.

Now consider the possibilities for $D_3[B]$. We see that $D_3[B]$ can be outside $D_3[\overline{Q}_0]$ in only one of the two possible D_3's, namely the right-hand one, and then only if $\mathrm{att}(B) = \{v_1, v_4\}$. But in this case B is just the edge v_1v_4, which is not in $\mathcal{N}$. So $D_3[B]$ is inside $D_3[\overline{Q}_0]$. It now follows from this and the previous paragraphs that $\mathrm{att}(B) \subseteq \{v_1\} \cup r_5$.

Putting this information into Π, we see that the only possibility for B, which is embedded in $\mathfrak{D}$ and not in $\mathfrak{M}$, is that $B = v_1v_5$.

In the case D_2 is the right-hand possibility in Figure 8.1, D_2 shows that $\mathrm{att}(B) \subseteq \{v_6\} \cup r_9\, r_0$. Since $v_6v_9 \notin \mathcal{N}$, $B \neq v_6v_9$, so $D_3[B]$ is not outside $D_3[\overline{Q}_0]$. Therefore, D_3 shows $\mathrm{att}(B) \subseteq \{v_6\} \cup r_0$.

Again we recall that B is embedded in $\mathfrak{D}$ in $\mathbb{R}P^2$. If B is embedded in the face bounded by $[a, r_9, v_0, s_0, v_5, r_5, b, \alpha, a]$, then $b = v_6$ and the only other possible attachment for B is v_0, as required. If B is embedded in the face bounded by $[b, r_5, v_6]r_6\, r_7\, r_8[v_9, r_9, a, \alpha, b]$, then $a = v_0$ and again this is the only possible attachment other than v_6, as required. ■

Let N be the graph $\displaystyle\bigcup_{B \in \mathcal{N}} B$.

LEMMA 8.10. *Let* $((G, H, \Pi, \gamma))$ *be an fsq. Then there are not disjoint* $(N \cap r^*)(N \cap r^*_{+5})$*-paths in* N*. In particular, if* $Q^* = \overline{Q}_0$ *and* $|\mathcal{N}| \geq 2$*, then either every* $B \in \mathcal{N}$ *has only* v_0 *as an attachment in* $r_9\, r_0$ *or every* $B \in \mathcal{N}$ *has only* v_5 *as an attachment in* $r_4\, r_5$*.*

Proof. Suppose by way of contradiction that P_1 and P_2 are disjoint $r^*r^*_{+5}$-paths in N, with, for $j = 1, 2$, P_j having the end p_j in r^* and the end q_j in r^*_{+5}. Choose the labelling so that, in r^*, p_1 is closer to v_9 than p_2 is. There are three possibilities for how P_1 and P_2 are embedded by Π: both in the (closed) disc contained in $\mathfrak{D}$ bounded by $[a, r_9, v_0]r_0\, r_1\, r_2\, r_3\, r_4[v_5, r_5, b]\alpha$ (recall that $\alpha = \gamma \cap \mathfrak{D}$); both in the disc in $\mathfrak{D}$ bounded by $[b, r_5, v_6]r_6\, r_7\, r_8[v_9, r_9, a]\alpha$; or one in each of these discs. In all cases, we conclude that q_1 is closer in r^*_{+5} to v_6 than q_2 is. Summarizing, we have the following.

Fact 1 *Any two disjoint* $r^*r^*_{+5}$*-paths in* N *overlap on* Q^**.*

For $Q^* = Q_4$ we are done: Corollary 8.7 implies $D_2[Q_4]$ bounds a face of $D_2[H - \langle s_2 \rangle]$. Both P_1 and P_2 have ends in both r^* and r^*_{+5}, so both must be inside $D_2[Q_4]$, yielding the contradiction that they cross in $D_2[Q_4]$.

Now suppose $Q^* = \overline{Q}_0$. For $i = 2, 3$, $D_i[\overline{Q}_0]$ is not self-crossing; thus Fact 1 implies that $D_i[P_1]$ and $D_i[P_2]$ are on different sides of $D_i[\overline{Q}_0]$. If $\overline{Q}_0$ is clean in D_i, then we have a contradiction, as no face of $D_i[H - \langle s_i \rangle]$ is incident with both r^* and r^*_{+5} except the ones bounded by Q_4 and Q_0.

Thus, $\overline{Q}_0$ is crossed in D_i. By Lemma 8.9, the one that is outside is one of v_0v_4, v_0v_6, v_1v_5, and v_5v_9. We treat in detail that this one is v_0v_4, as the other cases are completely analogous. It is in D_3 that v_0v_4 is outside $D_3[\overline{Q}_0]$.

Because q_1 is closer to v_6 than q_2 is, q_1 cannot be v_4; it follows that it is P_2 that is v_0v_4. Lemma 8.9 also implies that P_2, that is v_0v_4, is not outside $D_2[\overline{Q}_0]$ and, therefore, it is inside $D_2[\overline{Q}_0]$. Thus, P_1 is outside $D_2[\overline{Q}_0]$. By Lemma 8.9, P_1 is one of v_0v_6 and v_1v_5. By choice of the labelling, it cannot be that v_1 is an end of P_1, so $P_1 = v_0v_6$, which is not disjoint from $P_2 = v_0v_4$, a contradiction. We conclude that there are not such disjoint paths.

For the "in particular", there is a cut vertex u of N separating $N \cap (r_9\, r_0)$ and $N \cap (r_4\, r_5)$ in N, as claimed. As s_0 is a $([r_9\, r_0])([r_4\, r_5])$-path in N, we deduce $u \in s_0$. If B^0 is not the only member of $\mathcal{N}$, then any other element B of $\mathcal{N}$ shares the vertex u with B^0, so u is an attachment of both. But $u \in s_0$ implies $u \in \{v_0, v_5\}$. ∎

As a final preparatory remark, we have the following.

LEMMA 8.11. *Let $(\!(G, H, \Pi, \gamma)\!)$ be an fsq. Let B and B' be distinct elements of $\mathcal{N}$. Then:*

(1) B and B' do not overlap on Q^*; and
(2) either B overlaps M_{Q^*} on Q^* or $Q^* = Q_4$ and B is either v_4v_9 or v_0v_5.

Proof. In the case $Q^* = Q_4$, Corollary 8.7 and Lemma 8.8 imply B and B' are both drawn inside the face of $D_2[H - \langle s_2\rangle]$ bounded by Q_4 and, therefore, they do not overlap, yielding (1) for Q_4.

For $Q^* = \overline{Q}_0$, if both B and B' are in the same face of either $D_2[\overline{Q}_0]$ or $D_3[\overline{Q}_0]$, then they obviously do not overlap on $\overline{Q}_0$. Thus, we may assume one is outside $D_2[\overline{Q}_0]$ and the other is inside $D_2[\overline{Q}_0]$ and that one is outside $D_3[\overline{Q}_0]$ and the other is inside $D_3[\overline{Q}_0]$.

By Lemma 8.9, the one outside $D_2[\overline{Q}_0]$ is either v_1v_5 or v_0v_6, while the one outside $D_3[\overline{Q}_0]$ is either v_0v_4 or v_5v_9. Thus, we may assume $B \in \{v_1v_5, v_0v_6\}$ and $B' \in \{v_0v_4, v_5v_9\}$. But none of the four possibilities is an overlapping pair, which is (1) for $\overline{Q}_0$.

As for overlapping M_{Q^*}, we suppose first that B has an attachment x in the interior of one of r^* and r^*_{+5}. (The "in particular" part of Lemma 8.10 implies this is always the case when $Q^* = \overline{Q}_0$.) In this case, it is a simple exercise to see that x, together with any attachment of B in the other one of r^* and r^*_{+5}, are skew to at least one of the pairs of diagonally opposite corners of Q^* (in the case of Q_4 these pairs are $\{v_9, v_5\}$ and $\{v_4, v_0\}$; for $\overline{Q}_0$, they are $\{v_9, v_6\}$ and $\{v_4, v_1\}$). Thus, B overlaps M_{Q^*}.

In the remaining case, $Q^* = Q_4$ and $\mathrm{att}(B) \subseteq \{v_9, v_0, v_5, v_4\}$. If both v_9 and v_5 are attachments, then B is again skew to M_{Q^*}; the same happens if both v_0 and v_4 are attachments. The only remaining cases are: $\mathrm{att}(B) = \{v_4, v_9\}$ and $\{v_0, v_5\}$, as claimed. ∎

The next result contains the essence of the proof of Lemma 8.4.

LEMMA 8.12. *Let $(\!(G, H, \Pi, \gamma)\!)$ be an fsq. Suppose $B_1 \in \mathcal{N}$, $B_k = M_{Q^*}$, and $B_1, B_2, \dots, B_k$ is an induced cycle in $OD(Q^*)$. Then either*

(1) $Q^* = \overline{Q}_0$, $k = 3$, and $B_2 \in \{v_1v_4, v_6v_9\}$ or
(2) k is even and $B_{k-1} \in \mathcal{N} \cup \{v_1v_4, v_6v_9\}$.

Proof. Case 1. *k is odd.*

Theorem 5.23 implies $OD(Q_4)$ is bipartite. Therefore, $Q^* = \overline{Q}_0$ and s_0 is exposed in Π.

For $i = 2, 3$, let e_i be the edge of s_i incident with v_i and let D_i be a 1-drawing of $G - e_i$. Theorem 5.23 implies $\overline{Q}_i$ has BOD; Lemma 5.9 implies $\overline{Q}_i$ is crossed in D_i.

If, for some $i \in \{2, 3\}$, $\overline{Q}_0$ is clean in D_i, then Lemma 5.6 implies $\overline{Q}_0$ has BOD, yielding the contradiction that k is even. Therefore, $\overline{Q}_0$ is crossed in both D_2 and D_3.

Claim 1. If some B_i is either v_1v_4 or v_6v_9, then $i = 2$ and $k = 3$.

Proof. Since both v_1v_4 and v_6v_9 overlap $M_{\overline{Q}_0}$, neither is in $\mathcal{N}$, B_1 is in $\mathcal{N}$, and the cycle is induced, it must be that $i = k - 1$. For sake of definiteness, we suppose $B_{k-1} = v_1v_4$; the alternative is treated completely analogously.

Because $B_{k-1} = v_1v_4$, we deduce that D_2 is the left-hand one of the two drawings in Figure 8.1, while D_3 is the right-hand drawing in Figure 8.2; in both drawings, B_{k-1} is outside $\overline{Q}_0$.

We note that B^0 overlaps v_1v_4, so if B_1 is B^0, then $k = 3$, as claimed. Otherwise, $B_1 \in \mathcal{N} \setminus \{B^0\}$. By Lemma 8.10, either the only attachment of B_1 in $r_9\,r_0$ is v_0 or the only attachment of B_1 in $r_4\,r_5$ is v_5. For sake of definiteness, we assume the former; the latter is completely analogous. In order not to overlap v_1v_4, the only attachment for B_1 in $r_4\,r_5$ is v_4. Therefore, either $k = 3$ and we are done, or B_1 is just the edge v_0v_4. We show that $B_1 = v_0v_4$ is not possible.

Suppose that $B_1 = v_0v_4$. Because we know D_2, we see that $D_2[B_1] = D_2[v_0v_4]$ is inside $D_2[\overline{Q}_0]$, while $D_2[B_{k-1}] = D_2[v_1v_4]$ is outside. In D_3, both are outside. But this is impossible, as $B_1, B_2, B_3, \dots, B_{k-2}, B_{k-1}$ alternate sides of $\overline{Q}_0$ in both D_2 and D_3.

We conclude that $B_1 = v_0v_4$ is impossible and therefore $k = 3$, as claimed. $\square$

It remains to show that no other possibility can occur with k odd. So suppose no B_i is either v_1v_4 or v_6v_9. Suppose some B_i other than B_1 is in $\mathcal{N}$. As B_i overlaps $M_{\overline{Q}_0}$ and the cycle $B_1, B_2, \dots, B_k$ is induced, Lemma 8.11 implies $i = k - 1$. The same lemma implies $k \geq 5$. Therefore, Lemma 5.16 implies $B_1, B_2, \dots, B_{k-2}, B_{k-1}$ alternate sides of $\Pi[\overline{Q}_0]$. Since k is odd, B_1 and B_{k-1} are on different sides of $\Pi[\overline{Q}_0]$, contradicting the fact that both are in $\mathcal{N}$. Hence no other B_i is in $\mathcal{N}$.

By Lemma 8.9, for at least one $i \in \{2, 3\}$, $D_i[B_1]$ is inside $D_i[\overline{Q}_0]$. For the sake of definiteness, we consider the case $i = 2$ and D_2 is the left-hand drawing of $H - \langle s_2 \rangle$ in Figure 8.1; the remaining cases are completely analogous. Thus, either B_1 is B^0 or B_1 is either a Q_0- or a Q_1-bridge.

Since k is odd, B_{k-1} is on the other side of $D_2[\overline{Q}_0]$ from B_1. Therefore, B_{k-1} is outside $D_2[\overline{Q}_0]$. In order to understand how B_{k-1} can overlap $M_{\overline{Q}_0}$ in D_2, we analyze $D_2[M_{\overline{Q}_0}]$.

Let e be the edge of $M_{\overline{Q}_0}$ that is crossed in D_2. The end w of e outside $D_2[\overline{Q}_0]$ is in $\mathrm{Nuc}(M_{\overline{Q}_0})$. If the other end u of e is not in $\mathrm{Nuc}(M_{\overline{Q}_0})$, then $u = v_6$ and $[\times, r_6, v_6]$ is the only part of $M_{\overline{Q}_0}$ inside $D_2[\overline{Q}_0]$. Otherwise, $\mathrm{Nuc}(M_{\overline{Q}_0}) - \{e_2, e\}$ is not connected. Since $\mathrm{Nuc}(M_{\overline{Q}_0}) - e_2$ is connected, $\mathrm{Nuc}(M_{\overline{Q}_0}) - \{e_2, e\}$ consists of the component inside $D_2[\overline{Q}_0]$ and the component O outside. In particular, $M_{\overline{Q}_0} - \{e_2, e\}$ consists of two $\overline{Q}_0$-bridges in $G - \{e_2, e\}$. Let I be the one contained inside $D_2[\overline{Q}_0]$ and let O be the one outside. All attachments of $M_{\overline{Q}_0}$ are attachments of either I

or O, and possibly both. In the case $u = v_6$, we take I to be the portion of e from $\times$ to v_6.

We observe that D_2 shows that, except for one end of e, all the attachments of I are in Q_0. On the other hand, Theorem 7.1 implies that $M_{\overline{Q}_0}$, and, therefore I, has no attachment in $\langle s_0 \rangle$. The embedding Π shows that I has no attachment in $\langle r_0 \rangle$: otherwise, I is not just $[\times, e_6, v_6]$ and $u \neq v_6$. Thus, the simple closed curve $s_1\, r_1\, r_2\, r_3\, s_4[v_9, r_9, a]\,\alpha[b, r_5, v_6]$ bounds a closed disc in $\mathbb{R}P^2$ separating u from $\langle r_0 \rangle$ and is disjoint from $\mathrm{Nuc}(I) \cup \langle r_0 \rangle$. Unless $v_0 = a$, the same simple closed curve separates u from v_0; thus, if v_0 is an attachment of I, then $a = v_0$.

Because B_{k-1} is outside $D_2[\overline{Q}_0]$ and $\mathrm{att}(B_{k-1}) \subseteq \overline{Q}_0$, there are four candidates for the face of $D_2[H - \langle s_2 \rangle]$ that contains B_{k-1}. The one bounded by Q_3 is not possible: if B_{k-1} were in that face, it would not overlap $M_{\overline{Q}_0}$, as all the $M_{\overline{Q}_0}$ attachments there would be in s_4 and, therefore, all in O and not in I; both B_{k-1} and O being outside $D_2[\overline{Q}_0]$ shows they do not overlap.

The face of $D_2[H - \langle s_2 \rangle]$ incident with $[\times, r_0, v_1]$ is not a possibility for B_{k-1} for exactly the same reason: the only attachment of I there can be v_1 and v_1 is not part of a pair of attachments of $M_{\overline{Q}_0}$ that are skew to two attachments of B_{i-1}, which are all contained in $[\times, r_0, v_1]$.

The face of $D_2[H - \langle s_2 \rangle]$ incident with $r_8\, r_9$ is also not a possibility for B_{k-1}. To see this, v_0 is the only possible attachment of I in the boundary of this face. Thus, v_0 is an attachment of I and B_{k-1} must have attachments in each of $[v_9, r_9, v_0\rangle$ and $\langle v_0, r_0, \times]$. However, in Π we must have $a = v_0$ and then there is no way to embed B_{k-1}.

Therefore, B_{k-1} is in the face of $D_2[H - \langle s_2 \rangle]$ incident with $r_5\, s_1$.

By way of contradiction, suppose B_{k-1} is outside $D_3[\overline{Q}_0]$. Identical arguments as those just above show that B_{k-1} is in the face of $D_3[H - \langle s_2 \rangle]$ incident with $r_9\, s_4$. Because the previous paragraph shows $\mathrm{att}(B_{k-1}) \subseteq r_4\, r_5\, s_1$, it cannot overlap $M_{\overline{Q}_0}$ using an attachment of the portion of $M_{\overline{Q}_0}$ that is inside $D_3[\overline{Q}_0]$ and, therefore, it cannot overlap $M_{\overline{Q}_0}$ at all, a contradiction. Therefore, B_{k-1} is inside $D_3[\overline{Q}_0]$. This implies B_{k-1} is either a Q_0- or Q_4-bridge.

If B_{k-1} is a Q_4-bridge, then $\mathrm{att}(B_{k-1}) \subseteq r_4$ (because of D_2). Letting $\bar{r}$ denote the minimal subpath of r_4 containing $\mathrm{att}(B_{k-1})$, D_2 shows that no attachment of I is in $\langle \bar{r} \rangle$ and, because O and B_{k-1} do not overlap (in D_2), O also has no attachment in $\langle \bar{r} \rangle$. Consequently, B_{k-1} does not overlap $M_{\overline{Q}_0}$, a contradiction. Therefore, B_{k-1} is a Q_0-bridge.

Because B_{k-2} is inside $D_2[\overline{Q}_0]$, has no attachments in s_0, and overlaps B_{k-1} as $\overline{Q}_0$-bridges, we see that B_{k-2} is also a Q_0-bridge. Continuing back, we see that each of $B_{k-3}, \ldots, B_2$ is a Q_0-bridge and that B_1 is outside $D_3[\overline{Q}_0]$. By Lemma 8.9, B_1 is either v_0v_4 or v_5v_9. But neither of these overlaps B_2. This contradiction shows that, except for the case described in Claim 1, k is even.

Case 2. *k is even.*

For each $i = 2, 3, \ldots, k-2$, $B_i \cup Q^*$ has no non-contractible cycle in $\mathbb{R}P^2$. Thus, Lemma 5.16 implies B_1 and B_{k-1} are on the same side of Q^* in $\mathbb{R}P^2$; since B_1 is Q^*-exterior, we have that B_{k-1} is Q^*-exterior. If $\Pi[Q^* \cup B_{k-1}]$ has no non-contractible cycle, then Lemma 5.16 shows that it cannot overlap M_{Q^*}, a contradiction. In the case $Q^* = Q_4$, this implies that B_{k-1} is in $\mathcal{N}$, while if $Q^* = \overline{Q}_0$, then B_{k-1} is in $\mathcal{N} \cup \{v_1v_4, v_6v_9\}$. ■

Proof of Lemma 8.4. We show that any odd cycle C in $OD(\overline{Q}_0)$ contains either v_1v_4 or v_6v_9. Theorem 5.23 (3) implies that $OD(\overline{Q}_0)-M_{\overline{Q}_0}$ is bipartite. Therefore, C contains $M_{\overline{Q}_0}$. Lemma 8.12 shows that any odd cycle in $OD(\overline{Q}_0)$ containing $M_{\overline{Q}_0}$ and an element of $\mathcal{N}$ has length 3 and contains one of v_1v_4 and v_6v_9, as required.

Thus, we may suppose C avoids $\mathcal{N}\cup\{v_1v_4, v_6v_9\}$; let $C=(B_1,B_2,\dots,B_{2k},M_{\overline{Q}_0})$. For each $i=1,2,\dots,2k$, $\Pi[B_i\cup\overline{Q}_0]$ has no non-contractible cycles in $\mathbb{R}P^2$. Lemma 5.16 implies B_i and B_{i+1} are on different sides of $\Pi[\overline{Q}_0]$. From this, parity implies that B_1 and B_{2k} are on opposite sides of $\Pi[\overline{Q}_0]$. On the other hand, they are both on the side of $\Pi[\overline{Q}_0]$ not containing $M_{\overline{Q}_0}$, a contradiction. ∎

We are now prepared for the proof of Theorem 8.1.

Proof of Theorem 8.1. By Theorem 3.5, G has a representativity 2 embedding Π in $\mathbb{R}P^2$. For (1), if no spoke is exposed in Π, then we are done; thus, with the standard labelling, we may suppose that s_0 is exposed in Π. From Theorem 7.1, we know that the $\overline{Q}_0$-bridge B^0 containing s_0 is different from $M_{\overline{Q}_0}$. From Lemma 8.4, we know that $OD(\overline{Q}_0)-\{v_1v_4,v_6v_9\}$ is bipartite and from Theorem 5.23 (3), we know that $(M_{\overline{Q}_0})^{\#}$ is planar.

We need to modify Π so that the set $\mathcal{N}$ (Definition 8.5) becomes empty. We start with terminology that will be useful for the next claims.

Definition 8.13. Let L be a graph. A path $(v_1,v_2,\dots,v_k)$ in L is *chordless in L* if there is no edge v_iv_j of L that is not in P except possibly v_1v_k.

The following is a simple consequence of Lemma 8.12.

Claim 1. (1) If $Q^*=\overline{Q}_0$, then every $\mathcal{N}M_{\overline{Q}_0}$-path in $OD(\overline{Q}_0)$ of length at least two contains one of v_1v_4 and v_6v_9.

(2) If $Q^*=Q_4$, then every chordless $\mathcal{N}M_{Q_4}$-path in $OD(Q_4)$ of length at least two has length exactly two, one end is either v_4v_9 or v_0v_5, and that end does not overlap M_{Q_4}.

Proof. Suppose first that $Q^*=\overline{Q}_0$. Let P be any $\mathcal{N}M_{\overline{Q}_0}$-path in $OD(\overline{Q}_0)$ that has length at least 2. We may assume P is chordless: otherwise there is a shorter $\mathcal{N}M_{\overline{Q}_0}$-path P' of length at least 2 and $V(P')\subseteq V(P)$; if P' contains either v_1v_4 or v_6v_9, then so does P. By Lemma 8.11 (2), the ends of P are adjacent in $OD(\overline{Q}_0)$. Thus, P together with this edge of $OD(\overline{Q}_0)$ makes an induced cycle. As this cycle has only one vertex in $\mathcal{N}$, Lemma 8.12 implies the cycle has length 3 and contains one of v_1v_4 and v_6v_9.

Now suppose that $Q^*=Q_4$ and $P=(B_1,B_2,\dots,B_k,M_{Q_4})$ is a chordless $\mathcal{N}M_{Q_4}$-path in $OD(Q_4)$ of length at least 2. Then $B_1\in\mathcal{N}$. Since P is chordless and $B_k\notin\mathcal{N}$, Lemma 8.12 (2) implies B_1 does not overlap M_{Q_4}. Now Lemma 8.11 (2) implies B_1 is either v_4v_9 or v_0v_5. Thus, B_2 is skew to B_1. Since $\mathrm{att}(B_1)\subseteq\mathrm{att}(M_{Q_4})$, B_2 is also skew to M_{Q_4}. Since P is chordless, $k=2$, as required. □

If $Q^*=\overline{Q}_0$, then set $\mathcal{M}$ to be the set $\{M_{\overline{Q}_0},v_1v_4,v_6v_9\}$, while if $Q^*=Q_4$, then set $\mathcal{M}$ to be the set $\{M_{Q_4},v_4v_9,v_0v_5\}$. In either case, let $\mathcal{M}^-=\mathcal{M}\setminus\{M_{Q^*}\}$.

Let $\mathcal{N}^+$ be the set of Q^*-bridges B so that there is an $\mathcal{N}B$-path in $OD(Q^*)$ that is disjoint from $\mathcal{M}$. The next lemma shows that $\mathcal{N}^+$ consists of the members of $\mathcal{N}$, which have attachments in both r^* and r^*_{+5}, and other Q^*-bridges B that simply extend out along either r^* or r^*_{+5}. This structure is what will allow us to find natural "breaking points" a' and b' in r^* and r^*_{+5}, respectively, to allow us to "flip" the members of $\mathcal{N}$ into $\mathfrak{M}$, yielding the embedding with $H \subseteq \mathfrak{M}$ and $\mathcal{N} = \varnothing$.

Claim 2. If $B \in \mathcal{N}^+$, then $\operatorname{att}(B) \subseteq r^* \cup r^*_{+5}$. Furthermore, if $B \in \mathcal{N}^+ \setminus \mathcal{N}$, then either $\operatorname{att}(B) \subseteq r^*$ or $\operatorname{att}(B) \subseteq r^*_{+5}$.

Proof. Let P be a shortest $\mathcal{N}B$-path in $OD(Q^*)$ that is disjoint from $\mathcal{M}$. We proceed by induction on the length of P.

If $B \in \mathcal{N}$, then the result follows from Lemma 8.8. Otherwise, $B \notin \mathcal{N}$. The neighbour B' of B in P is closer to $\mathcal{N}$ than B is, so $\operatorname{att}(B') \subseteq r^* \cup r^*_{+5}$.

If B overlaps M_{Q^*}, then P extends to a chordless $\mathcal{N}M_{Q^*}$-path in $OD(Q^*)-\mathcal{M}^-$ of length at least 2. This contradicts Claim 1, showing B does not overlap M_{Q^*}.

Suppose by way of contradiction that B has an attachment x in the interior of some H-spoke s contained in Q^*. As B overlaps B' and $\operatorname{att}(B') \subseteq r^* \cup r^*_{+5}$, not all attachments of B can be in $[s]$. But any attachment y of B in $Q^* - [s]$ combines with x to show that B is skew to the ends of s and, therefore, overlaps M_{Q^*}. Therefore, $\operatorname{att}(B) \subseteq r^* \cup r^*_{+5}$.

Next suppose that B has an attachment in $\langle r^* \rangle$. If B also has an attachment in $Q^* - [r^*]$, then B overlaps M_{Q^*} (the two identified attachments of B are skew to the two ends of r^*). Thus, if B has an attachment in $\langle r^* \rangle$, then $\operatorname{att}(B) \subseteq r^*$. Likewise, if B has an attachment in $\langle r^*_{+5} \rangle$, then $\operatorname{att}(B) \subseteq r^*_{+5}$.

If B has an attachment in each of r^* and r^*_{+5}, then the preceding paragraph shows that $\operatorname{att}(B)$ consists of some of the four H-nodes that comprise the ends of r^* and r^*_{+5}. Because B overlaps B', $\operatorname{att}(B)$ cannot be just the two ends of one of the two H-spokes in Q^*. In the remaining case, B is skew to M_{Q^*}, a contradiction. Thus, either $\operatorname{att}(B) \subseteq r^*$ or $\operatorname{att}(B) \subseteq r^*_{+5}$. □

Let $OD^-(\overline{Q}_0) = OD(\overline{Q}_0) - \{v_1v_4, v_6v_9\}$ and let $OD^-(Q_4) = OD(Q_4)$. By Lemma 8.4 or Theorem 5.23 (1), $OD^-(Q^*)$ is bipartite; let (S,T) be a bipartition of $OD^-(Q^*)$, with $M_{Q^*} \in T$. We briefly treat separately the cases $Q^* = \overline{Q}_0$ and $Q^* = Q_4$.

For the former, every element of $\mathcal{N}$ overlaps $M_{\overline{Q}_0}$ and so $\mathcal{N} \subseteq S$. There is an embedding Φ of $(G - \{v_1v_4, v_6v_9\}) - \operatorname{Nuc}(M_{\overline{Q}_0})$ in the plane so that all the $\overline{Q}_0$-bridges in $\mathcal{N}$ are on the same side of $\Phi[\overline{Q}_0]$.

In the case of $Q^* = Q_4$, $\mathcal{N} \setminus \{v_4v_9, v_0v_5\} \subseteq S$. There is an embedding Φ of $G - \operatorname{Nuc}(M_{Q_4})$ in the plane so that all the Q_4-bridges in $\mathcal{N} \setminus \{v_4v_9, v_0v_5\}$ are on the same side of $\Phi[Q_4]$. Any of v_4v_9 and v_0v_5 that is also in S can also be embedded on that same side of $\Phi[Q_4]$.

Among the attachments of the elements of $\mathcal{N}^+$, let a_9 be the one in r^* nearest v_9 and let a_4 be the one in r^*_{+5} nearest v_4.

Claim 3. No Q^*-bridge not in $\mathcal{M}$ is skew to $\{a_4, a_9\}$.

Proof. It is clear that, in the case $Q^* = Q_4$, neither v_4v_9 nor v_0v_5 is skew to $\{a_4, a_9\}$. We show that a Q^*-bridge not in $\mathcal{M}$ that is skew to $\{a_4, a_9\}$ must overlap some Q^*-bridge in $\mathcal{N}^+$; this implies the contradiction that it is in $\mathcal{N}^+$.

By the Ordering Lemma 4.8, the elements of $\mathcal{N} \cap S$ occur in order on Q^* in Φ. Thus, there is one element B' of $\mathcal{N} \cap S$ that has both an attachment nearest to v_4 (relative to r^*) and an attachment nearest to v_9 (relative to r^*_{+5}). Let x'

and y' be the attachments of B' nearest v_4 in r^* and v_9 in r^*_{+5}, respectively. In the case $Q^* = \overline{Q}_0$, B^0 is a candidate for B', so, even in this case, we have that $x' \in [v_4, r_4, v_5]$ and $y' \in [v_9, r_9, v_0]$.

Suppose by way of contradiction that some Q^*-bridge B'' not in $\mathcal{M}$ has attachments x'' and y'' in the two components of $Q^* - \{a_4, a_9\}$. We note that, when $Q^* = Q_4$, $B'' \neq v_4v_9$ and $B'' \neq v_0v_5$.

If one of x'' and y'' is in the component of $Q^* - \{x', y'\}$ that is disjoint from $s_4 - \{x', y'\}$, then B'' overlaps B'. Since $B' \in \mathcal{N}$, Lemma 8.11 implies $B'' \notin \mathcal{N}$ and, therefore, $B'' \in \mathcal{N}^+$. But this contradicts the definition of either a_4 or a_9 and, therefore, both x'' and y'' are contained in the component of $Q^* - \{x', y'\}$ that contains $s_4 - \{x', y'\}$. In particular, we may assume $y'' \in \langle a_4, r_4, x'] \cup \langle a_9, r_9, y']$. For the sake of definiteness, we assume $y'' \in \langle a_9, r_9, y']$.

Some Q^*-bridge B^+ in $\mathcal{N}^+$ has a_9 as an attachment; since y'' is in $\langle a_9, r_9, y']$, $y' \neq a_9$ and, therefore, B^+ is not in $\mathcal{N}$. There is a shortest path $P = (B', B_1, \dots, B_n)$ in $OD^-(Q^*) - M_{Q^*}$ from B' to some element B_n of $\mathcal{N}^+$ so that B_n has an attachment y_n in $[a_9, r_9, y''\rangle$; choose y_n so that it is as close to a_9 in $[a_9, r_9, y''\rangle$ as possible.

The Q^*-bridge B_{n-1} is in $\mathcal{N}^+$ and so, by minimality of n, does not have an attachment in $[a_9, r_9, y''\rangle$. Since B_n overlaps B_{n-1}, there is an attachment z_n of B_n in $\langle y'', r_9, x']$. Since B'' is skew to $\{a_4, a_9\}$, there is an attachment z'' of B'' in $\langle a_9, r_9, v_9] s_4 [v_4, r_4, a_4\rangle$. But now z_n, y'', y_n, and z'' show B'' overlaps B_n. Since $B'' \notin \mathcal{M}$, B'' is in $\mathcal{N}^+$. But this contradicts the definition of a_4 or a_9. □

The following is immediate from Claim 3.

Claim 4. Each Q^*-bridge not in $\mathcal{M}$ has all its attachments in one of the two a_4a_9-subpaths of Q^*. □

The proof now bifurcates into the two cases. We consider first the case $Q^* = \overline{Q}_0$ and that s_0 is exposed in Π. The following is immediate from Claim 4.

Claim 5. The planar embedding Φ of $(G - \{v_1v_4, v_6v_9\}) - \mathrm{Nuc}(M_{\overline{Q}_0})$ has the property that there is a simple closed curve in the plane that meets $\Phi[(G - \{v_1v_4, v_6v_9\}) - \mathrm{Nuc}(M_{Q^*})]$ precisely at a_4 and a_9. □

We are now prepared to describe a representativity 2 embedding of G in $\mathbb{R}P^2$ so that all H-spokes are in $\mathfrak{M}$.

Let Ψ be an embedding of H in $\mathbb{R}P^2$ so that all H-spokes are contained in the Möbius band $\mathfrak{M}_\Psi$ bounded by $\Psi[R]$ and let γ_Ψ be a non-contractible, simple, closed curve that meets H in precisely the points a_4 and a_9. The claim is that this embedding extends to an embedding of G so that γ_Ψ meets G only at a_4 and a_9.

Claim 4 implies that we can add all the $\overline{Q}_0$-bridges other than v_1v_4, v_6v_9, and $M_{\overline{Q}_0}$ to Ψ so that there is no additional intersection with γ_Ψ. It remains to show that we may also add the at most three remaining $\overline{Q}_0$-bridges.

Claim 6. At most one of v_1v_4 and v_6v_9 is in G.

Proof. Suppose both are in G. We consider a 1-drawing D_2 of $G - \langle s_2\rangle$. As $\overline{Q}_2$ must be crossed in D_2 (it has BOD and s_2 is contained in a planar $\overline{Q}_2$-bridge; apply Lemma 5.9), we conclude that $r_0\,r_1\,r_2\,r_3$ crosses $r_5\,r_6\,r_7\,r_8$ in D_2. In particular, s_0 and s_4 cannot be exposed.

In order for v_1v_4 to be not crossed in D_2, we must have the crossing in r_0. Likewise, v_6v_9 implies the crossing is in r_5. But then neither $r_1\,r_2$ nor $r_6\,r_7$ is crossed, so $\overline{Q}_2$ is not crossed in D_2, a contradiction. □

We note that v_1v_4 and v_6v_9 are not symmetric: the embedding Π of G in $\mathbb{R}P^2$ distinguishes these two cases. However, it is easy to add either of these to Ψ so that the newly added edge is in the closed disc $\mathfrak{D}_\Psi$ bounded by $\Psi[R]$ in Ψ.

Finally, it remains to show that we may also add $M_{\overline{Q}_0}$ to Ψ. Here the argument depends slightly on which of v_1v_4 and v_6v_9 occurs in G. We will assume, for the sake of definiteness, that it is v_1v_4 that occurs; the argument in the other case is completely analogous. We shall simply import $\Pi[M_{\overline{Q}_0}]$ in $\mathbb{R}P^2$ as its embedding in Ψ.

To this end, let B be any H-bridge contained in $M_{\overline{Q}_0}$ so that $\Pi[B] \subseteq \mathfrak{D}$. We show that either $\operatorname{att}(B) \subseteq r_0\,r_1\,r_2\,r_3\,[v_4, r_4, a_4]$ or $\operatorname{att}(B) \subseteq r_5\,r_6\,r_7\,r_8\,[v_9, r_9, a_9]$.

We begin by observing that such a B cannot overlap v_1v_4 (as R-bridges), as both are are embedded in $\mathfrak{D}$ by Π. An analogous discussion applies if v_1v_4 is replaced by v_6v_9.

The embedding Π shows B cannot have an attachment in each of $\langle r_1\,r_2\,r_3\rangle$ and $\langle r_5\,r_6\,r_7\,r_8\,r_9\rangle$. Likewise, B cannot have an attachment in each of $\langle r_6\,r_7\,r_8\rangle$ and $r_0\,r_1\,r_2\,r_3\,r_4$. The next claim treats the remaining possibilities.

Claim 7. The H-bridge B does not have an attachment in each of $\langle r_1\,r_2\,r_3\rangle$ and $\langle a_4, r_4, v_5]$. Likewise, B does not have an attachment in each of $\langle r_6\,r_7\,r_8\rangle$ and either r_5 or $\langle a_9, r_9, v_0]$.

Proof. Suppose by way of contradiction that B has an attachment x in $\langle a_4, r_4, v_5]$ and an attachment $y \in \langle r_1\,r_2\,r_3\rangle$. Let P be an H-avoiding xy-path in B. Since a_4 is an attachment of some element of $\mathcal{N}^+$, there is a shortest path S in $OD(\overline{Q}_0) - \{v_1v_4, v_6v_9, M_{\overline{Q}_0}\}$ joining some $B_{\mathcal{N}}$ in $\mathcal{N}$ to a $\overline{Q}_0$-bridge $B_{\mathcal{N}^+}$ so that $B_{\mathcal{N}^+}$ has an attachment in $[v_4, r_4, x\rangle$.

If $B_{\mathcal{N}^+} \in \mathcal{N}$, then $B_{\mathcal{N}^+} \subseteq \mathfrak{D}$. Lemma 8.8 shows $B_{\mathcal{N}^+}$ has an attachment in each of r^* and r^*_{+5}; therefore, $B_{\mathcal{N}^+}$ is not contained in the closed disc bounded by P and a subpath of $r_1\,r_2\,r_3\,r_4$, $B_{\mathcal{N}^+}$ and P must cross in Π. Therefore, $B_{\mathcal{N}^+} \in \mathcal{N}^+ \setminus \mathcal{N}$.

The neighbour $B'_{\mathcal{N}^+}$ of $B_{\mathcal{N}^+}$ in S does not have an attachment in $[v_4, r_4, x\rangle$. Since $B_{\mathcal{N}^+}$ overlaps $B'_{\mathcal{N}^+}$, it follows that $B_{\mathcal{N}^+}$ has another attachment in $\langle x, r_4, v_5, r_5, b]$. In particular, the edge e of $[v_4, r_4, x]$ incident with x is H-green because of $B_{\mathcal{N}^+}$.

On the other hand, if either $x \neq v_5$ or $y \notin \langle r_1\rangle$, then P combines with the xy-subpath of $r_1\,r_2\,r_3[v_4, r_4, x]$ to make another H-green cycle containing e, contradicting Theorem 6.7. Therefore, $x = v_5$ and $y \in \langle r_1\rangle$. But then $\operatorname{att}(B) \subseteq \overline{Q}_0$, contradicting the fact that $B \subseteq M_{\overline{Q}_0}$.

The "likewise" statement has an analogous proof. □

We now see that Ψ may be extended to include $\Pi[M_{\overline{Q}_0}]$, completing the proof when $Q^* = \overline{Q}_0$.

The proof will be completed by now considering the case $Q^* = Q_4$. The only difference in how we proceed is to note that the H-bridges v_4v_9 and v_0v_5, if they exist, may be transferred to $\mathfrak{M}$ at the start. To see this, first observe that v_4v_9 and v_0v_5 overlap on R and so cannot both be embedded in $\mathfrak{D}$. If v_4v_9 is not contained in $\mathfrak{M}$, then we may consider H' to be $(H - \langle s_4\rangle) + v_4v_9$, relabel H' so that v_4v_9 — the exposed spoke — is s_0 and proceed as above to move v_4v_9 into $\mathfrak{M}$. ■

The following notions will be helpful for the duration of the work.

Definition 8.14. Let G be a graph, $V_{10} \cong H \subseteq G$ and let B be an H-bridge in G.

(1) If there is an $i \in \{0,1,2,3,4\}$ so that $\operatorname{att}(B) \subseteq Q_i$, then B is both a *local H-bridge* and a *Q_i-local H-bridge*.
(2) Otherwise, B is a *global H-bridge*.

Corollary 8.15. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$. Then there is no i so that $\overline{Q}_i$ has BOD and each edge of $r_{i-2}\, r_{i-1}\, r_i\, r_{i+1}$ is in an H-green cycle consisting of a global H-bridge and a path in R having at most two H-nodes other than v_i.*

Proof. By way of contradiction, suppose there is such an i. By Theorem 8.1, G has a representativity 2 embedding in $\mathbb{R}P^2$ so that $H \subseteq \mathfrak{M}$. Thus, s_i is in a $\overline{Q}_i$-bridge other than $M_{\overline{Q}_i}$.

By Lemma 6.6 (10), no edge of $r_{i-2}\, r_{i-1}\, r_i\, r_{i+1}$ can be crossed in any 1-drawing D of $G - \langle s_i \rangle$. By hypothesis, $\overline{Q}_i$ has BOD, so Lemma 5.9 implies $\overline{Q}_i$ is crossed in D, which further implies that some edge of $r_{i-2}\, r_{i-1}\, r_i\, r_{i+1}$ is crossed in D, a contradiction. ∎

CHAPTER 9

Parallel edges

In this very short chapter, we present some observations on how parallel edges can occur in 2-crossing-critical graphs. This will be used in later sections, especially Section 15, where we determine all the 3-connected, 2-crossing-critical graphs that do not have a subdivision of V_8. There are easy generalizations to k-crossing-critical graphs.

Definition 9.1. For an edge e of a graph G, $\mu(e)$ denotes the number of edges parallel to e (including e itself).

Observation 9.2. *Let G be a 2-crossing-critical graph and let e and e' be parallel edges of G. Then:*

(1) if $\overline{G}$ is the underlying simple graph, then $\overline{G}$ is not planar;
(2) the edge e' is crossed in any 1-drawing of $G - e$;
(3) $\mu(e) \leq 2$;
(4) if e' is an edge parallel to e, then $G - \{e, e'\}$ is planar;
(5) if $\mathrm{cr}(G) > 2$, then G is simple; and
(6) if $n \geq 4$ and $V_{2n} \cong H \subseteq G$, then one of e and e' is in the H-rim.

Proof. For (1), a planar embedding of $\overline{G}$ allows us to introduce all the parallel edges of G with no crossings, showing G is planar, a contradiction.

For (2)–(5), let D be a 1-drawing of $G - e$ and suppose e' is not crossed in D Then we may add e alongside $D[e']$ to obtain a 1-drawing of G, a contradiction. Since D has at most one crossing, it must be of e', which is (2). Adding e alongside $D[e']$ yields a 2-drawing of G. Thus we have (4) and (5). Also, (3) follows, since any other edge e'' parallel to e does not cross e' in D_e. Thus, e'' is not crossed in D_e, which contradicts the second sentence, with e'' in place of e'.

Finally, for (6), we may suppose e is not in H. Lemma 3.6 shows that the only edges that are in every non-planar subgraph of $G - e$ are those in the H-rim. Therefore, e' is in the H-rim. □

CHAPTER 10

Tidiness and global H-bridges

In this section, we show that, if $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, then there is a $V_{10} \cong H' \subseteq G$ with many useful additional characteristics that we call "tidiness". The main result is that a tidy subdivision of V_{10} has only very particular global bridges, each of which is an edge. We start with a slightly milder version of tidiness.

DEFINITION 10.1. Let Π be a representativity 2 embedding of G in $\mathbb{R}P^2$ and let $V_{10} \cong H \subseteq G$. Then H is Π-*pretidy* if:

(1) all H-spokes are embedded in $\mathfrak{M}$; and
(2) for every H-quad Q and for every Q-bridge B other than M_Q, $Q \cup B$ has no non-contractible cycle in Π.

The first step in this section is to find an embedding with a pretidy subdivision of V_{10}.

LEMMA 10.2. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$. Then G has a representativity 2 embedding Π in $\mathbb{R}P^2$ so that H is Π-pretidy.*

Proof. By Theorem 8.1, G has a representativity 2 embedding Π in $\mathbb{R}P^2$ so that all the H-spokes are contained in $\mathfrak{M}$ and so that, for any Q_4-bridge B other than M_{Q_4}, $\Pi[Q_4 \cup B]$ has no non-contractible cycle. We note that every global H-bridge is contained in $\mathfrak{D}$. We describe a particular representativity 2 embedding Π^* of G in $\mathbb{R}P^2$ for which H is Π^*-pretidy. Let γ be the non-contractible simple closed curve that meets $\Pi(G)$ at just the two points a and b.

The embedding Π^* is obtained by adjusting the local H-bridges; we do not adjust those that are Q_4-local. We start with Π^* being the same as Π on H and all the Q_4-bridges other than M_{Q_4}. Let Q be an H-quad other than Q_4. By Theorem 5.23, Q has BOD and all Q-bridges other than M_Q are planar. Let (S,T) be a bipartition of $OD(Q)$ labelled so that $M_Q \in T$. Let Π_Q be a planar embedding of Q and all the Q-bridges other than M_Q so that all the Q-bridges in $T \setminus \{M_Q\}$ are on one side of $\Pi_Q[Q]$ and all the Q-bridges in S are on the other side of $\Pi_Q[Q]$.

Extend Π^* to include all the Q-bridges other than M_Q by placing the Q-bridges in S into the H-face in Π^* bounded by $\Pi^*[Q]$, using Π_Q. As every Q-bridge in $T \setminus \{M_Q\}$ does not overlap M_Q, each of these has all its attachments on one of the four H-branches in Q and these may be embedded in Π^* on the other side of $\Pi^*[Q]$, and without crossing $M_Q \cup \gamma$.

The only concern here is that a local H-bridge can be local for distinct H-quads. Such an H-bridge B must have all its attachments on the same H-spoke s_i. We claim it is in T for one of Q_{i-1} and Q_i and in S for the other one of Q_{i-1} and Q_i.

As G is 3-connected, $OD(Q_i)$ is connected (see [**6**, Thm. 1], where this is proved for binary matroids). There is a shortest $M_{Q_i}B$-path $P = (B_0, B_1, \dots, B_n)$

in $OD(Q_i)$ (thus, $B_0 = M_{Q_i}$ and $B_n = B$). Let k be least so that B_k has an attachment in $\langle s_i \rangle$.

CLAIM 1. For $j > k$, $\text{att}(B_j) \subseteq s_i$, and $k \leq 1$.

PROOF. If, for some $j > k$, B_j has an attachment not in s_i, then $j < n$. If B_j has an attachment in $\langle s_i \rangle$, then B_j is skew to M_{Q_i} and P is not a shortest $M_{Q_i}B$-path, a contradiction. Thus, there is a least $j' > j$ so that $B_{j'}$ has an attachment in $\langle s_j \rangle$. Since $B_{j'}$ overlaps $B_{j'-1}$ and $B_{j'-1}$ has no attachment in $\langle s_i \rangle$, $B_{j'}$ has an attachment not in s_i. Again, $B_{j'}$ is skew to M_{Q_i}, so P is not a shortest $M_{Q_i}B$-path, a contradiction. Thus, for all $j > k$, $\text{att}(B_j) \subseteq s_i$.

If $k = 0$, then obviously $k \leq 1$, so we may assume $k \geq 1$. As B_k has an attachment in $\langle s_i \rangle$ and B_{k-1} does not, it follows that B_k has an attachment not in s_i. But then B_k is skew to M_{Q_i}. Because P is a shortest $M_{Q_i}B$-path, we deduce that $k \leq 1$. □

The claim shows that the Q_i-bridges $B_{k+1}, B_{k+2}, \ldots, B_n$ are also Q_{i-1}-bridges and, therefore, $(B_{k+1}, B_{k+2}, \ldots, B_n)$ is a path in $OD(Q_{i-1})$. Suppose first that $k = 0$. Then M_{Q_i} contains a vertex x in $\langle s_i \rangle$ so that x and v_{i+1} are skew to B_1. There is a shortest Q_i-avoiding path P in M_{Q_i} joining x to a vertex in $\text{Nuc}(M_{Q_i}) \cap H$. Since P is not in the face of $\Pi[Q_i]$ contained in $\mathfrak{M}$, we deduce that P is contained in the face of $\Pi[Q_{i-1}]$ contained in $\mathfrak{M}$. But then we conclude that P is contained in a Q_{i-1}-local H-bridge B', showing that B' is skew to both $M_{Q_{i-1}}$ and to B_1. We deduce that, in $OD(Q_i)$, M_{Q_i} and B_1 are on opposite sides of the bipartition of $OD(Q_i)$, while $M_{Q_{i-1}}$ and B_1 are on the same side of the bipartition of $OD(Q_{i-1})$. Since B_1 and $B = B_n$ have not changed their relative positions, we see that in one of $OD(Q_i)$ and $OD(Q_{i-1})$, B is on the same side of the bipartition as the corresponding Möbius bridge, while in the other B and the other corresponding Möbius bridge are on opposite sides of the bipartition.

The argument works exactly in reverse when $k = 1$. In this case, B_1 is skew to M_{Q_i} and B_2. Since $B_1 \subseteq M_{Q_{i-1}}$, we conclude that B_2 is skew to $M_{Q_{i-1}}$, and the result follows analogously to the argument in the preceding paragraph.

Finally, suppose B is a global H-bridge. Then, for each H-quad Q, $B \subseteq M_Q$, so B does not overlap any of the Q-local H-bridges already embedded in $\mathfrak{D}_{\Pi^*}$ and, since $\Pi[B] \subseteq \mathfrak{D}$, B can also be added to Π^*. ■

We are now ready to move to tidiness.

DEFINITION 10.3. Let $V_{10} \cong H \subseteq G$ and let Π be a representativity 2 embedding of G. Then H is Π-*tidy* if:

(1) $H \subseteq \mathfrak{M}$;
(2) every local H-bridge is contained in $\mathfrak{M}$;
(3) for each H-quad Q, no two Q-local H-bridges overlap; and
(4) there is no H-avoiding path P in $\mathfrak{D}$ and an index $i \in \{0, 1, 2, \ldots, 9\}$ so that P has both its ends in $\langle v_i, r_i, v_{i+1}, r_{i+1}, v_{i+2}, r_{i+2}, v_{i+3} \rangle$.

If $V_{10} \cong H \subseteq G$, then H is *tidy* if there is a representativity 2 embedding Π of G so that H is Π-tidy.

Our aim is the following result.

THEOREM 10.4. *Let $G \in \mathcal{M}_2^3$ have a subdivision of V_{10}. Then there exists a representativity 2 embedding Π in $\mathbb{R}P^2$ of G with a Π-tidy subdivision of V_{10}.*

The following concept is central to the proof.

DEFINITION 10.5. Let $V_{10} \cong H \subseteq G$. Then $\mathrm{Loc}(H)$ denotes the union of H and all the local H-bridges in G.

Proof of Theorem 10.4. For any $V_{10} \cong H \subseteq G$, Lemma 10.2 implies there is a representativity 2 embedding Π of G in $\mathbb{R}P^2$ so that H is Π-pretidy. Among all H for which $\mathrm{Loc}(H)$ is maximal and all Π so that H is Π-pretidy, we consider the pairs (H,Π) so that $G \cap \mathfrak{M}_{\Pi(H)}$ is maximal. Among all these pairs (H,Π), we choose one for which the number of edges of G in H-spokes in minimized. We claim that this H is Π-tidy. We note that (1) is satisfied by the fact that H is Π-pretidy.

If H and Π fail to satisfy either (2) or (4), then either there is an H-quad Q so that some Q-local H-bridge B is not embedded in $\mathfrak{M}_H$, or there is an H-avoiding path P contained in $\mathfrak{D}_H$ and an index $i \in \{0,1,2,\dots,9\}$ so that P has both ends in $\langle r_i\, r_{i+1}\, r_{i+2}\rangle$. In the first case, as $Q \cup B$ has no non-contractible cycles, the only possibility is that B has all its attachments in one of the H-rim branches of Q. Thus, the first case is a special case of the second; we now consider the second case.

Let P' be the subpath of $\langle r_i\, r_{i+1}\, r_{i+2}\rangle$ joining the ends u and w of P, with the labelling chosen so that u is nearer to v_i in P' than w is. Note that the cycle $P \cup P'$ is an H-green cycle and, therefore, bounds a face of G.

We construct a new subdivision H' of V_{10} in G. The H'-rim is obtained from the H-rim by replacing P' with P. The spokes s_i, s_{i+3}, and s_{i+4} of H' are also spokes of H'. The H-spokes s_{i+1} and s_{i+2} might need extension, using the subpaths of $r_i\, r_{i+1}\, r_{i+2}$ joining u and/or w to either v_{i+1} or v_{i+2} as necessary, to become spokes of H'. Evidently all H'-spokes are contained in $\mathfrak{M}_{H'}$, so $H' \subseteq G \cap \mathfrak{M}_{H'} \subseteq \mathrm{Loc}(H')$. Furthermore, if F is the (closed) face of G bounded by $P \cup P'$, then $\mathfrak{M}_{H'} = \mathfrak{M}_H \cup F$.

CLAIM 1. $\mathrm{Loc}(H) \subseteq \mathrm{Loc}(H')$.

PROOF. Let e be an edge of $\mathrm{Loc}(H)$. If $e \in \mathfrak{M}_{H'}$, then $e \in \mathrm{Loc}(H')$, so we may assume $e \notin \mathfrak{M}_{H'}$. Let B be the local H-bridge containing e. Since $e \notin \mathfrak{M}_{H'}$ and $\mathfrak{M}_H \subseteq \mathfrak{M}_{H'}$, we deduce that $B \subseteq \mathfrak{D}_H$, and so all attachments of B are in some H-rim branch (recall H is Π-pretidy). Thus, Corollary 5.15 implies B has precisely two attachments and therefore is just the edge e. Consequently, B is disjoint from P (it is not in $\mathfrak{M}_{H'}$), and so B is an H'-bridge, whence $e \in \mathrm{Loc}(H')$. □

If P is not contained in a local H-bridge, then, since $P \subseteq \mathrm{Loc}(H')$, we contradict maximality of $\mathrm{Loc}(H)$. Therefore, P is contained in, and therefore is, a local H-bridge B. But this implies that H' is Π-pretidy and that G has one more edge in $\mathfrak{M}_{H'}$ than it has in $\mathfrak{M}_H$, contradicting the maximality of $G \cap \mathfrak{M}_H$. Therefore, (2) and (4) hold for (H,Π).

It follows that, if H is not Π-tidy, then (3) is violated: there exists an H-quad Q and two Q-bridges B and B' in $(M_Q)^{\#}$ that overlap. As both B and B' are contained in $\mathfrak{M}$, one, say B, is Q-interior in Π, while B' is Q-exterior. This implies that $\mathrm{att}(B') \subseteq s$, for some H-spoke $s \subseteq Q$. Corollary 5.15 implies that B' is just an edge uw. We note that B has an attachment x in $\langle u,s,w\rangle$ and an attachment y not in $[u,s,w]$.

Let H'' be the subdivision of V_{10} obtained from H by replacing s with $(s - \langle u,s,w\rangle) \cup B'$. We note that H'' is Π-pretidy, $\mathrm{Loc}(H') = \mathrm{Loc}(H)$, and $\mathfrak{M}_{H''} = \mathfrak{M}_H$, so $G \cap \mathfrak{M}_{H''}$ is maximal. However, the H''-spokes have in total at least one fewer edge than the H-spokes, contradicting the choice of H. ■

We now turn our attention to the global H-bridges of a tidy H.

THEOREM 10.6. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$. If H is tidy, then any global H-bridge is just an edge, and, in particular, has one of the forms $v_i v_{i+2}$, $v_i v_{i+3}$, or has v_i as one end and the other end is in $\langle r_{i-3}\rangle \cup \langle r_{i+2}\rangle$.*

Proof. Let Π be a representativity 2 embedding of G for which H is Π-tidy. In particular, all H-spokes and all local H-bridges are in $\mathfrak{M}$, and, for each $i = 0, 1, 2 \dots, 9$, no global H-bridge has two attachments in $\langle r_i\, r_{i+1}\, r_{i+2}\rangle$.

Let B be a global H-bridge. We note that $B \subseteq \mathfrak{D}$.

CLAIM 1. If there is an i so that $\mathrm{att}(B) \subseteq r_i\, r_{i+1}\, r_{i+2}$, then either $B = v_i v_{i+2}$ or $B = v_{i+1}v_{i+3}$ or $B = v_i v_{i+3}$ or B has v_i as one end and the other end is in $\langle r_{i+2}\rangle$ or B has v_{i+3} as one end and the other end is in $\langle r_i\rangle$.

PROOF. Because H is tidy, no two attachments of B are in $\langle r_i\, r_{i+1}\, r_{i+2}\rangle$. Thus, at least one of v_i and v_{i+3} is an attachment of B; for the sake of definiteness, let it be v_i. Then tidiness implies no attachment of B can be in $\langle r_i\, r_{i+1}\rangle$. As tidiness also implies r_{i+2} has at most one, and therefore exactly one, attachment of B, the result follows. $\square$

CLAIM 2. If there is no i so that $\mathrm{att}(B) \subseteq r_i\, r_{i+1}\, r_{i+2}$, then either $\mathrm{att}(B) = \{v_0, v_5, z\}$, with $z \in \langle r_2\rangle \cup \langle r_7\rangle$ or $\mathrm{att}(B) = \{v_4, v_9, z\}$, with $z \in \langle r_1\rangle \cup \langle r_6\rangle$.

PROOF. We may assume that B is embedded in the $(H \cup \gamma)$-face contained in $\mathfrak{D}$ and incident with $v_0, v_1, \dots, v_4$. As H is tidy and B is H-global, there exist $i, j \in \{9, 0, 1, 2, 3, 4\}$ so that (taking 9 to be equal to -1) $i < j$, B has attachments x in $r_i - v_{i+1}$ and y in $r_j - v_j$, and $j - i \geq 3$; choose such i, j so that $j - i$ is as small as possible. By tidiness, there is no other attachment of B in

$$[r_{i-1}\, r_i\, r_{i+1}\rangle \cup \langle r_{j-1}\, r_j\, r_{j+1}]\ .$$

SUBCLAIM 1. Either $i = -1$ or $j = 4$.

PROOF. In the alternative, $i \geq 0$ and $j \leq 3$. As $j - i \geq 3$, we conclude that $i = 0$ and $j = 3$, so the six H-rim branches r_{i-1}, r_i, r_{i+1}, r_{j-1}, r_j, and r_{j+1} are all distinct and cover the entire ab-subpath in the boundary of $(H \cup \gamma)$-face containing B, with the possible exception of v_2, in which case both $x = v_0$ and $y = v_4$.

Let e be an edge in s_2 and let D be a 1-drawing of $G - e$. Theorem 5.23 implies $\overline{Q}_2$ has BOD; now Lemma 5.9 implies $\overline{Q}_2$ is crossed in D. In particular, $r_0\, r_1\, r_2\, r_3$ crosses $r_5\, r_6\, r_7\, r_8$ in D.

In the case v_2 is an attachment of B, let P and P' be H-avoiding v_0v_2- and v_2v_4-paths in B, respectively. Then the cycles $r_0\, r_1 [v_2, P, v_0]$ and $r_2\, r_3 [v_4, P', v_2]$ are both H-green. Lemma 7.2 (1) implies neither is crossed in D, yielding the contradiction that $r_0\, r_1\, r_2\, r_3$ is not crossed in D.

Thus, B is the edge xy. Note that B is not a local H-bridge and, therefore, not both v_0 and v_4 are attachments of B. As B is not crossed in D, we deduce that the xy-subpath of $r_0\, r_1\, r_2\, r_3$ is also not crossed in D. Therefore, either r_0 or r_3 is crossed in D. From this, we conclude that, since $\overline{Q}_2$ is crossed in D, $r_6\, r_7$ is crossed in D. Moreover, either s_1 or s_3 is exposed in D. By symmetry, we may assume s_1 is exposed in D.

If $x \neq v_0$, then the cycle $r_1\, r_2\, r_3\, s_3\, r_8\, r_9\, s_0\, r_5\, s_1$ is clean in D and separates $x \in \langle r_1\rangle$ from $y \in \langle v_3, r_3, v_4]$, so B must be crossed in D, a contradiction. If $y \neq v_4$, then the cycle $r_1\, r_2\, s_3\, r_8\, s_4\, r_4\, r_5\, s_1$ is clean in D and separates $x \in [v_0, r_1, v_1\rangle$ from $y \in \langle r_3\rangle$, and again B is crossed in D, a contradiction. □

Recall that -1 is equal to 9. The following is immediate from tidiness.

SUBCLAIM 2. (1) If $x \in [a, r_9, v_0\rangle$, then there is no attachment in $[v_0, r_0, v_1\rangle$.
(2) If $y \in \langle v_4, r_4, b]$, then there is no attachment in $\langle v_3, r_3, v_4]$. □

The next two subclaims are rather less trivial.

SUBCLAIM 3. (1) If $x \in [a, r_9, v_0\rangle$, then there is no attachment in $[v_2, r_2, v_3\rangle$.
(2) If $y \in \langle v_4, r_4, b]$, then there is no attachment in $\langle v_1, r_1, v_2]$.

PROOF. We prove (1); (2) is symmetric. For (1), suppose there is an attachment y' in $[v_2, r_2, v_3\rangle$. By tidiness, there is no attachment other than y' in $\langle r_0\, r_1\, r_2\, r_3\rangle$, and so minimality of $j - i$ implies $y' = y$.

The only other possible attachment is in $[v_4, r_4, b]$. If there is an attachment z in $[v_4, r_4, b]$, then either $y = v_2$ or $z = b = v_5$. Thus, either z does not exist and B is the edge xy, or z exists, B has exactly three attachments, namely x, y, and z, and Lemma 5.19 shows B is a $K_{1,3}$. Let P and P' be the xy- and yz-paths (the latter only if z exists) in B.

Suppose first that $y \neq v_2$. Then $x = v_9$, as otherwise $[y, P, x, r_9, v_0]\, r_0\, r_1 [v_2, r_2, y]$ is an H-green cycle with the three H-nodes v_0, v_1, v_2 in its interior, contradicting Lemma 6.6 (9).

Theorem 5.23 (6a) does not apply, as $x = v_9 = a$ implies $v_0 \neq a$. If Theorem 5.23 (6b) applies, then there is a second H-bridge B' attaching at $b = v_5$ and in $r_0\, r_1$. But then B and B' must cross in Π, a contradiction. Therefore, Theorem 5.23 (6) shows $\overline{Q}_1$ has BOD.

Let e be an edge of s_1 and let D be a 1-drawing of $G - e$. Lemma 5.9 implies $\overline{Q}_1$ is crossed in D. On the other hand, the presence of P and Lemma 7.2 (3a) and (2) imply $\overline{Q}_1$ cannot be crossed in D, the desired contradiction.

Therefore, $y = v_2$. Since $x, y \in r_9\, r_0\, r_1$, the hypothesis of the claim implies z must exist. The cycles $[x, P, v_2]\, r_1\, r_0 [v_0, r_9, x]$ and $[z, P', v_2]\, r_2\, r_3 [v_4, r_4, z]$ are H-green. Let e be an edge in s_2 and let D be a 1-drawing of $G - e$. Theorem 5.23 implies $\overline{Q}_2$ has BOD, so Lemma 5.9 implies $\overline{Q}_2$ is crossed in D. However, Lemma 7.2 (1) shows that r_0 and r_3 are not crossed. If $x \neq v_9$, then the same result shows r_1 is not crossed and likewise if $z \neq v_5$, then r_2 is not crossed. If, say, $x = v_9$, then Lemma 7.2 (3b) implies r_1 can only cross r_8. However, if $z \neq v_4$, then (2) shows r_8 cannot be crossed.

In the remaining case, $x = v_9$ and $z = v_4$. In this case, $a = x = v_9$. If $\overline{Q}_1$ does not have BOD, then Theorem 5.23 (6) implies $b = v_5$ and there is a $\overline{Q}_1$-bridge B' different from $M_{\overline{Q}_1}$, having attachments at b and in $r_0\, r_1$, and embedded in $\mathfrak{D}$. But then B' is an H-bridge different from B that overlaps B on R, while both are embedded in $\mathfrak{D}$, a contradiction. □

SUBCLAIM 4. (1) If $x \in [a, r_9, v_0\rangle$, then there is no attachment in $[v_3, r_3, v_4\rangle$.
(2) If $y \in \langle v_4, r_4, b]$, then there is no attachment in $\langle v_0, r_0, v_1]$.

PROOF. We prove (1); (2) is symmetric. For (1), suppose there is an attachment in $[v_3, r_3, v_4\rangle$. By minimality of $j - i$, Subclaim 3 and tidiness, this attachment is y. Also by tidiness, there is no other attachment in $\langle r_1\, r_2\, r_3\, r_4\rangle$.

Suppose there is also an attachment z in $[v_1, r_1, v_2\rangle$. The preceding paragraph shows $z = v_1$. Tidiness now implies that x is v_9 and, since $a \in r_9$ and $x \in [a, r_9, v_0\rangle$, $a = v_9$. Let P and P' be H-avoiding xz- and yz-paths in B, respectively.

Theorem 5.23 (6) implies $\overline{Q}_1$ has BOD. If D_1 is any 1-drawing of $G - \langle s_1 \rangle$, then Lemma 5.9 implies $\overline{Q}_1$ is crossed in D_1. But Lemma 7.2 implies (recall $z = v_1$) the two H-green cycles $[z, P, x, r_9, v_0, r_0, z]$ and $[y, P', z, r_1, v_2, r_2, v_3, r_3, y]$ are not crossed in D_1. Thus, $r_9\, r_0\, r_1\, r_2$ is not crossed in D_1 (since $x = v_9$), so $\overline{Q}_1$ is not crossed in D_1, a contradiction.

Therefore, there is no attachment in $[v_1, r_1, v_2\rangle$. Thus, we may assume that the only attachments in $[a, r_9, v_0]r_0\, r_1\, r_2\, r_3$ are $x \in [a, r_9, v_0\rangle$ and $y \in [v_3, r_3, v_4\rangle$. Tidiness further shows there is no attachment in $[v_4, r_4, v_5\rangle$, so the only other possible attachment of B is v_5, in which case $y = v_3$.

In each of the two cases $x \neq v_9$ and $x = v_9$, we show that $\overline{Q}_4$ has NBOD by showing that B, $M_{\overline{Q}_4}$, and the $\overline{Q}_4$-bridge B_4 containing s_4 are mutually overlapping. We remark that B and B_4 are in different faces of $\Pi[H]$, so $B \neq B_4$. Obviously, B_4 is skew to $M_{\overline{Q}_4}$.

Case 1. $x \neq v_9$.

The attachments x and y of B are skew to v_4 and v_9, so B and B_4 overlap. Also, x and y are skew to v_8 and v_0, so B and $M_{\overline{Q}_4}$ overlap, as required.

Case 2. $x = v_9$.

As $x, y \in Q_3$ and B is not Q_3-local, there is another attachment z of B. Our earlier remarks imply $z = v_5$ and $y = v_3$. Now y and z show B and B_4 are skew, while x and y show B and $M_{\overline{Q}_4}$ are skew.

We now resume our general discussion. Let P_{xy} be the xy-path in B. Since $x \in [a, r_9, v_0\rangle$, $v_0 \neq a$. Suppose some $\overline{Q}_1$-bridge B' has an attachment at $b = v_5$ and an attachment in $r_0\, r_1$. Since B is not a $\overline{Q}_1$-bridge and both B and B' are H-bridges, $B \neq B'$. Then P_{xy} and a $v_5\, [r_0\, r_1]$-path in B' would cross in Π, which is impossible. Therefore, Theorem 5.23 shows $\overline{Q}_1$ has BOD.

Let D_1 be a 1-drawing of $G - \langle s_1 \rangle$. Because $\overline{Q}_4$ has NBOD, Lemma 5.6 implies $D_1[\overline{Q}_4]$ is not clean in D_1. Since $\overline{Q}_1$ has BOD and s_1 is contained in a planar $\overline{Q}_1$-bridge, Lemma 5.9 implies $\overline{Q}_1$ is crossed in D_1. Therefore, s_0 is exposed in D_1. Thus $D_1[H - \langle s_1 \rangle]$ is one of two possible 1-drawings, depending on whether r_9 crosses $r_5\, r_6$ or r_4 crosses $r_0\, r_1$.

If $x \neq v_9$, then P_{xy} cannot be added to $D_1[H - \langle s_1 \rangle]$ without introducing a second crossing, which is impossible. If $x = v_9$, then the three attachments of B are not all on the same face of $D_1[H - \langle s_1 \rangle]$, so B cannot be added to $D_1[H - \langle s_1 \rangle]$ without introducing a second crossing, the final contradiction. □

We can now complete the proof of Claim 2. Subclaim 1 implies either $x \in [a, r_9, v_0\rangle$ or $y \in \langle v_4, r_4, b]$. By symmetry, we may assume the former. Subclaims 3 and 4 imply $y \in [v_4, r_4, b]$. If $y \neq v_4$, then Subclaims 2, 3 and 4 (all six statements) show that there is no other attachment of B. But then B is Q_4-local, a contradiction. Therefore, $y = v_4$, and, furthermore, there is an attachment z of B in $[v_1, r_1, v_2\rangle$.

If $x \neq v_9$, then both x and z are in $\langle r_9\, r_0\, r_1 \rangle$, contradicting tidiness. Thus, $x = v_9$.

The claim will be proved once we know $z \neq v_1$. By way of contradiction, suppose $z = v_1$. Consider any 1-drawing D_2 of $G - \langle s_2 \rangle$. By Theorem 5.23, $\overline{Q}_2$ has BOD. Thus, Lemma 5.9 implies $\overline{Q}_2$ is crossed in D_2. That is, $r_0\, r_1\, r_2\, r_3$ crosses $r_5\, r_6\, r_7\, r_8$ in D_2. In particular, neither s_0 nor s_4 is exposed in D_2.

Since B is global and has attachments at v_4 and v_9, it must be that $D_2[B]$ is in the face of $D_2[R \cup s_0 \cup s_4]$ incident with s_4 and the crossing. Since v_1 is an attachment of B, v_1 must be in the subpath of $r_0\, r_1\, r_2\, r_3$ between the crossing and v_4. But then s_3 is not exposed in D_2, implying B must cross s_3 in D_2, a contradiction that shows v_1 is not an attachment of B, completing the proof of the claim. □

To complete the proof of the theorem, by way of contradiction assume there is no i so that $\mathrm{att}(B) \subseteq r_i\, r_{i+1}\, r_{i+2}$. Claim 2 shows either $\mathrm{att}(B) = \{v_0, v_5, z\}$, with $z \in \langle r_2 \rangle \cup \langle r_7 \rangle$ or $\mathrm{att}(B) = \{v_4, v_9, z\}$, with $z \in \langle r_1 \rangle \cup \langle r_6 \rangle$. These are all the same up to the labelling of H, a, and b, so we may assume $\mathrm{att}(B) = \{v_0, v_5, z\}$, with $z \in \langle r_2 \rangle$. Let H' be the subdivision of V_{10} consisting of $H - \langle s_0 \rangle$, together with the $v_0 v_5$-path in B.

In order to apply Theorem 7.1, we show that Π is H'-friendly. If Π is not H'-friendly, then Lemma 6.5 (1) implies (since H and H' have the same nodes) $v_6 v_9$ is an edge and $\Pi[v_6 v_9]$ is contained in $\mathfrak{M}_{H'}$, which is the same as $\mathfrak{M}_H$. But v_6 and v_9 are not incident with the same H-face in $\mathfrak{M}_H$ and, therefore, this is impossible. Thus, Π is H'-friendly. However, H' violates Theorem 7.1, a contradiction.

Therefore, there is an i so that $\mathrm{att}(B) \subseteq r_i\, r_{i+1}\, r_{i+2}$. Claim 1 implies B has one of the three desired forms. ■

We can go somewhat further in our analysis of the global H-bridges of a tidy $V_{10} \cong H \subseteq G$.

Definition 10.7. Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. Let B be a global H-bridge with attachments x and y.

(1) The *span of* B is the xy-subpath R with the fewest H-nodes.
(2) An edge or subpath of R is *spanned by* B if it is in the span of B.
(3) B is: a *2-jump* if, for some i, its attachments are v_i and v_{i+2}; a *3-jump* if, for some i, its attachments are v_i and v_{i+3}; or else is a *2.5-jump*.

We remark that Theorem 10.6 implies that, in the case of a 2.5-jump, there is an i so that v_i is one attachment and the other attachment is in $\langle r_{i-3} \rangle \cup \langle r_{i+2} \rangle$. Theorem 10.6 further implies a global H-bridge has precisely two attachments and its span has at most four H-nodes. It follows from Definition 6.2 that every global H-bridge combines with its span to form an H-green cycle.

Lemma 10.8. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. For each $i \in \{0, 1, 2, 3, 4\}$, either $\overline{Q}_i$ has BOD or one of $v_{i-1} v_{i-4}$ and $v_{i+1} v_{i+4}$ is a global H-bridge.*

Proof. Let Π be an embedding of G in $\mathbb{R}P^2$ so that H is Π-tidy. Suppose neither of the edges $v_{i-1} v_{i-4}$ and $v_{i+1} v_{i+4}$ occurs in G. The $\overline{Q}_i$-bridges that are $\overline{Q}_i$-exterior consist of $M_{\overline{Q}_i}$, those that are contained in $\mathfrak{M}$ and, therefore, attach along either s_{i-1} or s_{i+1}, and those that are contained in $\mathfrak{D}$. Since H is Π-tidy, these latter must be global. By Theorem 10.6 they are 2-, 2.5-, and 3-jumps.

Consider any global H-bridge. It is embedded in $\mathfrak{D}$ so that it, together with its spanned path in R, bounds a face of G. In particular, if we are considering a 2-jump B that is a $\overline{Q}_i$-bridge, the 2-jump is either $v_{i-1}v_{i+1}$ or $v_{i+4}v_{i+6}$. In this case, $\overline{Q}_i \cup B$ has no non-contractible cycle in $\mathbb{R}P^2$ and so, by Lemma 5.16, B does not overlap any other $\overline{Q}_i$-exterior $\overline{Q}_i$-bridge.

It is not possible for a 2.5-jump to be a $\overline{Q}_i$-bridge. The only 3-jumps that can be a $\overline{Q}_i$-bridge are $v_{i+1}v_{i+4}$ and $v_{i-4}v_{i-1}$, and these are assumed not to be in G. We conclude that the $\overline{Q}_i$-exterior $\overline{Q}_i$-bridges do not overlap and, therefore, $\overline{Q}_i$ has BOD. ■

LEMMA 10.9. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. Then:*

(1) no two global H-bridges have an H-node in common;
(2) at most one global H-bridge is a 3-jump;
(3) there is no i so that v_iv_{i+3} is a 3-jump and some 2.5-jump has an end in $\langle v_{i-1}, r_{i-1}, v_i]$;
(4) if B_1 and B_2 are global H-bridges, then, for every $i \in \{0, 1, 2, 3, 4\}$, there is some edge of $\overline{Q}_i \cap R$ that is not spanned by either B_1 or B_2; and
(5) for each $i \in \{0, 1, 2, 3, 4\}$, at most one of $\langle r_i \rangle$ and $\langle r_{i+5} \rangle$ can contain an end of a 2.5-jump.

Proof. We start with (1).

CLAIM 1. No two global H-bridges have an H-node in common.

PROOF. suppose by way of contradiction that the two global H-bridges B_1 and B_2 have the H-node v_i in common. For $j = 1, 2$, let P_j be the subpath of R spanned by B_j. Then each of $B_j \cup P_j$ is a green cycle; therefore, Theorem 6.7 implies P_1 and P_2 are edge disjoint. We choose the labelling so that $r_i \cup r_{i+1} \subseteq P_1$ and $r_{i-2} \cup r_{i-1} \subseteq P_2$. We treat various cases.

SUBCLAIM 1. At least one of B_1 and B_2 is not a 3-jump.

PROOF. Suppose to the contrary that B_1 and B_2 are both 3-jumps, so $B_1 = v_iv_{i+3}$ and $B_2 = v_{i-3}v_i$, respectively. Then there is a 1-drawing D_i of $(H - s_i) \cup B_1 \cup B_2$; Lemma 10.8 implies $\overline{Q}_i$ has BOD, so Lemma 5.9 implies $\overline{Q}_i$ is crossed in D_i.

Because of B_1, Lemma 7.2 (3a) implies r_{i+1} and r_{i+2} are not crossed in D_i, while (3b) of the same lemma implies that if r_i were crossed, it would cross r_{i+3}. However, (2) shows r_{i+3} is not crossed. Therefore, no edge of $r_i\, r_{i+1}\, r_{i+2}$ is crossed in D_i. Analogously, no edge of $r_{i-3}\, r_{i-2}\, r_{i-1}$ is crossed in D_i. These two assertions show $\overline{Q}_i$ cannot be crossed in D_i, a contradiction. □

SUBCLAIM 2. Neither B_1 nor B_2 is a 3-jump.

PROOF. By Claim 1, not both B_1 and B_2 are 3-jumps. So suppose for sake of definiteness that B_1 is the 3-jump v_iv_{i+3} and B_2 is a global H-bridge with one end at v_i and one end in $\langle v_{i-3}, r_{i-3}, v_{i-2}]$.

The embedding in $\mathbb{R}P^2$ shows that $v_{i+2}v_{i+5}$ is not an edge of G (it would cross B_1) and Claim 1 shows $v_{i-3}v_i$ is not an edge of G. Therefore, Lemma 10.8 implies $\overline{Q}_{i+1}$ has BOD. Thus, in any 1-drawing D_{i+1} of $G - \langle s_{i+1} \rangle$, Lemma 5.9 implies $\overline{Q}_{i+1}$ is crossed in D_{i+1}.

By Lemma 7.2 (3a) (when B_2 is a 2.5-jump) or (1) (when B_2 is a 2-jump), r_{i-1} is not crossed in D_{i+1}. Likewise, (1) shows that none of r_i, r_{i+1}, and r_{i+2} is crossed in D. But then $\overline{Q}_{i+1}$ is not crossed in D_{i+1}, a contradiction. □

By Claim 2, we know that neither B_1 nor B_2 is a 3-jump. By Theorem 6.7, neither $v_{i-1}v_{i-4}$ nor $v_{i+1}v_{i+4}$ can occur in G; Lemma 10.8 implies $\overline{Q}_i$ has BOD. Let D_i be a 1-drawing of $G - \langle s_i \rangle$. By Lemma 5.9, $\overline{Q}_i$ is crossed in D_i.

Lemma 7.2 (1) shows that P_1 and P_2 are both not crossed in D_i. This implies that $r_{i-2}\, r_{i-1}\, r_i\, r_{i+1}$ is not crossed in D and, therefore, $\overline{Q}_i$ is not crossed in D_i, a contradiction that completes the proof of the claim. □

We move on to (2).

Claim 2. There is at most one global H-bridge that is a 3-jump.

Proof. Suppose there are distinct 3-jumps. Claim 1 implies that, up to relabelling, they are either v_iv_{i+3} and $v_{i+4}v_{i+7}$ or v_iv_{i+3} and $v_{i+5}v_{i+8}$. Theorem 6.7 and Claim 1 imply that there cannot be a third 3-jump. Thus, Lemma 10.8 implies $\overline{Q}_{i+1}$ has BOD.

Let C_1 and C_2 be the two H-green cycles containing these 3-jumps. Lemma 5.9 implies $\overline{Q}_{i+1}$ is crossed in a 1-drawing D_{i+1} of $G - \langle s_{i+1} \rangle$. But Lemma 7.2 (1) implies that neither $r_i\, r_{i+1}$ nor $r_{i+5}\, r_{i+6}$ is crossed in D_{i+1}, a contradiction proving the claim. □

We next turn to (3).

Claim 3. There is no i so that v_iv_{i+3} is a 3-jump and some 2.5-jump has an end in $\langle v_{i-1}, r_{i-1}, v_i]$.

Proof. Suppose to the contrary that there is such an i. From Claim 1, the 2.5-jump has an end $w \in \langle v_{i-1}, r_{i-1}, v_i \rangle$. Its other end is v_{i-3}. Lemma 10.8 and Claim 2 imply that $\overline{Q}_{i+2}$ has BOD. Let D_{i+2} be a 1-drawing of $G - \langle s_{i+2} \rangle$. Lemma 5.9 implies $\overline{Q}_{i+2}$ is crossed in D_{i+2}.

By Lemma 7.2 (2), r_{i+3} is not crossed in D_{i+2}. The same lemma (1) implies $r_i\, r_{i+1}\, r_{i+2}$ is not crossed in D_{i+2}. Consequently, $\overline{Q}_{i+2}$ is not crossed in D_{i+2}, contradicting the preceding paragraph and proving the claim. □

Now we prove (4).

Claim 4. If B_1 and B_2 are global H-bridges, then, for every $i \in \{0, 1, 2, 3, 4\}$, some edge of $\overline{Q}_i \cap R$ is not spanned by either B_1 or B_2.

Proof. Suppose by way of contradiction that the global H-bridge B_1 spans the side $r_i \cup r_{i+1}$ of $\overline{Q}_{i+1}$ and a second global H-bridge B_2 spans $r_{i+5} \cup r_{i+6}$. To see that $\overline{Q}_{i+1}$ has BOD, by Lemma 10.8 it suffices to show that neither of the 3-jumps v_iv_{i-3} and $v_{i+2}v_{i+5}$ is in G. For the former, Theorem 6.7 implies v_i is an attachment of B_1, contradicting Claim 1. For the latter, v_{i+2} is an attachment of B_2, with the same contradiction. Therefore $\overline{Q}_{i+1}$ has BOD.

Lemma 5.9 implies that, for any 1-drawing D_{i+1} of $G - \langle s_{i+1} \rangle$, $\overline{Q}_{i+1}$ is crossed in D_{i+1}. However, Lemma 7.2 (1) implies that neither $r_i\, r_{i+1}$ nor $r_{i+5}\, r_{i+6}$ is crossed in D_{i+1}, showing $\overline{Q}_{i+1}$ is not crossed in D_{i+1}, a contradiction proving the claim. □

Finally, we prove (5). Suppose, for $j \in \{i, i+5\}$, $\langle r_j \rangle$ contains an end of the 2.5-jump B_j. We may use the symmetry to assume that $B_i = wv_{i-2}$. If B_{i+5} has v_{i+3} as an end, then we contradict Claim 4. Therefore, B_{i+5} has $v_{i+8} = v_{i-2}$ as an end, contradicting Claim 1. ■

We conclude this section with two observations about local bridges of a tidy subdivision of V_{10}.

LEMMA 10.10. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. Then no H-bridge has all its attachments in one H-spoke.*

Proof. By way of contradiction, suppose B is an H-bridge and s is an H-spoke so that $\mathrm{att}(B) \subseteq s$. By Corollary 5.15, B has precisely two attachments, so B is just an edge uw. Choose B so that no other H-bridge has all its attachments in a proper subpath of $[u, s, w]$. If $[u, s, w]$ has no interior vertex, then B and $[u, s, w]$ are parallel edges not in the H-rim, contradicting Observation 9.2 (6). Thus, some H-bridge B' has an attachment x in $\langle u, s, w \rangle$.

Let Π be an embedding of G in $\mathbb{R}P^2$ for which H is Π-tidy. Since $H \subseteq \mathfrak{M}$, B' is a local H-bridge. Moreover, Corollary 5.15 and the choice of B show that not all attachments of B' can be in $[u, s, w]$, so B has an attachment y not in $[u, s, w]$. But then, for at least one of the two H-quads Q containing s, B and B' are overlapping Q-bridges, contradicting the definition of tidiness. ■

LEMMA 10.11. *Let $G \in \mathcal{M}_2^3$, $V_{10} \cong H \subseteq G$, with H tidy. For any H-spoke s, if B is an H-bridge having an attachment in $\langle s \rangle$, then B has no other attachment in $[s]$.*

Proof. Suppose B is an H-bridge and s an H-spoke so that B has attachments x, y in s, with $x \in \langle s \rangle$. Let Π be an embedding of G in $\mathbb{R}P^2$ for which H is Π-tidy. Then Π shows B is not a global H-bridge. By Lemma 10.10, B has a third attachment z not in $[s]$. Let Q be the unique H-quad containing all of x, y, and z.

If y is not an H-node, then let r be an H-rim branch of Q not containing z. Then x, y, and z are all contained in $Q - [r]$, contradicting Corollary 5.15. Thus, y is an H-node v_i. We choose the labelling so that $r_i \subseteq Q$. Corollary 5.15 shows that z is not in $Q - [r_{i+5}]$ and, therefore, z is in r_{i+5}. Furthermore, Corollary 5.15 now shows that B can have no other attachment, so Theorem 8.2 implies B is isomorphic to $K_{1,3}$. Let w be the vertex in $\mathrm{Nuc}(B)$.

CLAIM 1. The cycles $[y, B, w, B, x, s, y]$ and $[z, B, w, B, x, Q - y, z]$ bound faces of $\Pi[G]$.

PROOF. For the latter, $[z, B, w, B, x, Q - y, z]$ is an H-green cycle, so the result follows from Lemma 6.6. The former, call it C, has just one vertex in R, so Lemma 5.20 implies it has BOD and every one of its bridges other than the one containing $H - \langle s \rangle$ is planar. If it has a second bridge B', then C is clean in any 1-drawing of $B'^{\#}$, contradicting Lemma 5.9. □

The chosen labelling shows that Q_{i-1} is the other H-quad containing s.

CLAIM 2. There is no Q_{i-1}-local H-bridge that has an attachment in $\langle s \rangle$.

PROOF. Suppose B'' is a Q_{i-1}-local H-bridge having an attachment x' in $\langle s \rangle$. Lemma 10.10 implies B'' has an attachment z' not in $[s]$. If z' is in the same H-rim branch r_{i-1} contained in Q_{i-1} as y, then $[x', B'', z', r, y, B, w, x, s, x']$ is an H-green cycle C. As the edge of s incident with y is C-interior, C does not bound a face of $\Pi[G]$. If z' is not in r_{i-1}, then $\left[z', B'', x', s, x, B, w, B, z, \overline{Q}_i - y, z'\right]$ is a non-facial H-green cycle. Both conclusions contradict Lemma 6.6 (8). □

We conclude that s has length 2 and that B is the only H-bridge attaching in $\langle s \rangle$. Let D be a 1-drawing of $G - wy$. Then $D[s \cup (B - wy)]$ is clean in D and we may extend D to a 1-drawing of G by adding in wy alongside $[w, B, x, s, y]$. ■

CHAPTER 11

Every rim edge has a colour

In this section we introduce, for a tidy subdivision H of V_{10} in G, H-yellow edges. The main result is that every H-rim edge has a colour: H-green, H-yellow, or red. This is a major step on the route. In the next section, we will analyze red edges, with the main result being that there are red edges.

Definition 11.1. Let H be a subdivision of V_{10} in a graph G.

(1) A 3-*rim path* is a path contained in the union of three consecutive H-rim branches.
(2) The *closure* $\mathrm{cl}(Q)$ of an H-quad Q is the union of Q and all Q-local H-bridges.
(3) Let H be tidy in G. A cycle C in G is H-*yellow* if C may be expressed as the composition $P_1P_2P_3P_4$ of four paths so that:
  (a) P_2 and P_4 are R-avoiding (recall R is the H-rim) and have length at least 1;
  (b) P_1 and P_3 are 3-rim paths and $P_1 \cup P_3$ is not contained in a 3-rim path; and
  (c) there is an H-green cycle C' so that $P_1 \subseteq \langle C' \cap R\rangle$.
(4) An H-rim edge e is H-*yellow* if it is not H-green and is in an H-yellow cycle.

We remark that the H-rim edges that are H-yellow are those in P_3. The next result elucidates the nature of an H-yellow cycle.

Lemma 11.2. *Let $G \in \mathcal{M}_2^3$, $V_{10} \cong H \subseteq G$, with H tidy. Let C be an H-yellow cycle, with decomposition $P_1P_2P_3P_4$ into paths as in Definition 11.1, and let C' be the witnessing H-green cycle. Then:*

(1) $C' - \langle C' \cap R\rangle$ is a global H-bridge;
(2) for $i \in \{2,4\}$, P_i is either H-avoiding or decomposes as $P_i^1P_i^2$, where P_i^1 is contained in some H-spoke, including an incident H-node, and P_i^2 is H-avoiding;
(3) there is only one C-bridge in G; and
(4) there is an $i \in \{0,1,2,3,4\}$ so that $C \subseteq \mathrm{cl}(Q_i)$.

Proof. Let Π be an embedding of G in $\mathbb{R}P^2$ for which H is Π-tidy; in particular, every H-green cycle bounds a face of $\Pi[G]$.

For (1), the alternative is that C' is contained in $\mathrm{cl}(Q)$, for some H-quad Q. Lemma 6.6 (8) shows that C' bounds a face of G in $\mathbb{R}P^2$, so P_2 and P_4 are contained in global H-bridges. Each of P_2 and P_4 is in an H-green cycle (as is every global H-bridge) and, since P_2 has an end in $\langle C' \cap R\rangle$, some edge of $C' \cap R$ is in two H-green cycles, contradicting Theorem 6.7.

For (2), let $i \in \{2,4\}$. Since P_i has positive length, the end u_i of P_i in P_1 is distinct from the end w_i of P_i in P_3. Because C' bounds a face of G and is contained in $\mathfrak{D}$, we see that the edges of P_i incident with u_i is in $\mathfrak{M}$. Since P_i is R-avoiding, P_i is contained in $\mathfrak{M}$, with only its ends in R.

Now suppose P_i has an edge e not in H. Choose e to be as close to u_i in P_i as possible. As w_i is in H, there is a first vertex y of P_i after e that is in H. If $y = w_i$, then we are done, so we may assume $y \neq w_i$. Since P_i is R-avoiding, we see that y must be in the interior of some spoke s. Let z be the vertex of P_i incident to e so that e is in $[z, P_i, y]$.

As P_i is contained in $\mathfrak{M}$, we see that $[u_i, P_i, y]$ is contained in a closed $\Pi[H]$-face bounded by some H-quad Q. Also, $[z, P_i, y]$ is H-avoiding and so is contained in some Q-local H-bridge B_i. By Lemma 10.11, y is the only attachment of B_i in $[s]$. Since $z \neq y$ and both are attachments of B_i, we have that $z \notin [s]$.

The path $[u_i, P_i, z]$ is R-avoiding and contained in H. Therefore, either it is trivial or it is contained in some H-spoke s'. In the latter case, $z \neq y$ implies $s' \neq s$. In the former case, $u_i = z$, so $u_i \notin s$. In both cases, $[u_i, P_i, y] \cup Q$ contains an H-green cycle that contains an H-rim edge incident with u_i, contradicting Theorem 6.7 and completing the proof of (2).

For (3), we start by noting that there exist i and j so that $P_1 \subseteq r_i\, r_{i+1}, \dots, r_j$ and $i - 1 \leq j \leq i + 2$; we assume P_1 has one end in $[v_i, r_i, v_{i+1}\rangle$, one end in $\langle v_j, r_j, v_{j+1}]$, and that $j = i - 1$ only if P_1 is just the single H-node v_i. Item 2 implies P_2 is contained in $\mathrm{cl}(Q_{i-1}) \cup \mathrm{cl}(Q_i)$ and that P_4 is contained in $\mathrm{cl}(Q_j) \cup \mathrm{cl}(Q_{j+1})$. It follows that P_3 has its ends in $r_{i+4}\, r_{i+5}$ and $r_{j+5}\, r_{j+6}$. There are at most $(j + 6) - (i + 3) \leq 5$ H-rim branches $r_{i+4}\, r_{i+5}\, \dots\, r_{j+6}$, so P_3, being a 3-rim path, must be contained in this path. It follows that C is disjoint from either s_{i-2} or s_{i+2}.

Let s be an H-spoke disjoint from C and let M_C denote the C-bridge containing s.

Set $R' = (R - \langle C' \cap R\rangle) \cup (C' - \langle C' \cap R\rangle)$. Then $R' \cup s$ contains a non-contractible cycle C'' disjoint from C. Lemma 5.20 shows C is contractible, has BOD, and every C-bridge other than M_C is planar.

Suppose there is a C-bridge B other than M_C; let D be a 1-drawing of $B^{\#}$. Lemma 5.9 implies $D[C]$ is crossed. Let s, s', and s'' be the three H-spokes disjoint from $\langle C' \cap R\rangle$. Then $R \cup s \cup s' \cup s''$ is a subdivision of V_6 in $B^{\#}$ that is edge-disjoint from both P_2 and P_4; this shows that some edge of $P_1 \cup P_3$ is crossed in D.

But now $R' \cup s \cup s' \cup s''$ is another subdivision of V_6 in $B^{\#}$. Therefore, the crossing in D must involve two edges of $R' \cup s \cup s' \cup s''$. In particular it does not involve an edge of $C' \cap R$, and, since $P_1 \subseteq C' \cap R$, no edge of P_1 is crossed in D.

Likewise, let R'' be obtained from R' by replacing P_3 with $P_2 P_1 P_4$. Now $R'' \cup s \cup s' \cup s''$ is a third subdivision of V_6 in $B^{\#}$ that is disjoint from P_3. Thus, the crossing in D does not involve an edge of P_3. Thus, none of P_1, P_2, P_3, and P_4 is crossed in D, contradicting the fact that C is crossed in D. We conclude that there is no C-bridge other than M_C, as claimed.

Finally, for (4), suppose first that P_1 is not contained in a single H-rim branch. Then there is an H-node v_i in the interior of P_1. However, P_1 is incident on one side with the face bounded by C', so the edge of s_i incident with v_i is on the other side of P_1. Since C is contractible, we conclude that there are at least two C-bridges, contradicting (3). Therefore, there is an $i \in \{0, 1, 2, 3, 4\}$ so that $P_1 \subseteq r_i$.

If both P_2 and P_4 are contained in $\mathrm{cl}(Q_i)$, then so is P_3, as it is a 3-rim path. Therefore, by symmetry, we may assume that P_2 has some edge not in $\mathrm{cl}(Q_i)$. As we traverse P_2 from its end in P_1, we come to a first edge e that is not in $\mathrm{cl}(Q_i)$. One end of e is the vertex u that is in either s_i or s_{i+1}; for the sake of definiteness, we assume the former. Then (2) implies $[v_i, s_i, u] \subseteq P_2$ and that the remainder of P_2 consists of an H-avoiding uw-path, with w an end of P_3. It follows that $w \in r_{i+4}$. Let $\hat{e}$ be the edge of s_i incident with u and not in P_2.

Switching paths, we know that P_4 has an end x in r_i. If $x \neq v_{i+1}$, then (2) implies $P_4 \subseteq \mathrm{cl}(Q_i)$. In this case, $\hat{e}$ is in a C-bridge other than M_C, contradicting (3). Otherwise $x = v_{i+1}$, in which case $P_1P_2[w, r_{i+4}, v_{i+5}, r_{i+5}, \;\; v_{i+6}, s_{i+1}, v_{i+1}]$ is an H-yellow cycle $\widehat{C}$. There is a $\widehat{C}$-bridge other than $M_{\widehat{C}}$ containing $\hat{e}$, also contradicting (3) for $\widehat{C}$. ■

We now turn our attention to the all-important red edges. We comment that, if $n \geq 4$ and $V_{2n} \cong H \subseteq G$, then any red edge of G is in the H-rim.

The remainder of this section is devoted to proving the following.

THEOREM 11.3. *Let $G \in \mathcal{M}_2^3$ and let $V_{10} \cong H \subseteq G$. If H is tidy, then every H-rim edge is one of H-green, H-yellow, and red.*

We start with an easy observation.

LEMMA 11.4. *Let $G \in \mathcal{M}_2^3$ and let $V_{10} \cong H \subseteq G$. If H is tidy and the H-rim edge e is either H-green or H-yellow, then e is not red.*

Proof. Suppose first that e is H-green and let C be the H-green cycle containing e. There are three H-spokes s, s', and s'' disjoint from $\langle C \cap R \rangle$. Thus, $(R - \langle C \cap R \rangle) \cup (C - \langle C \cap R \rangle)$ together with s, s', and s'' is a subdivision of V_6 contained in $G - e$, showing e is not red.

Now suppose e is H-yellow and let C be the H-yellow cycle containing e. Let C' be the H-green cycle and $P_1P_2P_3P_4$ the decomposition of C as in Definition 11.1. Then e is in P_3 and there are three H-spokes s, s', and s'' disjoint from $C \cup \langle C' \cap R \rangle$. In this case, $(R - (\langle C' \cap R \rangle \cup \langle P_3 \rangle)) \cup (C' - \langle C' \cap R \rangle) \cup P_2P_1P_4$, together with s, s', and s'' is a subdivision of V_6 contained in $G - e$, showing e is not red. ■

The following concepts and lemma play a central role in the proof of Theorem 11.3.

DEFINITION 11.5. Let $V_{10} \cong H \subseteq G$. Let e and f be two edges of the H-rim R. Then e and f are *R-separated in G* if G has a subdivision H' of V_8 so that the H'-rim is R and e and f are in disjoint H'-quads.

The following two observations are immediate from the definition.

OBSERVATION 11.6. *Let $V_{10} \cong H \subseteq G$ and suppose e and f are two edges of the H-rim R that are R-separated in G.*

1. *If D is a 1-drawing of G, then e and f do not cross each other in D.*
2. *If H' is a V_8 in G witnessing the R-separation of e and f, then there are two H'-spokes that have all their ends in the same component of $R - \{e, f\}$.* ■

The following is a kind of converse of Observation 11.6 (1).

Lemma 11.7. *Let $G_0 \in \mathcal{M}_2^3$ be a graph and let $V_{10} \cong H \subseteq G_0$, with H tidy. Suppose $G \subseteq G_0$ with $H \subseteq G$. Let $e \in r_i$ and $f \in r_{i+4} \cup r_{i+5} \cup r_{i+6}$ be edges that are both neither H-green nor H-yellow. If e and f are not R-separated in G, then there is a 1-drawing of G in which e crosses f.*

Proof. Let Π be an embedding of G in $\mathbb{R}P^2$ so that H is Π-tidy.

We may write $r_i = [v_i, \dots, x_e, e, y_e, \dots, v_{i+1}]$ and, by symmetry, we may assume f is in

$$r_{i+5} \cup r_{i+6} = [v_{i+5}, r_{i+5}, \dots, x_f, f, y_f, \dots, r_{i+6}, v_{i+7}] \,.$$

If $f \in r_{i+5}$, then let $J_{e,f} = \mathrm{cl}(Q_i)$ and $Q = Q_i$, while if $f \in r_{i+6}$, then let $J_{e,f} = \mathrm{cl}(Q_i) \cup \mathrm{cl}(Q_{i+1})$ and $Q = \overline{Q}_{i+1}$. The two H-spokes contained in Q are s_i and $s_{e,f}$, which is either s_{i+1} or s_{i+2}.

Claim 1. There are not totally disjoint $s_i s_{e,f}$-paths in $J_{e,f} - e$.

Proof. Because H is Π-tidy, $\Pi[J_{e,f}]$ is contained in the closed disc bounded by $\Pi[Q]$. Therefore, one of a pair of totally disjoint $s_i s_{e,f}$-paths in $J_{e,f}$ would be disjoint from $r_{i+5}\, r_{i+6}$ and it, together with a subpath of $Q - r_{i+5}\, r_{i+6}$ yields the contradiction that e is H-green. □

Let w_e be a cut-vertex in $J_{e,f} - e$ separating s_i from $s_{e,f}$. Then $J_{e,f} - e$ has a separation (H_e, K_e) with $s_i \subseteq H_e$, the other H-spoke $s_{e,f}$ contained in Q is contained in K_e, and $H_e \cap K_e = \|w_e\|$. Clearly, $w_e \in r_{i+5}\, r_{i+6}$.

There is also a separation (H_f, K_f) of $J_{e,f} - f$, so that $H_f \cap K_f$ is a single vertex w_f, $s_i \subseteq H_f$, and $s_{e,f} \subseteq K_f$. For $x \in \{e, f\}$, there is a face F_x of $\Pi[J_{e,f}]$ incident with both x and w_x. If $F_e = F_f$, then any vertex of $r_i\, r_{i+1}$ in the boundary cycle C of F_e may be selected as w_f. Similarly, w_e may be any vertex of $r_{i+5}\, r_{i+6}$ that is in C. We choose w_e and w_f so that they are in different components of $C - \{e, f\}$. Thus, whether $F_e = F_f$ or not, the cycle Q has the form $[w_e, \dots, e, \dots, w_f, \dots, f, \dots]$. In particular, e and w_e are in the same component of $Q - \{w_f, f\}$, while f and w_f are in the same component of $Q - \{w_e, e\}$. By interchanging the roles of e and f and exchanging the labels of v_j and v_{j+5}, for $j = 0, 1, 2, 3, 4$, we may assume Q has the form

$$[w_e, \dots, v_{i+5}, s_i, v_i \dots, e, \dots, w_f, \dots, s_{e,f}, \dots, f, \dots] \,.$$

For technical reasons, we choose w_e as close as possible to f in $r_{i+5}\, r_{i+6}$ and w_f as close as possible to e in $r_{i+5}\, r_{i+6}$, while respecting the ordering that was just described of these four elements of Q.

Set $N = K_e \cap H_f$. Then $J_{e,f} - \{e, f\} = H_e \cup N \cup K_f$, $H_e \cap N = \|w_e\|$, and $K_f \cap N = \|w_f\|$. See Figure 11.1.

Claim 2. N does not have disjoint paths both with ends in the two components of $N \cap R$.

Proof. Such paths, together with the H-rim and the H-spokes s_{i-1} and s_{i+3}, would show e and f are R-separated. □

Let w be a cut-vertex in N separating the two components of $N \cap R$, and let (N_i, N_{i+5}) be a separation of N so that, for $j \in \{i, i+5\}$, $N_j \cap R$ is contained in $r_j \cup r_{j+1}$ and $N_i \cap N_{i+5} = \|w\|$. We proceed to describe a new 2-representative embedding of G in $\mathbb{R}P^2$ that shows that G has a 1-drawing.

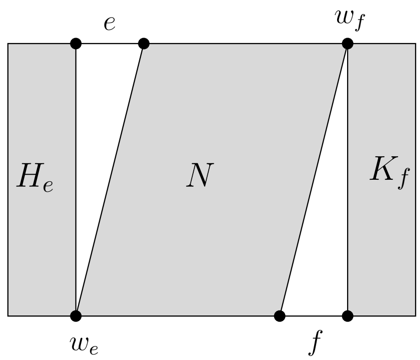

FIGURE 11.1. The locations of e, f, w_e, w_f, H_e, N, and K_f.

Let G' be the subgraph of G obtained by deleting all the vertices and edges of N that are not in $N \cap R$. There is a face of $\Pi[G']$ contained in $\mathfrak{M}$ and incident with both e and f.

CLAIM 3. No global H-bridge has a vertex in $\langle N_i \cap R\rangle \cup \langle N_{i+5} \cap R\rangle$ in its span.

PROOF. For sake of definiteness, suppose some vertex of $\langle N_i \cap R\rangle$ is in the span of the global H-bridge B. If the H-node $v_{e,f}$ in $r_i\, r_{i+1}$ incident with $s_{e,f}$ is in the interior of the span of B, then the cycle bounding F_f is H-yellow, contradicting the fact that f is not H-yellow. Letting z be the vertex of N_i nearest e in $r_i\, r_{i+1}$, we conclude that B has an attachment in $\langle z, r_i\, r_{i+1}, v_{e,f}]$, and B does not span any edge of r_{i+2}.

By Theorem 10.6, B is either a 2-, 2.5-, or 3-jump. It follows from the preceding paragraph that e is contained in the span of B, yielding the contradiction that e is H-green. □

We can now easily complete the proof of the lemma. By Claim 3, we can separately embed N_i and N_{i+5} in the face outside of $\mathfrak{M}$. As no global H-bridge can attach on both paths in $R-\{e,f\}$ without making at least one of e and f H-green, we can join the two copies of w together to obtain a representativity 2 embedding Π' of G in $\mathbb{R}P^2$ having a non-contractible simple closed curve meeting $\Pi'[G]$ only in the interiors of e and f. This implies that G has a 1-drawing, as required. ■

We further investigate the detailed structure of H-rim edges.

LEMMA 11.8. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. If $v_i v_{i+3}$ is a global H-bridge, then, for $j \in \{i-1, i+3\}$ there is an edge $e_j \in r_j$ that is neither H-yellow nor H-green.*

Proof. The two sides are symmetric, so it suffices to prove the existence of e_{i-1}. Lemmas 10.8 and 10.9 (2) imply that $\overline{Q}_{i+1}$ has BOD. Let D be a 1-drawing of $G - \langle s_{i+1}\rangle$. Lemma 5.9 implies $\overline{Q}_{i+1}$ is crossed in D.

However, the cycle C consisting of $v_i v_{i+3}$ and the path it spans is H_1-close, for $H_1 = R \cup s_{i-1} \cup s_i \cup s_{i+3}$. Therefore, Lemmas 5.3 and 5.4 imply that C is not crossed in D. We conclude from the nature of 1-drawings of V_8 that r_{i-1} crosses $r_{i+5} \cup r_{i+6}$; let e be the edge in r_{i-1} that is crossed in D.

Suppose, by way of contradiction, that there is a global H-bridge B spanning e. Theorem 10.6 implies B is either a 2-, 2.5- or 3-jump, while Theorem 6.7 implies B does not span any edge of r_i (such an edge is already spanned by v_iv_{i+3}). Lemma 10.9 (1) implies v_i is not an attachment of B, so B must be a 2.5-jump with one end in $\langle r_{i-1}\rangle$, contradicting Lemma 10.9 (3). Thus, e is not spanned by a global H-bridge.

It follows that, if e is in an H-green cycle C', then $C' \subseteq \mathrm{cl}(Q_{i-1})$. But such a C' is H_2-close, for $H_2 = R \cup s_i \cup s_{i+2} \cup s_{i+3}$. By Lemmas 5.3 and 5.4, C' is not crossed in any 1-drawing of $G - \langle s_{i+1}\rangle$. This contradicts the fact that e is crossed in D. We conclude that e is not H-green.

So now we suppose e is in the H-yellow cycle C' and that C'' is a witnessing H-green cycle. Then $C' \subseteq \mathrm{cl}(Q_{i-1})$ and C'' contains a global H-bridge B that spans an edge in r_{i+4}. This implies $B \neq v_iv_{i+3}$, so Lemma 10.9 (2) shows that B is not a 3-jump.

Moreover, (3) of the same lemma shows B cannot have an attachment in $[v_{i+3}, r_{i+3}, v_{i+4}\rangle$, while (4) shows B cannot have v_{i+7} as an attachment. Therefore, B is a 2- or 2.5-jump $v_{i+4}w$, with $w \in [v_{i+6}, r_{i+6}, v_{i+7}\rangle$.

The cycle $(R - \langle C'' \cap R\rangle) \cup B$, together with the H-spokes s_{i-1}, s_{i+2}, and s_{i+3} is a subdivision H_3 of V_6 for which C' is H_3-close, showing that e is not crossed in any 1-drawing of $G - \langle s_{i+1}\rangle$. This contradicts the fact that e is crossed in D and, therefore, e is not H-yellow. ∎

The proof of Theorem 11.3 will also depend on the following new concepts.

Definition 11.9. Let G be a graph and let $V_{10} \cong H \subseteq G$, with H tidy. Let Π be an embedding of G in $\mathbb{R}P^2$ so that H is Π-tidy and has the standard labelling relative to γ. For $i \in \{0, 1, 2, \dots, 9\}$:

(1) $\overleftarrow{P}_i = r_{i-2}\, r_{i-1}$, $\underleftarrow{P}_i = r_{i+3}\, r_{i+4}$, $\overrightarrow{P}_i = r_{i+1}\, r_{i+2}$, and $\underrightarrow{P}_i = r_{i+6}\, r_{i+7}$.

(2) the *spines* $\sqsupset_i$ and ${}_i\sqsubset$ of Q_i consist of the paths $\overleftarrow{P}_i \cup s_i \cup \underleftarrow{P}_i$ and $\overrightarrow{P}_i \cup s_{i+1} \cup \underrightarrow{P}_i$, respectively (see Figure 11.2);

(3) the *scope* K_i of Q_i consists of $\mathrm{cl}(Q_i) \cup \sqsupset_i \cup {}_i\sqsubset \cup \mathcal{B}_i$, where $\mathcal{B}_i$ consists of all global H-bridges having both attachments either in $\overleftarrow{P}_i \cup \overrightarrow{P}_i$ or in $\underleftarrow{P}_i \cup \underrightarrow{P}_i$; and

(4) the *complement* $K_i^\natural$ of K_i is obtained from M_{Q_i} by deleting the edges (but not their incident vertices) that comprise the H-bridges in $\mathcal{B}_i$.

(5) The two vertices v_{i-2} and v_{i+3} are the *trivial* $\sqsupset_i\,{}_i\sqsubset$-paths in K_i. Any other $\sqsupset_i\,{}_i\sqsubset$-path in K_i is *non-trivial*.

We note that $\sqsupset_i \cap {}_i\sqsubset$ is equal to $\|\{v_{i-2}, v_{i+3}\}\|$. For our purposes, these are not "useful" $\sqsupset_i\,{}_i\sqsubset$-paths.

We observe that, for each $i \in \{0, 1, 2, 3, 4\}$, $G = K_i \cup K_i^\natural$.

The following lemma plays an important role in the rest of this section.

Lemma 11.10. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. Let e be an edge of R and let i be such that $e \in r_i$. Then $G - e$ has a subdivision of V_6 if and only if there are disjoint non-trivial $\sqsupset_i\,{}_i\sqsubset$-paths in $K_i - e$.*

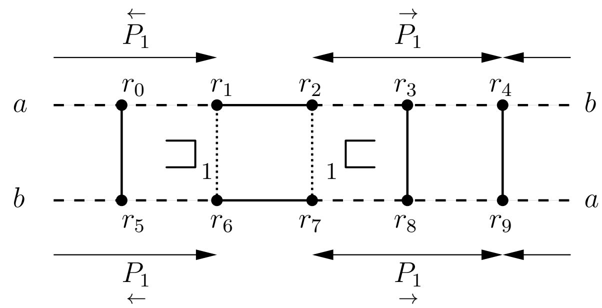

FIGURE 11.2. The paths with small dashes are $\overleftarrow{P}_1$, $\underleftarrow{P}_1$, $\overrightarrow{P}_1$, and $\underrightarrow{P}_1$. The spine $\sqsupset_1$ is the path $r_9\, r_0\, s_1\, r_5\, r_4$, while ${}_1\sqsubset$ is $r_3\, r_2\, s_2\, r_7\, r_8$.

There is some subtlety here; 2-criticality is important. Suppose we have a subdivision H of V_{10} embedded in $\mathbb{R}P^2$ with representativity 2 so that all the H-spokes are in $\mathfrak{M}$. Give H the usual labelling relative to γ. Now delete $\langle r_1\rangle$ and $\langle r_6\rangle$, and then add the 2.5-jump av_2 and the 3-jump v_6v_9. Then there are disjoint non-trivial $\sqsupset_{1\,1}\sqsubset$-paths in the union H' of $(H - \langle r_1\rangle) - \langle r_6\rangle$ and the two jumps, but H' is planar.

We shall need the following.

LEMMA 11.11. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. Let e be an edge of R and let i be such that $e \in r_i$. If there are disjoint non-trivial $\sqsupset_{i\,i}\sqsubset$-paths in $K_i - e$, then there are two such paths so that at least one of them is contained in $\mathrm{cl}(Q_i)$ and the other contains at most one global H-bridge.*

In the proof, we consider many possibilities for the two disjoint $\sqsupset_{i\,i}\sqsubset$-paths. For a given i, some possibilities might not occur because of limitations imposed by Π. In principle, for $i = 2$, all of the considered possibilities can occur, while for $i = 4$, several of the considered possibilities cannot occur.

Proof. Let P_1 and P_2 be the hypothesized disjoint paths.

CLAIM 1. If there is a $\sqsupset_{i\,i}\sqsubset$-path in $K_i - e$ disjoint from r_{i+5}, then there are disjoint $\sqsupset_{i\,i}\sqsubset$-paths so that one of them is contained in $\mathrm{cl}(Q_i)$ and the other contains at most one global H-bridge.

PROOF. Suppose that P and r_{i+5} are disjoint paths. If P contains two (or more) global H-bridges, then they must be 2.5-jumps having an end in $\langle r_i\rangle$. By Theorem 6.7, they must be of the form $v_{i-2}w_1$ and w_2v_{i+3}, with w_1 being no further from v_i in r_i than w_2 is. By symmetry, we may assume e is not in $[v_i, r_i, w_1]$. Now $[v_i, r_i, w_1](P - v_{i-2})$ and r_{i+5} are the desired disjoint $\sqsupset_{i\,i}\sqsubset$-paths in $K_i - e$. $\square$

Thus, we may assume both P_1 and P_2 intersect r_{i+5}.

CLAIM 2. If either of P_1 and P_2 contains two global H-bridges, then there are disjoint $\sqsupset_{i\,i}\sqsubset$-paths in $K_i - e$ so that one of them is contained in $\mathrm{cl}(Q_i)$ and the other contains at most one global H-bridge.

PROOF. We may assume P_1 contains two global H-bridges B_1 and B_2. Both B_1 and B_2 are 2.5-jumps. Both have ends in $\langle r_i\rangle \cup \langle r_{i+5}\rangle$. By Lemma 10.9 (5), they both have an end in the same one of $\langle r_i\rangle$ and $\langle r_{i+5}\rangle$. We choose the labelling so that (B_1, B_2) is either $(v_{i-2}w_1, w_2v_{i+3})$ or $(v_{i+3}w_1, w_2v_{i+8})$. We treat these cases separately.

Suppose $(B_1, B_2) = (v_{i-2}w_1, w_2v_{i+3})$. Assume first that $e \notin [w_1, r_1, w_2]$. Then $B_1 \cup [w_1, r_1, w_2] \cup B_2$ is disjoint from r_{i+5}, and we are done by Claim 1. Therefore, we may assume $e \in [w_1, r_1, w_2]$.

In this case, P_1 consists of B_1, B_2, and a w_1w_2-path P_1' contained in $\mathrm{cl}(Q_i)$. We know that P_1' contains a vertex in r_{i+5}. Lemma 10.9 (5) implies that P_2 consists of a global H-bridge with no vertex in $\langle r_{i+5}\rangle$. Therefore, we may choose $[v_i, r_i, w_1] \cup P_1' \cup [w_2, r_i, v_{i+1}]$ and P_2 as the desired paths.

We conclude the proof of this claim by considering the case $(B_1, B_2) = (v_{i+3}w_1, w_2v_{i+8})$. First, by way of contradiction suppose P_2 is not contained in $\mathrm{cl}(Q_i)$. Lemma 10.9 (5) implies that P_2 consists of a global H-bridge having both ends in $\overleftarrow{P}_i \cup \overrightarrow{P}_i$. But then P_2 is disjoint from r_{i+5} and we are done by Claim 1. Thus, we may assume $P_2 \subseteq \mathrm{cl}(Q_i)$.

If P_2 is disjoint from either $[v_{i+5}, r_{i+5}, w_1\rangle$ or $\langle w_2, r_{i+5}, v_{i+6}]$, then we may replace either B_1 with the former or B_2 with the latter, and we are done again. Otherwise, there is a $[v_{i+5}, r_{i+5}, w_1\rangle\,\langle w_2, r_{i+5}, v_{i+6}]$-path P_2' contained in P_2 that is r_{i+5}-avoiding; let its ends be $w_3 \in [v_{i+5}, r_{i+5}, w_1\rangle$ and $w_4 \in \langle w_2, r_{i+5}, v_{i+6}]$.

If P_2' is r_i-avoiding, then $P_2' \cup [w_3, r_{i+5}, w_4]$ is an H-green cycle. Since B_1 together with the subpath of R it spans is also H-green, the edge of $[v_{i+5}, r_{i+5}, w_1]$ incident with w_1 is in two H-green cycles, contradicting Theorem 6.7.

Therefore, P_2' is not r_i-avoiding and so contains two subpaths, one being a w_3r_i-path $P_2'^1$ and the other being an r_iw_4-path $P_2'^2$. For $k = 1, 2$, let u_k be the vertex of $P_2'^k$ in r_i. If $e \in [v_i, r_i, u_1]$, then the paths $[v_{i+5}, r_{i+5}, w_3] \cup P_2'^1 \cup [u_1, r_i, v_{i+1}]$ and $B_1 \cup [w_1, r_{i+5}, v_{i+6}]$ constitute the required disjoint paths. Otherwise, $[v_i, r_i, u_1] \cup [u_1, P_2', w_4, r_{i+5}, v_{i+6}]$ and $[v_{i+5}, r_{i+5}, w_2] \cup B_2$ constitute the required disjoint paths. $\square$

To complete the proof of the lemma, we may now assume that, for each $j = 1, 2$, P_j contains a unique global H-bridge B_j.

We first suppose, by way of contradiction, that both B_1 and B_2 have an end in $\langle r_i\rangle \cup \langle r_{i+5}\rangle$. Lemma 10.9 (5) shows that such ends are in the same one of $\langle r_i\rangle$ and $\langle r_{i+5}\rangle$; let $i' \in \{i, i+5\}$ be such that, for $j = 1, 2$, B_j has an end $w_j \in \langle r_{i'}\rangle$. We may assume $B_1 = v_{i'-2}w_1$ and $B_2 = w_2v_{i'+3}$.

Theorem 6.7 implies w_1 is closer to $v_{i'}$ in $r_{i'}$ than w_2 is. The paths $P_1 - v_{i'-2}$ and $P_2 - v_{i'+3}$ are both in $\mathrm{cl}(Q_i)$; the former is a w_1s_{i+1}-path, with end $x_1 \in s_{i+1}$, and the latter is a w_2s_i-path, with end $x_2 \in s_i$.

Recall that $\Pi[\mathrm{cl}(Q_i)]$ is a planar embedding of $\mathrm{cl}(Q_i)$ with Q_i bounding a face. The vertices w_1, w_2, x_1, x_2 occur in this cyclic order in Q_i, so the disjoint paths $P_1 - v_{i'-2}$ and $P_2 - v_{i'+3}$ must cross in $\Pi[\mathrm{cl}(Q_i)]$, a contradiction. Therefore, at most one of B_1 and B_2 has an end in $\langle r_i\rangle \cup \langle r_{i+5}\rangle$, while the other is equal to the path among P_1 and P_2 that contains it.

We may choose the labelling so that P_2 consists only of B_2. Theorem 6.7 implies no edge of $r_i \cup r_{i+5}$ is spanned by both B_1 and B_2; since B_2 spans one of r_i and r_{i+5} completely, one of B_1 and B_2 spans edges in r_i and the other spans edges in r_{i+5}. If either B_j spans all of r_i, then, as it is disjoint from r_{i+5}, we are done by

Claim 1. In particular, B_2 spans r_{i+5}, edges spanned by B_1 are in r_i, and B_1 does not span all of r_i.

Therefore, B_1 is a 2.5-jump with one end w_1 in $\langle r_i\rangle$. We may assume the other end of B_1 is v_{i+3}. If $e \notin [v_i, r_i, w_1]$, then $[v_i, r_i, w_1] \cup B_1$ is disjoint from r_{i+5}, and we are done by Claim 1. If $e \in [v_i, r_i, w_1]$, then $(P_1 - v_{i+3})\ [w_1, r_i, v_{i+1}]$ and P_2 are the desired paths. ∎

Proof of Lemma 11.10. The following claim settles one direction.

Claim 1. If there are not disjoint non-trivial $\sqsupset_i {}_i\sqsubset$-paths in $K_i - e$, then $G - e$ is planar.

Proof. For this proof, we need to apply Menger's Theorem; in order to do so, we treat the copies of v_{i-2} and v_{i+3} in $\sqsupset_i$ as different from their copies in ${}_i\sqsubset$. Let u be a cut-vertex of $K_i - e$ separating $\sqsupset_i$ and ${}_i\sqsubset$. Let $\overleftarrow{K}_i$ be the union of the $\|u\|$-bridges in $K_i - e$ that have an edge in $\sqsupset_i$ and let $\overrightarrow{K}_i$ be the union of the remaining $\|u\|$-bridges in $K_i - e$. Then $K_i - e = \overleftarrow{K}_i \cup \overrightarrow{K}_i$ and $\overleftarrow{K}_i \cap \overrightarrow{K}_i$ is just $\|u\|$.

Let Π be an embedding of G in $\mathbb{R}P^2$ so that H is Π-tidy. Since $r_{i+5} \subseteq K_i - e$, $u \in r_{i+5}$. Because $K_i - \{e, u\}$ is not connected, there is a non-contractible, simple closed curve in $\mathbb{R}P^2$ that meets $\Pi[G-e]$ only at u. Thus, there is no non-contractible cycle in $G - \{e, u\}$, showing that $G - e$ is planar. □

For the converse, Lemma 11.11 shows there are disjoint non-trivial $\sqsupset_i {}_i\sqsubset$-paths P_1 and P_2 in $K_i - e$ so that $P_1 \subseteq \mathrm{cl}(Q_i)$. In particular, P_1 is an $s_i s_{i+1}$-path. It follows from the embedding $\Pi[K_i]$ that P_2 is disjoint from either r_i or r_{i+5}.

In every case, we find our V_6 by adding three spokes to the cycle contained in $\big(R - (\langle r_i\rangle \cup \langle r_{i+5}\rangle)\big) \cup P_1 \cup P_2 \cup s_i \cup s_{i+1}$ and containing $\big(R - (\langle r_i\rangle \cup \langle r_{i+5}\rangle)\big) \cup P_1 \cup P_2$.

If P_2 contains no global H-bridges, then s_{i+2}, s_{i+3}, and s_{i+4} may be chosen as the spokes.

If P_2 contains precisely one global H-bridge B_2, then B_2 is one of:

(1) $v_{i-2}v_{i+1}$ (symmetrically, $v_i v_{i+3}$);
(2) $v_{i-1}v_{i+2}$;
(3) $v_{i-2}w$ and w is in $\langle r_i\rangle$ (symmetrically, wv_{i+3});
(4) wv_{i+1} and w is in $\langle r_{i-2}\rangle$ (symmetrically, $v_i w$, with $w \in \langle r_{i+2}\rangle$);
(5) $v_{i-1}w$ and w is in $\langle r_{i+1}\rangle$ (symmetrically, wv_{i+2}, with $w \in \langle r_{i-1}\rangle$);
(6) $v_{i-1}v_{i+1}$ (symmetrically, $v_i v_{i+2}$);
(7) and the comparable jumps with ends in $r_{i+3}\, r_{i+4}\, r_{i+5}\, r_{i+6}\, r_{i+7}$.

We choose, in all cases, s_{i-2} and s_{i+2} as two of the spokes, with third spoke (taking the cases in the same order):

(1) the P_1P_2-subpath of s_{i+1} (symmetrically, the P_1P_2-subpath of s_i);
(2) s_{i-1};
(3) the P_1P_2-subpath of s_{i+1} (symmetrically, the P_1P_2-subpath of s_i);
(4) the P_1P_2-subpath of s_{i+1} (symmetrically, the P_1P_2-subpath of s_i);
(5) s_{i-1} (symmetrically, the same);
(6) s_{i-1} (symmetrically, the same); and
(7) these cases are symmetric to the preceding ones.

In every case, we have found a V_6 in $G - e$, as required. ∎

We conclude this section by proving that every rim edge is either red, H-green, or H-yellow.

Proof of Theorem 11.3. Let e be an edge in the H-rim. There is an i so that $e \in r_i$. By Lemma 11.10, G is red if and only if there are no disjoint non-trivial $\sqsupset_i\,{}_i\sqsubset$-paths in $K_i - e$.

Now suppose there are disjoint non-trivial $\sqsupset_i\,{}_i\sqsubset$-paths P_1 and P_2 in $K_i - e$. By Lemma 11.11, we may assume P_1 is contained in $\mathrm{cl}(Q_i)$, while P_2 contains at most one global H-bridge. If P_1 is disjoint from r_{i+5}, then every maximal r_i-avoiding subpath of P_1 is contained in an H-green cycle. The edge e is in one of these H-green cycles, as required.

Thus, we may assume P_1 contains a vertex in r_{i+5}. If $P_2 \subseteq \mathrm{cl}(Q_i)$, then the planar embedding of $\mathrm{cl}(Q_i)$ shows P_2 is disjoint from r_{i+5} and the preceding paragraph, with P_2 in place of P_1, shows e is H-green. Consequently, we may further assume P_2 contains a global H-bridge B_2.

Case 1: *B_2 has its ends in $\overleftarrow{P}_i \cup r_i \cup \overrightarrow{P}_i$.*

In this case, if e is spanned by B_2, then there is an H-green cycle containing e, namely the cycle consisting of B_2 and the subpath of R that it spans. The only other possibility in this case is that B_2 is a 2.5-jump with an end w_2 in $\langle r_i \rangle$ and that e is in the one of $[v_i, r_i, w_2]$ and $[w_2, r_i, v_{i+1}]$ not spanned by B_2. For the sake of definiteness, we suppose $B_2 = v_{i-2}w_2$ and that e is in $[w_2, r_i, v_{i+1}]$.

Since $P_1 \subseteq \mathrm{cl}(Q_i)$, we see that, in this case, P_2 is disjoint from r_{i+5} and, therefore, we may assume $P_1 = r_{i+5}$. We replace P_2 with $[v_i, r_i, w_2]\ (P_2 - v_{i-2})$ so that there are disjoint $\sqsupset_i\,{}_i\sqsubset$-paths contained in $\mathrm{cl}(Q_i) - e$; a situation resolved in the paragraph preceding this case.

Case 2: *B_2 has its ends in $\underleftarrow{P}_i \cup r_{i+5} \cup \underrightarrow{P}_i$.*

In this case, either P_2 is B_2 or, up to symmetry, B_2 is a 2.5-jump wv_{i+8}, with $w \in \langle r_{i+5} \rangle$, and P_2 is $[v_{i+5}, r_{i+5}, w] \cup B_2$. On the other hand, P_1 is an $s_i s_{i+1}$-path in $\mathrm{cl}(Q_i)$ intersecting r_{i+5}.

Let x be the first vertex in r_{i+5} as we traverse P_1 from s_i and let P_1' be the $s_i x$-subpath of P_1. We note that P_2 prevents x from being in $[v_{i+5}, r_{i+5}, w]$, so $x \in \langle w, r_{i+5}, v_{i+6}]$. Let y be the end of P_1' in s_i. The cycle $P_1'\ [x, r_{i+5}, v_{i+6}]\ s_{i+1}\, r_i\ [v_i, s_i, y]$ is H-yellow, as witnessed by the H-green cycle containing B_2. Therefore, e is either H-yellow or H-green (in Definition 11.1, an H-yellow edge is not H-green). ■

CHAPTER 12

Existence of a red edge and its structure

In this section, we prove that if G is a 3-connected, 2-crossing-critical graph containing a tidy subdivision H of V_{10}, then some edge of the H-rim is red. Furthermore, we prove that each red edge e has an associated special cycle we call Δ_e. These "deltas" will be the glue that hold successive tiles together and so form a vital element of the tile structure.

The argument for proving the existence of a red edge depends on whether or not there is a global H-bridge that is either a 2.5- or 3-jump. Once these cases are disposed of, matters become simple. However, with the knowledge of the Δ's, it turns out we can show that there is no 3-jump. This will be our first aim and so, since we need the Δ's to complete the elimination of 3-jumps, we shall begin by determining the structure of the Δ of a red edge.

THEOREM 12.1. *Let $G \in \mathcal{M}_2^3$, $V_{10} \cong H \subseteq G$, with H tidy. Let $e = uw$ be a red edge of G and let $i \in \{0, 1, 2, \dots, 9\}$ be such that $e \in r_i$. Then there exists a vertex $x_e \in [r_{i+5}]$ and internally disjoint $x_e u$- and $x_e w$-paths A_u and A_w, respectively, in $\mathrm{cl}(Q_i)$ so that, letting $\Delta_e = (A_u \cup A_w) + e$:*

(1) there are at most two Δ_e-bridges in G;

(2) there is a Δ_e-bridge M_{Δ_e} so that $H \subseteq M_{\Delta_e} \cup \Delta_e$, while the other Δ_e-bridge, if it exists, is one of two edges in a digon incident with x_e; and

(3) when there are two Δ_e-bridges, let u^e and w^e be the attachments of the one-edge Δ_e-bridge, labelled so that $u^e \in A_u$ and $w^e \in A_w$; otherwise let $u^e = w^e = x_e$. In both cases, $\Delta_e - e$ contains unique uu^e- and ww^e-paths P_u and P_w, each containing at most one H-rim edge, which, if it exists, is in the span of a global H-bridge and, therefore, is H-green.

Proof. Let Π be an embedding of G in $\mathbb{R}P^2$ for which H is Π-tidy. We may assume $r_i = [v_i, r_i, u, e, w, r_i, \ v_{i+1}]$. Lemma 11.10 implies $K_i - e$ has a cut vertex x_e separating $\sqsupset_i$ and ${}_i\sqsubset$ (again adopting the perspective that v_{i-2} and v_{i+3} are split into different copies in $\sqsupset_i$ and ${}_i\sqsubset$). As r_{i+5} is a $\sqsupset_i\,{}_i\sqsubset$-path in $K_i - e$, x_e is in r_{i+5}.

Because $\mathrm{cl}(Q_i)$ is 2-connected and $\Pi[\mathrm{cl}(Q_i)]$ has Q_i bounding the exterior face, there is a face F_e of G in $\mathbb{R}P^2$ that is in the interior of Q_i and incident with both e and x_e. As G is 3-connected and non-planar, F_e is bounded by a cycle C_e and $C_e - e$ consists of a ux_e-path A_u and a wx_e-path A_w.

For (1) and (2), we begin by noticing that $C_e \subseteq \mathrm{cl}(Q_i)$. Thus, there is a C_e-bridge M_{C_e} containing the three H-spokes not in Q_i.

CLAIM 1. Each of $C_e \cap s_i$, $C_e \cap s_{i+1}$, and $C_e \cap r_i$ is connected. Either $C_e \cap r_{i+5}$ is connected or it has two components that are joined by an edge e' of r_{i+5} and C_e has an edge parallel to e'. In particular, each of s_i, s_{i+1}, r_i, and $r_{i+5} - e'$ is contained in $C_e \cup M_{C_e}$.

Proof. Suppose by way of contradiction that $C_e \cap s_i$ is not connected. As C_e bounds a face of Π, it follows that there is a Q_i-local H-bridge having all its attachments in s_i, contradicting Lemma 10.10. Thus, $C_e \cap s_i$ is connected.

It follows that any part of s_i that is not in C_e is in the same C_e-bridge as either r_{i-1} or r_{i+4}. That is, it is in M_{C_e}, and therefore, $s_i \subseteq M_{C_e} \cup C_e$.

Symmetry shows that this also holds for s_{i+1}.

Now suppose $C_e \cap r_i$ is not connected. Then there is a Q_i-local H-bridge B having all attachments in r_i. Corollary 5.15 implies B has precisely two attachments x and y, and so Lemma 5.19 implies B is just the edge xy. Thus, $[x, r_i, y]B$ is an H-green cycle C. Lemma 6.6 (8) shows C bounds a face of $\Pi[G]$.

By symmetry, we may assume that x and y are both in $[v_i, r_i, u]$. Suppose that z is any vertex in $\langle x, r_i, u]$.

Suppose first that $z \neq u$. As G is 3-connected, z has a neighbour z' not in $[x, r_i, y]$. If zz' is in the interior of Q_i, it must be parallel to an edge in r_i, as any other edge would go into one of the faces of $\Pi[G]$ bounded by C_e and C. Therefore, zz' is outside $\mathfrak{M}$ and, so is an edge of another H-green cycle. But then one of the edges of $[x, r_i, y]$ incident with z is in two H-green cycles, contradicting Theorem 6.7.

This same argument, however, also applies if $z = u$, with the small variation that, by Lemma 11.4, zz' cannot span the red edge e, giving the contradiction that the edge of $[x, r_i, u]$ incident with u is in two H-green cycles. Thus, $C_e \cap r_i$ is connected. As it did for s_i, this implies that $r_i \subseteq M_{C_e} \cup C_e$.

Finally, we consider $C_e \cap r_{i+5}$. Proceeding as we did for r_i, if $C_e \cap r_{i+5}$ is not connected, there is (up to symmetry) a Q_i-local H-bridge B having all attachments in $[v_{i+5}, r_{i+5}, x_e]$; B is a single edge and is in an H-green cycle. One end of B is x_e, and the H-green cycle containing B consists of two parallel edges.

Thus, there are at most two such H-bridges B, each of which is an edge parallel to an edge in r_{i+5}. If they both exist, then the 3-connection of G implies x_e has another neighbour, which, as above, is adjacent to x_e by an edge not in $\mathfrak{M}$, showing one of the edges of r_{i+5} incident with x_e is in two H-green cycles, contradicting Theorem 6.7. □

We can now define Δ_e. If $C_e \cap r_{i+5}$ is connected, then $\Delta_e = C_e$. Otherwise, Δ_e is obtained from C_e by replacing the edge of C_e incident with x_e and not in r_{i+5} with its parallel mate that is in r_{i+5}. Notice that the Δ_e- and C_e-bridges are the same, except for these exchanged edges incident with x_e. Set M_{Δ_e} to be M_{C_e}. The following is evident from what has just preceded.

Claim 2. $H \subseteq M_{\Delta_e} \cup \Delta_e$ and $\Delta_e \cap r_{i+5}$ is connected. □

Consider again $r_i \cap \Delta_e$. It is connected, so if it is more than just $[u, e, w]$, the symmetry shows we may assume it contains an edge xu other than e. The 3-connection of G implies that u is adjacent with a vertex y other than x and w. The edge uy is not interior to Q_i, as then it would be in the face of G bounded by C_e.

Thus, uy is not in $\mathfrak{M}$, and, as uw is red, Lemma 11.4 implies uy spans xu. The vertex x is seen to be H-green by the H-green cycle C_y containing uy. Since x has at least three neighbours in G, there is a neighbour of x different from the two neighbours of x in $r_{i-1}\, r_i$. Because C_y bounds a face of G (Lemma 6.6 (8)), every edge incident with x and not in $r_{i-1}\, r_i$ is in $\mathfrak{M}$. There is a unique neighbour z of

x so that z is not in $r_{i-1}\,r_i$ and xz is an edge of Δ_e. This shows that x is one end of $r_i \cap \Delta_e$. These observations easily yield the following claim.

CLAIM 3. Each of $A_u \cap r_i$ and $A_w \cap r_i$ has at most one edge. $\square$

We now turn our attention to r_{i+5}.

CLAIM 4. (1) No edge of $r_{i+5} \cap \Delta_e$ is H-yellow.
(2) No global H-bridge has x_e in the interior of its span.

PROOF. For (1), suppose by way of contradiction there were an H-yellow edge in $r_{i+5} \cap \Delta_e$. Then Lemma 11.2 (3) shows the witnessing H-yellow cycle must be Δ_e. However, the witnessing H-green cycle must have $\Delta_e \cap r_i$ in the interior of its span, yielding the contradiction that e is H-green.

For (2), suppose by way of contradiction that there is a global H-bridge xy with x_e in the interior of the span of xy. Then $xy \cup (r_{i+5} - x_e)$ contains a $\sqsupset_i\, {}_i\sqsubset$-path in $K_i - \{e, x_e\}$, contradicting Lemma 11.10. $\square$

CLAIM 5. (1) If $[v_{i+5}, r_{i+5}, x_e] \cap \Delta_e$ contains three vertices x, y, and x_e of r_{i+5}, then (choosing the labelling of x and y appropriately) $[v_{i+5}, r_{i+5}, x_e] \cap \Delta_e = [x, xy, y, yx_e, x_e]$, y and x_e are joined by a digon, and y is incident with a global H-bridge that spans x.
(2) If $[v_{i+5}, r_{i+5}, x_e] \cap \Delta_e$ does not contain three consecutive vertices of r_{i+5}, but has a vertex x other than x_e, then either x and x_e are joined by a digon, or x_e is incident with a global H-bridge that spans x.

The symmetric statements also hold for $[x_e, r_{i+5}, v_{i+6}] \cap \Delta_e$.

PROOF. For (1), the fact that $\Delta_e \cap r_{i+5}$ is connected implies that there are vertices x and y so that $[x, xy, y, yx_e, x_e] \subseteq [v_{i+5}, r_{i+5}, x_e]$. Because G is 3-connected, y is adjacent to a vertex z other than x and x_e. The edge yz cannot be in $\mathfrak{M}$, as then it would be in the face of G bounded by C_e, a contradiction. Therefore, it is a 2.5-jump. Claim 4 (2) shows yz does not span x_e.

As G is 3-connected, x has a neighbour x' different from the two neighbours of x in R. If the edge xx' is in $\mathfrak{D}$, then it is in the face bounded by the H-green cycle containing yz, a contradiction. Therefore, xx' is in $\mathfrak{M}$ and, in particular, for that x' giving the edge nearest to xy in the cyclic rotation about x, xx' is in Δ_e and, therefore, no other vertex of $[v_{i+5}, r_{i+5}, x_e]$ is in Δ_e.

Since yx_e is not R-separated from e in G, Lemma 11.7 implies yx_e is either H-yellow or H-green. Claim 4 (1) implies it is not H-yellow; we conclude that yx_e is H-green and let C_{yx_e} be the witnessing H-green cycle.

As pointed out in the first paragraph of the proof, C_{yx_e} cannot contain a global H-bridge that spans x_e. On the other hand, xy is H-green by the global H-bridge yz. By Theorem 6.7, this is the only H-green cycle containing xy. Thus, the only H-rim edge contained in C_{yx_e} is yx_e. It follows that C_{yx_e} is contained in $\mathrm{cl}(Q_i)$. Claim 1 implies C_{yx_e} is a digon.

For (2), the fact that $[v_{i+5}, r_{i+5}, x_e] \cap \Delta_e$ is connected implies that $[v_{i+5}, r_{i+5}, x_e] \cap \Delta_e = [x, xx_e, x_e]$. Lemma 11.7 implies that xx_e is either H-yellow or H-green, and Claim 4 (1) shows it is not H-yellow. Therefore, it is H-green.

Claim 4 (2) shows any global H-bridge spanning xx_e has x_e as an attachment. Otherwise, the H-green cycle C_{xx_e} containing xx_e is contained in $\mathrm{cl}(Q_i)$. Again, Claim 1 shows C_{xx_e} is a digon. $\square$

There is one more observation to make before we complete the proof of the theorem. From Claim 5 (1), it seems possible that both $[v_{i+5}, r_{i+5}, x_e] \cap \Delta_e$ and $[v_{i+5}, r_{i+5}, x_e] \cap \Delta_e$ have three vertices. However, this is not possible, as x_e must have a neighbour z different from its neighbours in R. But now $x_e z$ cannot be in $\mathfrak{M}$, as then it would be in the face bounded by C_e, and it cannot be in $\mathfrak{D}$, as then it is a global H-bridge and one of the digons incident with x_e is also spanned by $x_e z$, contradicting Theorem 6.7. Therefore, $r_{i+5} \cap \Delta_e$ has at most three edges, and all such edges are H-green.

If there are no edges, then $r_{i+5} \cap \Delta_e$ is just x_e. If no edge of $r_{i+5} \cap \Delta_e$ is in a digon, then u^e and w^e are defined in (3) of the statement to be x_e. In this case, Claim 5 (1) implies there can be at most one edge of $r_{i+5} \cap \Delta_e$ on each side of x_e, but any such edge is spanned by a global H-bridge. If there is a digon, then it is $u^e w^e$, each of u^e and w^e is incident with at most one other edge in $r_{i+5} \cap \Delta_e$, and any such edge is spanned by a global H-bridge.

Finally, By Lemma 10.9 (4), not both u and u^e, for example, can be incident with such global H-bridges, so P_u has at most one H-rim edge. ■

Definition 12.2. Let $G \in \mathcal{M}_2^3$, $V_{10} \cong H \subseteq G$, with H tidy, and e a red edge of G with ends u and w. With u^e and w^e as in the statement of Theorem 12.1, the *peak* of Δ_e is the subgraph of G induced by u^e and w^e. If the peak has just one vertex, then Δ_e is *sharp*.

The following observations are given to summarize important points from Theorem 12.1.

Corollary 12.3. *Let $G \in \mathcal{M}_2^3$, $V_{10} \cong H \subseteq G$, with H tidy, and e a red edge of G. Then the peak of Δ_e is either a single vertex or a digon and no edge of the peak is in the interior of the span of a global H-bridge.*

Proof. That the peak is either a single vertex or a digon is a rephrasing of Theorem 12.1 (2) and (3). In the case the peak is a digon, neither u^e nor w^e can be in the interior of the span of a global H-bridge, since then the H-rim edge in the digon is in two H-green cycles, contradicting Theorem 6.7.

So suppose the peak is just the vertex $u^e = w^e$, let B be a global H-bridge with u^e in the interior of its span, and let i be such that $e \in r_i$. If $\Delta_e \cap r_{i+5}$ has an edge e', then e' is incident with u^e and, moreover, is H-green by a global H-bridge B' incident with u^e. But then B provides a second H-green cycle containing e', contradicting Theorem 6.7. So $\Delta_e \cap r_{i+5}$ is just u^e, in which case B provides a witnessing H-green cycle that shows Δ_e is H-yellow. But then e is H-yellow, contradicting Lemma 11.4. ■

Our next goal is to eliminate 3-jumps. For this the next two lemmas are helpful.

Lemma 12.4. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. Suppose C is an H-yellow cycle and C' is the witnessing H-green cycle. Let e be an edge of G not in $C \cup C' \cup R$. Suppose either C' does not contain a 3-jump or e is in one of the four spokes containing an H-node spanned by C'. Then no H-yellow edge in C is crossed in any 1-drawing of $G - e$.*

Proof. There are at least four H-spokes contained in $G - e$. By hypothesis, at least one of these has no end in C' and, therefore, no end in $C \cup C'$. Therefore, Lemma 7.2 (2) applies. □

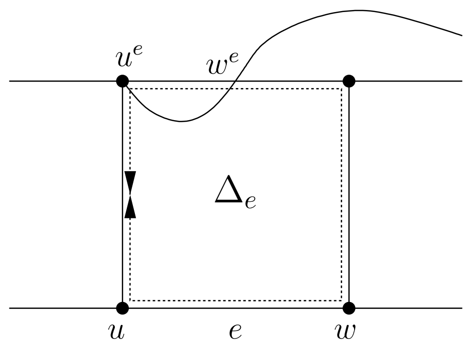

FIGURE 12.1. One of several examples of a Δ.

LEMMA 12.5. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. Suppose C is an H-green cycle in G. Suppose that C does not contain a 3-jump, e is an edge of G not in $R \cup C$ and D is a 1-drawing of $G - e$. If an edge e' of C is crossed in D, then C contains a 2.5-jump with an end in $\langle r_i \rangle$, for some i, and e' is in r_i.*

Proof. This is a straightforward consequence of Lemma 7.2 (3a and 3b). ■

THEOREM 12.6. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. Then no global H-bridge is a 3-jump.*

Proof. The proof begins by showing that if $v_{i-3}v_i$ is a global H-bridge that is a 3-jump, then there is a red edge in r_i. The next step is to show that the edge of r_i incident with v_i is red. The final step is to show that, if e^* is the edge of s_i incident with v_i, then $\mathrm{cr}(G - e^*) \geq 2$, contradicting the criticality of G. Let Π be an embedding of G in $\mathbb{R}P^2$ so that H is Π-tidy.

CLAIM 1. There is a red edge of G in r_i.

PROOF. Lemma 10.9 (2) implies neither $v_{i+5}v_{i-2}$ nor v_iv_{i+3} is in G. Thus, Lemma 10.8 implies $\overline{Q}_{i-1}$ has BOD.

Let D_{i-1} be a 1-drawing of $G - \langle s_{i-1} \rangle$. Lemma 5.9 implies $\overline{Q}_{i-1}$ is crossed in D_{i-1}. Let H' be the subdivision of V_6 consisting of the H-rim R and the three spokes s_i, s_{i-3}, and s_{i+1}. Lemma 7.2 implies the cycle $r_{i-3}\, r_{i-2}\, r_{i-1}\, [v_i, v_{i-3}v_i, v_{i-3}]$ is clean in D_{i-1}. In particular, the crossing must be of an edge in $r_{i+3} \cup r_{i+4}$ and an edge e in r_i.

We prove e is red in G by proving it is neither H-green nor H-yellow. Lemma 10.9 (1) and (3) imply that no global H-bridge other than $v_{i-3}v_i$ has an end in $[v_i, r_i, v_{i+1}\rangle$. Therefore, no H-green cycle containing e can contain a global H-bridge. Thus, any H-green cycle C containing e is contained in $\mathrm{cl}(Q_i)$. Lemma 12.5 implies C is not crossed in D_{i-1}, contradicting the fact that the edge e is in C and is crossed in D_{i-1}. We conclude that e is not H-green.

So suppose C is an H-yellow cycle containing e and let $P_1P_2P_3P_4$ be the decomposition of C into paths as in Definition 11.1. By Lemma 11.2, there is a global H-bridge B so that the interior of the span of B contains P_1. Lemma 10.9 (2) says there is at most one 3-jump in G, so B is either a 2- or 2.5-jump.

That e is not H-yellow is an immediate consequence of Lemma 12.4. □

We now aim to show that the edge of r_i incident with v_i is red. By Claim 1, there is a red edge in r_i; let e_1 be the red edge nearest to v_i in r_i. Let r_i' be the component of $r_i - e_1$ containing v_i and let u be the end of e_1 in r_i'.

Claim 2. No edge of r_i' is H-yellow.

Proof. Suppose some edge e' of r_i' is H-yellow and let C and C' be the witnessing H-yellow and H-green cycles, respectively. Lemma 11.2 (1) implies C' contains a global H-bridge B. We note that Lemma 10.9 (1) and (3) imply (because $v_{i-3}v_i$ is present and $v_{i-3} = v_{i+7}$) that B has no vertex in $\langle v_{i+6}, r_{i+6}, v_{i+7}]$. On the other hand, to make C H-yellow, B must have one end in $\langle v_{i+5}, r_{i+5}, v_{i+6}]$.

Due to the presence of $v_{i-3}v_i$, Lemma 10.9 (4) implies v_{i+3} is not in B. Therefore, Theorem 10.6 implies B has v_{i+6} as one end and its other end is in $\langle v_{i+3}, r_{i+3}, v_{i+4}]$. Theorem 12.1 (3) implies the edge e of $\Delta_{e_1} - e_1$ incident with u is not in H; by Theorem 12.1, it is in $\mathrm{cl}(Q_i)$.

Let D be a 1-drawing of $G - e$. By Theorem 5.23, Q_i has BOD, so Lemma 5.9 implies Q_i is crossed in D. Lemma 7.2 implies no edge in $r_{i+4}\, r_{i+5}$ is crossed in D, so the crossing in D is of r_i with r_{i+6}.

Lemmas 12.4 and 12.5 combine with Theorem 11.3 to show that the edge e'' of r_{i+6} crossed in D is red in G. Lemma 11.7 implies e'' and e_1 are R-separated in G and we conclude that they are also R-separated in $G-e'$; in fact, e'' is R-separated from $r_i'[u, e_1, w]$. It follows that the edge f of r_i crossed in D is in $[w, r_i, v_{i+1}]$.

Lemmas 12.4 and 12.5 combine with Theorem 11.3 to show that f is red in G; however, e_1 and f are not R-separated in $G - e'$ and, therefore, not separated in G, contradicting Lemma 11.7. It follows that no edge of r_i' is H-yellow, as required. $\square$

Claim 3. $u = v_i$.

Proof. By way of contradiction, suppose that $u \neq v_i$. By definition of e_1, no edge of r_i' is red, and Claim 2 shows no edge of r_i' is H-yellow. Theorem 11.3 shows that every edge of r_i' is H-green. Because of $v_{i-3}v_i$, Lemma 10.9 (1) and (3) shows no edge of r_i' is H-green by a global H-bridge.

Let e be the edge of $\Delta_{e_1} - e_1$ incident with u; Theorem 12.1 and the fact that e_1 is not incident with v_i imply that e is not in H. Let D be a 1-drawing of $G - e$. Note that e is in a Q_i-local H-bridge. Since Q_i has BOD (Theorem 5.23), it is crossed in D (Lemma 5.9). Every edge of r_{i-1} is H-green in $G - e$; thus, Lemma 6.6 (10) implies the following.

Subclaim 1. No edge in r_{i-1} is crossed in D. $\square$

We next rule out another possibility.

Subclaim 2. No edge in r_{i+1} is crossed in D.

Proof. Suppose some edge e_i^D of r_{i+1} is crossed in D. Since Q_i is crossed in D, the other crossed edge ${e'}_i^D$ is in r_{i+5}. By Lemma 12.4, no H-yellow edge in $r_{i+1} \cup r_{i+5}$ can be crossed in D. Since $H \subseteq G - e$, Lemma 6.6 (10) implies no H-green cycle not containing e can be crossed in D; in particular, no H-green edge in $r_{i+1} \cup r_{i+5}$ can be crossed in D. Now Theorem 11.3 implies e_i^D and ${e'}_i^D$ are both red in G.

Suppose first that ${e'}_i^D$ is in $[v_{i+5}, r_{i+5}, u^{e_1}]$. (Recall that u^{e_1} is the vertex in the peak of Δ_{e_1} nearest u in $\Delta_{e_1} - e_1$.) Lemma 11.7 implies ${e'}_i^D$ and e_1 are R-separated in G; this implies that $\Delta_{e'^D_i}$ is disjoint from Δ_{e_1}. One of the $r_i r_{i+5}$-paths in $\Delta_{e'^D_i}$, s_{i+1}, s_{i+2}, and s_{i+3} combine with R to show that e'^D_i is R-separated in $G-e$ from every edge in r_{i+1}, a contradiction.

If, on the other hand, e'^D_i is not in $[v_{i+5}, r_{i+5}, u^{e_1}]$, then Lemma 11.7 shows e^D_i and e'^D_i are R-separated in G and there is a subdivision of V_8 that both witnesses this separation and does not contain e (the spokes are s_{i+2}, s_{i+3}, and the "nearer" $(r_i\, r_{i+1})(r_{i+5}\, r_{i+6})$-paths in $\Delta_{e^D_i}$ and $\Delta_{e'^D_i}$). This shows that e^D_i and e'^D_i are R-separated in $G-e$, a contradiction. □

Since Q_i is crossed in D, Subclaims 1 and 2 imply that some edge e^D_i of r_i is crossed in D.

Subclaim 3. $e^D_i \in r'_i$.

Proof. If e^D_i is not in r'_i, then let e'^D_i be the edge of $r_{i+4}\, r_{i+5}\, r_{i+6}$ that is crossed in D. Then e^D_i and e'^D_i are not R-separated in $G-e$. Observe that Δ_{e_1} shows no H-green or H-yellow cycle containing e^D_i can also contain e. Therefore, e^D_i is red in G and, consequently is R-separated from e'^D_i in G. In particular, e is in every subdivision of V_8 that contains R and witnesses the R-separation of e^D_i and e'^D_i. This implies that e'^D_i is in $[v_{i+5}, r_{i+5}, u^{e_1}]$.

As e_1 and e'^D_i are both red in G, by Lemma 11.7 there is a subdivision K of V_8 containing R and witnessing the R-separation of e_1 and e'^D_i. There is an $r'_i r_{i+5}$-path P in K that is disjoint from Δ_{e_1}. Moreover, $P \subseteq \mathrm{cl}(Q_i)$. But now, P together with the $r_i r_{i+5}$-path in $\Delta_{e_1} - u$, s_{i+2}, and s_{i+3} make the four spokes of a subdivision of V_8 containing R and witnessing the R-separation of e^D_i and e'^D_i in $G-e$, a contradiction. □

We now locate the edge e'^D_i. To this end, let $\hat{e}$ be the edge of s_{i-1} incident with v_{i-1} and let $\widehat{D}$ be a 1-drawing of $G-\hat{e}$. By Lemmas 10.8, 10.9 (2), and 5.9, $\overline{Q}_{i-1}$ must be crossed in $\widehat{D}$. However, Lemma 7.2 shows that none of $r_{i-3}\, r_{i-2}\, r_{i-1}$ can be crossed in $\widehat{D}$. Since the edges in r'_i are all H-green and none of the witnessing H-green cycles contains a global H-bridge, Lemma 12.5 implies that no edge of r'_i is crossed in $\widehat{D}$. Thus, some edge of $r_{i+3}\, r_{i+4}$ crosses an edge of $r_i - \langle r'_i \rangle$ in $\widehat{D}$.

Subclaim 4. Every edge in r_{i+4} is H-green in G and no edge in r_{i+4} is crossed in $\widehat{D}$.

Proof. If $e' \in r_{i+4}$ is H-yellow, then $v_{i-3}v_i$ is in the witnessing H-green cycle and, therefore, the edge of s_{i-1} incident with v_{i+4} is in the interior of an H-yellow cycle containing s_{i-2}; this contradicts Lemma 11.2, so e' is not H-yellow.

Now we eliminate the possibility that e' is red. To do this, it will be helpful to know that no H-green edge in r_{i+4} is crossed in $\widehat{D}$: fortunately, this is just Lemma 12.5, combined with Lemma 10.9 (1) and (4) to eliminate the possibility of a 2.5-jump.

Choose e' to be the red (in G) edge in r_{i+4} that is nearest in r_{i+4} to v_{i+5}. Lemma 11.7 implies e' is R-separated from e_1 in G; we may choose the witnessing subdivision K of V_8 to contain s_{i-2} and s_{i+2}; in particular, K avoids $\hat{e}$. Therefore, e' is R-separated from e_1 in $G-\hat{e}$. Since the edges in r_{i+4} between e' and v_{i+5} are neither red (choice of e') nor H-yellow (two paragraphs preceding), they are

H-green (Theorem 11.3), we know they are not crossed in $\widehat{D}$ (preceding paragraph). The subgraph K shows that none of the rest of $r_{i+3}\, r_{i+4}$ can be crossed in $\widehat{D}$, which is a contradiction. Therefore, no edge of r_{i+4} is red in G; since none is H-yellow by the preceding paragraph, Theorem 11.3 shows they are all H-green. $\square$

It follows that an edge of r_{i+3} is crossed in $\widehat{D}$ and it must cross some edge in $[u, r_i, v_{i+1}]$. This further implies that the uu^{e_1}-subpath P_u of $\Delta_{e_1} - e_1$ intersects s_i as otherwise each edge of $[u, r_i, v_{i+1}]$ is R-separated from r_{i+3} in $G - \hat{e}$.

We now return to consideration of D. No edge in r_{i+4} is red in G and, because P_u intersects s_i, every edge (if there are any) of $[v_{i+5}, r_{i+5}, u^{e_1}]$ is H-green. This combines with Lemma 10.9 (1) and (4) to show that no edge in $r_{i+4}[v_{i+5}, r_{i+5}, u^{e_1}]$ is in the span of a global H-bridge; therefore, Lemmas 12.4 and 12.5 imply that no edge of $r_{i+4}[v_{i+5}, r_{i+5}, u^{e_1}]$ is crossed in D. Thus, the edge ${e'}_i^D$ that crosses e_i^D in D is in $[u^{e_1}, r_{i+5}, v_{i+6}]r_{i+6}$.

Because of $v_{i-3}v_i$, no edge in $[u^{e_1}, r_{i+5}, v_{i+6}]r_{i+6}$ is in the span of a global H-bridge. Therefore, Lemmas 12.4 and 12.5 imply ${e'}_i^D$ is red in G. But now Lemma 11.7 implies ${e'}_i^D$ is R-separated in G from e_1; there is a witnessing subdivision K of V_8 that contains s_{i-1}, s_i, and the nearer $(r_i\, r_{i+1})(r_{i+5}\, r_{i+6})$-paths in Δ_{e_1} and $\Delta_{{e'}_i^D}$. Note that the path taken from Δ_{e_1} does not contain e. Therefore, K is also contained in $G - e$; Observation 11.6 (1) shows that these edges cannot be crossed in D, the final contradiction that proves the claim. $\square$

We now move into the final phase of the proof that there is no 3-jump. Let e^* be the edge of s_i incident with v_i and let D^* be a 1-drawing of $G - e^*$. Lemma 10.9 (2) implies $v_{i-3}v_i$ is the only 3-jump of G, so Lemma 10.8 implies $\overline{Q}_i$ has BOD. Lemma 5.9 implies $\overline{Q}_i$ is crossed in D^*. In particular, there is an edge e in $r_{i+3}\, r_{i+4}\, r_{i+5}\, r_{i+6}$ that is crossed in D^*. Lemma 7.2 shows that r_{i+3} is not crossed in D^*.

Claim 4. e is red in G.

Proof. If e is H-yellow in G, then Lemma 12.4 shows that e is not crossed in D^*. Thus, e is not H-yellow.

Suppose e is H-green in G, and let C be the witnessing H-green cycle. Lemma 10.9 (2) implies C does not contain a 3-jump and Lemma 7.2 implies both that it does not contain a 2-jump and is not contained in the union of some Q_j together with a Q_j-local H-bridge. Therefore, C contains a 2.5-jump b and Lemma 7.2 implies e is in the H-rim branch that contains the end x of b that is not an H-node.

The edge e has already been shown to be in $r_{i+4}\, r_{i+5}\, r_{i+6}$. Suppose e is in r_{i+4}. If $b = v_{i+2}x$, then we contradict Lemma 10.9 (4) — $v_{i-3}v_i$ and b span the opposite sides of $\overline{Q}_{i-2}$, a contradiction. The other alternative is that $b = xv_{i-3}$, which violates Lemma 10.9 (1). Thus, $e \notin r_{i+4}$.

If $e \in r_{i+5}$, then either $b = xv_{i-2}$ or $b = xv_{i+3}$. The former does not occur, as otherwise the edges of r_{i-3} are all in two H-green cycles, contradicting Theorem 6.7. If the latter occurs, then we contradict Lemma 10.9 (4) — $v_{i-3}v_i$ and b span the opposite sides of $\overline{Q}_{i-1}$. Thus, $e \notin r_{i+5}$.

So $e \in r_{i+6}$. In this case b is either xv_{i-1} or xv_{i+4}. For the former, the edges of $r_{i-3}\, r_{i-2}$ are all in two H-green cycles, contradicting Theorem 6.7. For the latter, the edge e_1 of r_i incident with v_i is red by Claim 3. The existence of b shows Q_i is

H-yellow, contradicting the fact that e_1 is red. This is the final contradiction that shows e is red. □

Recall that the edge e is in $r_{i+3}\,r_{i+4}\,r_{i+5}\,r_{i+6}$, since it is involved in a crossing with $\overline{Q}_i$. We have already observed that e is not in r_{i+3}.

Suppose first that $e \in r_{i+4}$. Lemma 11.7 implies e and e_1 are R-separated in G; in particular, v_i is not in Δ_e. But then $v_{i-3}v_i$ shows $\Delta_e \subseteq \text{cl}(Q_{i-1}) - v_i$ to be an H-yellow cycle, contradicting the fact that e is red.

Therefore, $e \in r_{i+5}\,r_{i+6}$. Let $\hat{e}$ be the edge crossed by e in D^*. Since Lemma 7.2 implies r_{i-2} is not crossed in D^*, $\hat{e} \notin r_{i-2}$. Since e and e_1 are both red in G, Lemma 11.7 implies they are R-separated in G; there is a witnessing subdivision K of V_8 that contains s_{i-1} and s_{i-2}. This K does not contain e^*, and so is contained in $G - e^*$. Therefore, K separates e from r_{i-1} in $G - e^*$, and so, in D^*, e does not cross r_{i-1}. Thus, $\hat{e}$ is not in r_{i-1}.

Therefore, $\hat{e} \in r_i\,r_{i+1}$. Lemma 10.9 (4) implies there is no 2.5-jump xv_{i+4} — it and $v_{i-3}v_i$ would span the opposite sides of $\overline{Q}_{i-2}$. Also, Lemma 10.9 (3) implies there is no 2.5-jump xv_{i+3} with $x \in \langle r_i \rangle$.

It follows from Lemmas 12.4 and 12.5 (the preceding pararaph is used here) that the edge $\hat{e}$ crossed by e in D^* is red in G. This implies that e and $\hat{e}$ are R-separated in G and this in turn implies that e and $\hat{e}$ are R-separated in $G - e^*$, the final contradiction. ■

Corollary 12.7. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. Then every H-hyperquad has BOD.*

Proof. By Theorem 12.6, no global H-bridge is a 3-jump. By Lemma 10.8, every H-hyperquad has BOD. ■

We are now prepared for the main result of this section.

Theorem 12.8. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. Then there is a red edge in the H-rim.*

Proof. We prove this by first considering the case there is a global H-bridge. By Theorem 12.6, there is no 3-jump. By Theorem 10.6, a global H-bridge is either a 2.5- or a 2-jump.

Claim 1. If G has a 2.5-jump, then G has a red edge.

Proof. By symmetry, we may assume wv_{i+2} is a 2.5-jump with $w \in \langle r_{i-1} \rangle$. By way of contradiction, we assume that G has no red edge. We first treat two special cases.

Case 1: *there is a 2.5-jump $v_{i-3}w'$, with $w' \in \langle r_{i-1} \rangle$.*

In this case, let D be a 1-drawing of $G - \langle s_{i+2} \rangle$. Corollary 12.7 and Lemma 5.9 show that $\overline{Q}_{i+2}$ is crossed in D. Lemma 7.2 implies each of the cycles consisting of one of these two 2.5-jumps and the subpath of R it spans is clean in D. The same lemma implies that neither r_{i+3} nor r_{i+5} is crossed in D. The combination of facts imply that some edge e_2 in r_{i+2} crosses some edge e_6 in r_{i+6}.

Since G has no red edge, Theorem 11.3 implies each of e_2 and e_6 is either H-yellow or H-green in G. There is complete symmetry between them (relative to the two 2.5-jumps), so we treat e_6. If e_6 is H-yellow in G, then it is in some witnessing H-yellow cycle C for which there is a witnessing H-green cycle C'. The only possibility is that C' contains wv_{i+2}.

We have that $C \subseteq \mathrm{cl}(Q_{i+1}) - v_{i+2}$. Let $C = P_1P_2P_3P_4$ be the composition of paths showing C is H-yellow, as in Definition 11.1. Since $P_1 \subseteq \langle C' \cap R \rangle$, we have $P_1 \subseteq r_{i+1} - v_{i+2}$. Choose the labelling of P_2 and P_4 so the r_{i+1}-end of P_2 is nearer v_{i+2} in r_{i+1} than the r_{i+1}-end of P_4 is.

If P_2 is not disjoint from $\langle s_{i+2} \rangle$, then the edge of r_{i+1} incident with v_{i+2} is in two H-green cycles, contradicting Theorem 6.7. Therefore, $C \cup C'$ is disjoint from $\langle s_{i+2} \rangle$. But then Lemma 12.4 implies e_6 is not crossed in D and, therefore, e_6 is not H-yellow. Likewise, e_2 is not H-yellow.

Therefore, e_6 is H-green, so Lemma 12.5 implies e_6 is spanned by some 2.5-jump J_6 and, moreover, is not in either H-rim branch fully contained in the span of J_6. By Theorem 6.7, no H-rim edge is in two H-green cycles. Thus, the only possibility for the 2.5-jump J_6 spanning e_6 is $v_{i+4}w_6$, with $w_6 \in \langle r_{i+6} \rangle$. An analogous argument applies to e_2, so e_2 is spanned by the 2.5-jump J_2 w_2v_{i+5}, with $w_2 \in \langle r_{i+2} \rangle$. But now we have that every edge of r_{i+4} is in the distinct H-green cycles containing J_2 and J_6, contradicting Theorem 6.7, completing the proof in Case 1.

Case 2: *There is a 2.5-jump $v_{i-4}w'$, with $w' \in r_{i-2}$.*

Let D_1 be a 1-drawing of $G - \langle s_{i+1} \rangle$. Corollary 12.7 and Lemma 5.9 imply $\overline{Q}_{i+1}$ is crossed in D. Lemma 7.2 (1) shows none of $[w, r_{i-1}v_i]$, r_i, r_{i+1}, and r_{i+6} is crossed in D, while (2) of the same lemma shows r_{i+2} is not crossed. It follows that some edge $e_5 \in r_{i+5}$ crosses an edge $e_9 \in [v_{i+9}, r_{i+9}, w]$.

Since e_9 is not red, Theorem 11.3 shows it is either H-yellow or H-green. If e_9 is H-yellow as witnessed by the H-yellow cycle C and the H-green cycle C', then the global H-bridge J in C' is a 2- or 2.5-jump (Theorems 10.6 and 12.6) and $C \subseteq \mathrm{cl}(Q_{-1})$ (Lemma 11.2 (4)). Lemma 12.4 implies that e_9 is not crossed in D, a contradiction.

Likewise, if e_9 is H-green, the Lemma 12.5 shows it is not crossed in D, the final contradiction completing the proof in Case 2.

Case 3: *All the remaining cases.*

Let e_i be the edge of s_i incident with v_i and let D_i be a 1-drawing of $G - e_i$. Corollary 12.7 and Lemma 5.9 imply $\overline{Q}_i$ is crossed in D_i.

Since G (in particular, r_{i-2}) has no red edge, Lemma 12.4 shows any H-yellow edge in r_{i-2} is not crossed in D_i, while Lemma 12.5 implies that, as we are not in Case 2, no H-green edge of r_{i-2} is crossed in D_i. Lemma 7.2 (1) implies no edge of $[w, r_{i-1}, v_i] r_i \, r_{i+1}$ is crossed in D_i. Therefore, it must be that some edge e_{i-1} of $[v_{i-1}, r_{i-1}, w]$ is crossed in D_i.

As e_{i-1} is not red in G, Theorem 11.3 implies e_{i-1} is either H-green or H-yellow. If it is H-green, then, because we are not in Case 1, Lemma 12.5 implies e_{i-1} is in an H-green cycle C contained in $\mathrm{cl}(Q_{i-1})$ and $e_i \in C$. But then every edge in $[w, r_{i-1}, v_i]$ is in two H-green cycles, contradicting Theorem 6.7.

We conclude that e_{i-1} is H-yellow. Let C and C' be the witnessing H-yellow and H-green cycles, respectively, and let B be the global H-bridge contained in C'. Lemma 12.4 implies $e_i \in C$. Moreover, v_{i+5} is in the span of B, as otherwise B attaches at v_{i+2}, contradicting Lemma 10.9 (1). By Lemma 10.9 (4), v_{i+7} is not in the span of B.

If B has an end in $\langle r_{i+2}\rangle$, then the other end of B is v_{i+5}. The R-avoiding path (one of P_2 and P_4 in the decomposition of the H-yellow cycle as in Definition 11.1) in C containing e_i contains a positive-length H-avoiding subpath joining a vertex of $\langle s_i\rangle$ to a vertex of $[v_{i+4}, r_{i+5}, v_{i+5}\rangle$. This yields the contradiction that the edge of r_{i+4} incident with v_{i+5} is in two H-green cycles. Therefore, B has one attachment in $[r_{i+5}\, r_{i+6}\rangle$ and one attachment in r_{i+3}.

Let D_{i+1} be a 1-drawing of $G - \langle s_{i+1}\rangle$. Lemma 12.4 implies no H-yellow edge in either r_{i-1} or r_{i+2} is crossed in D_{i+1}. An H-green edge of r_{i+2} is not spanned by a global H-bridge (there is no room for such a jump between B and wv_{i+2}), so Lemma 12.5 implies no H-green edge of r_{i+2} is crossed in D_{i+1}. Because we are not in Case 1 and there is no 3-jump, Lemma 12.5 implies no H-green edge of either r_{i-1} or r_{i+2} is crossed in D_{i+1}.

Lemma 7.2 (1) implies no edge of $r_i\, r_{i+1}$ is crossed in D_{i+1}. Thus, none of $r_{i-1}\, r_i\, r_{i+1}\, r_{i+2}$ is crossed in D_{i+1}, and therefore $\overline{Q}_{i+1}$ cannot be crossed in D_{i+1}. However, Corollary 12.7 and Lemma 5.9 imply that $\overline{Q}_{i+1}$ is crossed in D_{i+1}. This contradiction completes the proof that G has a red edge when there is a 2.5-jump. □

At this point, we may assume G has no 2.5-jump and no 3-jump.

Claim 2. If G has a 2-jump v_iv_{i+2}, then either r_{i-1} or r_{i+2} has a red edge.

Proof. In this case, let D_{i+1} be a 1-drawing of $G - \langle s_{i+1}\rangle$. Corollary 12.7 and Lemma 5.9 imply that $\overline{Q}_{i+1}$ is crossed in D_{i+1}. Lemma 7.2 (1) shows no edge of $r_i\, r_{i+1}$ is crossed in D_{i+1}. Therefore, some edge of $r_{i-1} \cup r_{i+2}$ must be crossed in D_{i+1}. Lemmas 12.4 and 12.5 imply that no H-yellow or H-green edge in $r_{i-1} \cup r_{i+2}$ is crossed in D_{i+1}. Therefore, Theorem 11.3 shows some edge in $r_{i-1} \cup r_{i+2}$ is red. □

In the final case, there are no global H-bridges. Therefore, there are no H-yellow cycles and every H-green cycle is contained in $\mathrm{cl}(Q_i)$, for some i. For $j \in \{0,1,2,3,4\}$, let e_j be the edge in s_j incident with v_j and let D_j be a 1-drawing of $G - e_j$. Corollary 12.7 and Lemma 5.9 imply that $\overline{Q}_j$ is crossed in D_j, so some edge in $r_{j+3}\, r_{j+4}\, r_{j+5}\, r_{j+6}$ is crossed in D_j. Since e_j cannot be in any H-green cycle containing an edge in $r_{j+3}\, r_{j+4}\, r_{j+5}\, r_{j+6}$, Lemma 12.5 implies no H-green edge in $r_{j+3}\, r_{j+4}\, r_{j+5}\, r_{j+6}$ can be crossed in D_j. Therefore the edge in $r_{j+3}\, r_{j+4}\, r_{j+5}\, r_{j+6}$ crossed in D_j is red in G. ■

We conclude this section with the technical lemma (12.14) below that will be used in the next section. We start with four lemmas leading to a more refined understanding of R-separation in cases of interest for us. The first three are primarily used in the proof of the fourth. (Recall that an RR-path is an R-avoiding path with both ends in R.)

Lemma 12.9. *Let $G \in \mathcal{M}_2^3$ and let $V_{10} \cong H \subseteq G$, with H tidy, witnessed by the embedding Π. Let P be an RR-path in G. If B is a global H-bridge so that one end of P is in the interior of the span of B, then there is an H-quad Q so that $P \subseteq \mathrm{cl}(Q)$ and the two cycles in $R \cup P$ containing P are non-contractible in $\mathbb{R}P^2$.*

Proof. As P is R-avoiding, Theorem 6.7 implies P is not contained in $\mathfrak{D}$. If P is just an H-spoke, then both conclusions are obvious. Otherwise, as we traverse P from an end u in the interior of the span of B, there is a first edge e that is not in

H. Since $P \subseteq \mathfrak{M}$, there is an H-quad Q so that $e \in \mathrm{cl}(Q)$. Let P' be the H-bridge in $H \cup P$ containing e. Then P' is an H-avoiding path with both ends in H, so $P' \subseteq \mathrm{cl}(Q)$.

Since P is R-avoiding, if both ends of P' are in R, then $P' = P$ and $P \subseteq \mathrm{cl}(Q)$, as claimed. Otherwise, one end w of P' is in the interior of some H-spoke s_i. Our two claims eliminate many possibilities for the other end x of P'. We choose the labelling so that $u \in r_{i-1}\, r_i$.

Claim 1. x is not in $\langle s_{i-1}\, r_{i-1}\, r_i\, s_{i+1}\rangle$.

Proof. Suppose first that u is an end of P'. The choice of e implies $P' = [u, P, w]$ is just the edge e. If u is an end of s_i, then e is an H-bridge having all its attachments in s_i, contradicting Lemma 10.10. If u is not an end of s_i, then there is an H-green cycle that contains an edge f of R incident with u. But then f is in two H-green cycles, contradicting Theorem 6.7. Thus, u is not an end of P'.

If $x \in \langle s_{i-1}\, r_{i-1}\, r_i\, s_{i+1}\rangle - u$, then $u = v_i$ and $P' = [w, P, x]$ is contained in either $\mathrm{cl}(Q_{i-1}) - r_{i+4}$ or $\mathrm{cl}(Q_i) - r_{i+5}$. In this case, we again have the contradiction that some edge of R incident with u is in two H-green cycles. □

Claim 1 implies $u = v_i$ and $[u, P, w] \subseteq s_i$. Moreover, x is in $Q_{i-1} \cup Q_i$ and either $P' \subseteq \mathrm{cl}(Q_{i-1})$ or $P' \subseteq \mathrm{cl}(Q_i)$. The next claim eliminates another possibility for x.

Claim 2. $x \notin \langle s_i \rangle$.

Proof. Suppose by way of contradiction that $x \in \langle s_i \rangle$. Let B' be the H-bridge containing P'. Observe that B' is H-local and that w and x are both attachments of B' in $\langle s_i \rangle$. Corollary 5.15 implies that these are the only attachments of B', contradicting Lemma 10.10. □

We conclude from Claims 1 and 2 that x is in $r_{i+4}\, r_{i+5}$. Evidently, P is in $\mathrm{cl}(Q_{i-1})$ or $\mathrm{cl}(Q_i)$, respectively, as required for the first conclusion. Furthermore, both cycles in $\Pi[R \cup P]$ that contain P are non-contractible in $\mathbb{R}P^2$. ■

Lemma 12.10. *Let $G \in \mathcal{M}_2^3$ and let $V_{10} \cong H \subseteq G$, with H tidy, witnessed by the embedding Π. For $i \in \{0,1,2,3,4\}$ and $j \in \{i+3, i+4, i+5\}$, let $e \in r_i$ and $f \in r_j$ be edges that are not H-green. Suppose P is an RR-path in $\mathfrak{M}$ having both ends in the component R' of $R - \{e, f\}$ containing $r_{i+6}\, r_{i+7}\, r_{i+8}\, r_{i+9}$ and so that the cycle in $\Pi[R' \cup P]$ is non-contractible. Then*

$$P \subseteq \Big(\mathrm{cl}(Q_j) - [v_j, s_j, v_{j-5}\rangle\Big) \cup \Big(\bigcup_{j-5<k<i} \mathrm{cl}(Q_k)\Big) \cup \Big(\mathrm{cl}(Q_i) - \langle v_{i+6}, s_{i+1}, v_{i+1}]\Big).$$

Proof. Choose the labelling u and w of the ends of P so that u is nearer in R' to the end incident with f than w is.

Let γ be a non-contractible curve meeting $\Pi[G]$ in just the two points a and b; we note that u and w are on different ab-subpaths of R (allowing a or b to be an end of P). We may choose the labelling of a and b so that $a \in R'$, and if both a and b are in R', then a is closer to the end of R' incident with f than b is.

Claim 1. (1) If v_j and w are on the same ab-subpath of R, then $P \cap \langle s_j \rangle$ is empty.

(2) If v_{i+1} and u are in the same ab-subpath of R, then $P \cap \langle s_{i+1} \rangle$ is empty.

PROOF. The statements are symmetric, so it suffices to prove the first. Suppose to the contrary that $P \cap \langle s_j \rangle$ is not empty. As we traverse P from w (which is not incident with s_j), let x be the first vertex in $\langle s_j \rangle$ and let P' denote the wx-subpath of P. Evidently, P' is contained in one component $\mathfrak{M}'$ of $\mathfrak{M} \setminus (\gamma \cup s_j)$. On the other hand, f is between v_j and w, and so f is in $\mathfrak{M}'$. If $w \in r_j$, then P' and f are in an H-green cycle, a contradiction.

Otherwise, $v_{j+1} \in \mathfrak{M}'$ and P' intersects $\langle s_{j+1} \rangle$. In this case, some $\langle s_{j+1} \rangle \langle s_j \rangle$-subpath of P' is in an H-green cycle with f, also a contradiction. $\square$

If both v_{i+1} and w are in the same ab-subpath of R and both v_{i+4} and u are in the same ab-subpath of R, then Claim 1 implies P is trapped between s_j and s_{i+1}, as required. By symmetry, we may assume that v_{i+1} is not in the same ab-subpath of R as w. Let R_w denote the ab-subpath of R containing w and let R_{i+1} denote the other ab-subpath of R, so $v_{i+1} \in R_{i+1}$.

This implies that $v_{i+1}, v_{i+2}, \ldots, v_j$ are all in R_{i+1}. We noted above that $u \notin R_w$, so u is also in R_{i+1}. From Claim 1 (1), we conclude that P is disjoint from $\langle s_j \rangle$. Thus, P is contained in the component of $\mathfrak{M} - s_j$ disjoint from v_{i+2}.

It follows from the fact that all the H-spokes are in $\mathfrak{M}$ that v_{j-5} is on the same ab-subpath as w. This combines with the fact that v_{i+1} is not in that ab-subpath and the fact that P meets γ at most in a to tell us that

$$P \subseteq \Big(\mathrm{cl}(Q_j) - \langle s_{i-1} \rangle \Big) \cup \left(\bigcup_{j-5<k<i} \mathrm{cl}(Q_k) \right) \cup \Big(\mathrm{cl}(Q_i) - \langle s_{i+1} \rangle \Big),$$

as required.

The additional fact that P cannot include v_j and v_{i+1} follows from the knowledge that these vertices are not in R'. ■

In a similar vein, we have the following.

LEMMA 12.11. *Let $G \in \mathcal{M}_2^3$ and let $V_{10} \cong H \subseteq G$, with H tidy as witnessed by the embedding Π. Suppose $e \in r_i$, $f \in r_{i+3}\, r_{i+4}$ and P is an RR-path with both ends in the component of $R - \{e, f\}$ containing $r_{i+1}\, r_{i+2}$. If e is not H-green, then both cycles in $\Pi[R \cup P]$ containing P are contractible.*

Proof. Let R' be the component of $R - \{e, f\}$ containing $r_{i+1}\, r_{i+2}$ and let C be the cycle in $R \cup P$ that contains P and is contained in $R' \cup P$. Since R is contractible, the other cycle in $R \cup P$ containing P is homotopic to C; thus, it suffices to show C is contractible.

Let u_e be the end of R' incident with e. Suppose there is a $([u_e, r_i, v_{i+1}]\, s_{i+1})s_i$-path P' in P contained in $\mathrm{cl}(Q_i)$. Since C is disjoint from r_{i+1}, P' is contained in an H-green cycle containing e, a contradiction.

Thus, there is no $([u_e, r_i, v_{i+1}]\, s_{i+1})s_i$-path in P contained in $\mathrm{cl}(Q_i)$. Since C is disjoint from r_{i+5}, there is an arc in the disc bounded by $\Pi[Q_i]$ joining a point of $[v_i, r_i, u_e\rangle$ to r_{i+5} that is disjoint from C; this shows that C is contractible, as required. ■

Our next lemma takes us one step closer to the useful description of R-separation.

Lemma 12.12. *Let $G \in \mathcal{M}_2^3$ and let $V_{10} \cong H \subseteq G$, with H tidy. Suppose $e \in r_i$ and $f \in r_{i+3}\, r_{i+4}$ are R-separated as witnessed by the subdivision H' of V_8. If e is not H-green, then the component of $R - \{e, f\}$ containing both ends of some H'-spoke is the one containing $r_{i+5}\, r_{i+6}\, r_{i+7}\, r_{i+8}\, r_{i+9}$.*

Proof. Let Π be an embedding of G in $\mathbb{R}P^2$ so that H is Π-tidy. Recall that R is also the H'-rim. Observation 11.6 (2) shows that two of the four H' spokes have all their ends in the same component of $R - \{e, f\}$. Of the four H'-spokes, at most one can be in $\mathfrak{D}$. Thus, of the two that have both ends in the same component R' of $R - \{e, f\}$, there is at least one, call it s, that is in $\mathfrak{M}$.

In particular, the two cycles in $R \cup s$ containing s are non-contractible. Now Lemma 12.11 shows the two ends of the RR-path s are not in the component of $R - \{e, f\}$ containing $r_{i+1}\, r_{i+2}$ and so must be in the component containing $r_{i+5}\, r_{i+6}\, \dots\, r_{i+9}$, as claimed. ■

Our next lemma in the series gives a quite refined description of R-separation.

Lemma 12.13. *Let $G \in \mathcal{M}_2^3$ and let $V_{10} \cong H \subseteq G$, with H tidy. Let $e \in r_i$ and $f \in r_{i+4}\, r_{i+5}$ be edges that are both not H-green. If e and f are R-separated in G, then there is a witnessing subdivision H' of V_8 having s_{i+2} and s_{i+3} as H'-spokes and the other two H'-spokes are in* $\mathrm{cl}(Q_{i-1}) \cup \mathrm{cl}(Q_i)$.

Proof. Let Π be an embedding of G in $\mathbb{R}P^2$ for which H is Π-tidy. Let H_1 be a subdivision of V_8 witnessing the R-separation of e and f. Let s be an H_1 spoke having both ends in the same component R' of $R - \{e, f\}$.

Claim 1. The cycles in $\Pi[R \cup s]$ containing s are non-contractible.

Proof. Suppose first by way of contradiction that $\Pi[s]$ in not contained in $\mathfrak{M}$. Since H is Π-tidy, s is a global H-bridge. Theorems 10.6 and 12.6 show s is either a 2- or a 2.5-jump. By hypothesis, it is not possible for both e and f to be in the span of s and, therefore, neither is. On the other hand, each of the other three H_1-spokes has precisely 1 end in the span of s, and is contained in $\mathfrak{M}$. Let these spokes appear in the order t_1, t_2, t_3 in the span of s.

We claim that the t_i imply the existence of an H-yellow cycle that does not bound a face of $\Pi[G]$, contradicting Lemma 11.2 (3). Let P be the span of s and, for $i = 1, 2, 3$, let u_i be the end of t_i that is not in P. Because $\Pi[s \cup P]$ bounds a closed disc, both cycles in $\Pi[R \cup t_i]$ containing t_i are non-contractible. Thus, t_i has an end in each of the ab-subpaths of R.

Lemma 12.9 implies that each t_i is contained in an H-quad. Thus $t_1 \cup t_2 \cup t_3$ is contained the the union of the closures of the H-quads that have an edge in P. In particular, u_1, u_2, and u_3 occur in a 3-rim path P_1 having u_1 and u_3 as ends. Letting P_3 be the minimal subpath of P containing the ends of the t_i, we see that $P_1\, t_1\, P_3\, t_3$ is an H-yellow cycle C. However, $\Pi[C]$ bounds a face of $\Pi[G]$; the contradiction is that t_2 and s are on different sides of $\Pi[C]$.

Thus, s is contained in $\mathfrak{M}$. Since s is one of four H_1-spokes, the two cycles in $\Pi[R \cup s]$ that contain s are non-contractible. □

In particular, s has an end in each of the two ab-subpaths of R determined by the standard labelling of $\Pi[G]$.

In the case $f \in r_{i+5}$, we may, if necessary, use the reflective symmetry $j \leftrightarrow 4-j$ (for $0 \le j \le 4$), to arrange that the end s_f of s is, in $\Pi[R']$, between the end u_f

of f in R' and a, say, while the other end s_e of s is between a and the end u_e of e. In particular, v_{i+1}, v_{i+2}, v_{i+3}, and v_{i+4} are not in R'. Lemma 12.12 shows this always holds when $f \in r_{i+4}$.

Let s' be the other H'-spoke having both ends in R'. The arguments above for s apply equally well to s'. Lemma 12.10 shows that $(s \cup s') \subseteq \mathrm{cl}(Q_{i-1}) \cup \mathrm{cl}(Q_i)$. In particular, s and s' are disjoint from s_{i+2} and s_{i+3}, so these H-spokes may replace the two H_1-spokes having ends in both components of $R - \{e, f\}$, as required. ∎

The final technical lemma of this section will be used in the next.

Lemma 12.14. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. If e and e' are red edges in the same H-rim branch, then Δ_e and $\Delta_{e'}$ are disjoint.*

Proof. We may choose the labelling of e and e' so that $e = uw$ and $e' = xy$ are such that $r_i = [v_i, r_i, u, w, r_i, x, y, r_i, v_{i+1}]$. As we follow $\Delta_e - e$ from w, there is a first edge f that is not in R. In fact, Theorem 12.1 (3) implies f is incident with w, as there can be no global H-bridge spanning e'.

Observe that f is not in H, so $H \subseteq G - f$. Moreover, if f is in an H-yellow cycle, then either e or e' is H-yellow, a contradiction. Thus, Lemmas 12.4 and 12.5 imply the colours of an edge of R are the same in G and $G - f$, unless the edge is in an H-green cycle in G that contains f. Such an edge is necessarily in $[w, r_i, x]$.

Let D be a 1-drawing of $G - f$ and let e_1 and e_2 be the edges of $G - f$ crossed in D. Since f is incident with $w \in \langle r_i \rangle$, Theorem 5.23 and Lemma 5.9 imply that Q_i is crossed in D, so we may assume $e_1 \in r_{i-1}\, r_i\, r_{i+1}$ and $e_2 \in r_{i+4}\, r_{i+5}\, r_{i+6}$. Moreover, no H-green cycle containing e_2 contains f, so e_2 is red in G. In particular, Lemma 11.7 implies e_2 is R-separated from both e and e'.

Let u^e and w^e be the first vertices in r_{i+5} as we traverse $\Delta_e - e$ from u and w, respectively. Likewise, we have $x^{e'}$ and $y^{e'}$ in $r_{i+5} \cap (\Delta_{e'} - e')$.

Claim 1. $e_2 \in \left[u^e, r_{i+5}, y^{e'}\right]$.

Proof. Suppose by way of contradiction that $e_2 \in r_{i+4}\, [v_{i+5}, r_{i+5}, u^e]$; a similar argument will treat the case $e_2 \in \left[y^{e'}, r_{i+5}, v_{i+6}\right] r_{i+6}$.

If $e_1 \in r_{i-1}\, [v_i, r_i, u]$, then e_1 is red in G, so e_1 and e_2 are R-separated in G. Note that either $e_1 \in r_i$ or $e_2 \in r_{i+5}$. Lemma 12.13 implies there is a witnessing subdivision H' of V_8 that contains s_{i+2} and s_{i+3}, while the other two spokes are in $\mathrm{cl}(Q_{i-1}) \cup \mathrm{cl}(Q_i)$. Furthermore, Δ_e shows that $f \notin H'$; therefore, $H' \subseteq G - f$ shows that e_1 and e_2 are R-separated in $G - f$, and therefore cannot cross in D, a contradiction.

The other possibility is that $e_1 \in [u, r_i, v_{i+1}]\, r_{i+1}$. Since e and e_2 are both red in G, Lemma 11.7 implies e_2 is R-separated from e in $G - f$. As in the preceding paragraph, we may choose the witnessing subdivision H' of V_8 to contain s_{i+2} and s_{i+3}, while the other two spokes are in $\mathrm{cl}(Q_{i-1}) \cup (\mathrm{cl}(Q_i) - f)$. Again H' witnesses the R-separation of e_1 and e_2 in $G - f$, a contradiction. □

Theorem 12.1 (2) shows that any edge in either $\Delta_e \cap r_{i+5}$ or $\Delta_{e'} \cap r_{i+5}$ is in a digon in G and so is not e_2. Thus, e_2 is further restricted to be in $\left[w^e, r_{i+5}, x^{e'}\right]$. Lemma 11.7 implies Δ_e and Δ_{e_2} are disjoint, as are Δ_{e_2} and $\Delta_{e'}$, which further implies that Δ_e and $\Delta_{e'}$ are disjoint, as required. ∎

CHAPTER 13

The next red edge and the tile structure

We now know that there are red edges and every red edge comes equipped with a Δ. The tiles are determined by what is between "consecutive" red edges. In this section, we explain what "consecutive" means, show that consecutive red edges determine one of the tiles, and complete the proof of our main result, Theorem 2.14, by demonstrating that every red edge has a consecutive red edge on each side.

DEFINITION 13.1. Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. Let $e = uw$ be a red edge in r_i, labelled so that $r_i = [v_i, r_i, u, e, w, r_i, v_{i+1}]$. A red edge e_w is *w-consecutive for e* if:

(1) $e_w \in [w^e, r_{i+5}, v_{i+6}] r_{i+6}\, r_{i+7}$ (recall that w^e is the vertex in the peak of Δ_e nearest w in $\Delta_e - e$);
(2) there is no red edge in $[w^e, r_{i+5}, v_{i+6}] r_{i+6}\, r_{i+7}$ between w^e and e_w;
(3) there is no red edge in $[w, r_i, v_{i+1}] r_{i+1}\, r_{i+2}$ between w and the peak of Δ_{e_w};
(4) if e^w is the edge of P_w nearest w that is not in R, then there is a 1-drawing D of $G - e^w$ in which e crosses e_w.
(5) There is an analogous definition for *u-consecutive*.

Our first main goal is, therefore, the following.

THEOREM 13.2. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. Let $e = uw$ be red in G. Then there is a w-consecutive red edge and a u-consecutive red edge for e.*

The next lemma will be helpful in the proof.

LEMMA 13.3. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. Let $e = uw$ and $\hat{e}$ be red edges in G, with $e \in r_i$ and the labelling chosen so that $r_i = [v_i, r_i, u, e, w, r_i, v_{i+1}]$ and $\hat{e} \in [w^e, r_{i+5}, v_{i+6}] r_{i+6}\, r_{i+7}$. If e^w is the w-nearest edge of P_w that is not in R and e and $\hat{e}$ are not R-separated in $G - e^w$, then e has a w-consecutive red edge.*

Proof. Suppose there is a red edge e' in $r_i\, r_{i+1}\, r_{i+2}$ between w and the peak of $\Delta_{\hat{e}}$. Then e' is R-separated from $\hat{e}$ in both G and $G - e^w$, showing that e and $\hat{e}$ are R-separated in $G - e^w$, a contradiction. Thus, no such red edge exists.

Let $\hat{e}'$ be the w^e-nearest red edge in $[w^e, r_{i+5}, v_{i+6}] r_{i+6}\, r_{i+7}$. Lemma 11.7 implies $\hat{e}'$ is R-separated from e in G; if $\hat{e}'$ were also R-separated from e in $G - e^w$, then so would $\hat{e}$, which contradicts the hypothesis. But now Lemma 11.7 implies there is a 1-drawing of $G - e^w$ in which e crosses $\hat{e}'$, as required. ■

And now the final major proof needed to prove Theorem 2.14.

Proof of Theorem 13.2. It obviously suffices to prove the existence of a w-consecutive red edge for e. Let r_i be the H-rimbranch containing e. Let e^w be the edge of P_w nearest w and not in R. There are two principal cases.

Case 1: *e^w is incident with w.*

We note that e^w is contained in a $\overline{Q}_{i+1}$-bridge that is not $M_{\overline{Q}_{i+1}}$. Let D be a 1-drawing of $G - e^w$. Corollary 12.7 and Lemma 5.9 show that $\overline{Q}_{i+1}$ is crossed in D.

Let

- f be the edge of $r_{i+4}\, r_{i+5}\, r_{i+6}\, r_{i+7}$ that is crossed in D and
- f' be the other edge crossed in D; thus, $f' \in r_{i-1}\, r_i\, r_{i+1}\, r_{i+2}$.

CLAIM 1. If f is not red in G, then there is a w-consecutive red edge for e.

PROOF. Because we are in Case 1, no global H-bridge has w in the interior of its span and, therefore, e^w is not in any H-yellow cycle that could witness the H-yellowness of any edge in $r_{i+4}\, r_{i+5}\, r_{i+6}\, r_{i+7}$, (in particular, the H-yellowness of f). Therefore, Lemma 12.4 shows f is not H-yellow. Since f is not red, Theorem 11.3 implies f is H-green. Lemma 12.5 implies there is a 2.5-jump J that spans f and so that f is in the H-rim branch whose interior contains an end of J. We note that if v_{i+6} is in the span of J, then Lemma 7.2 (1) shows no edge in the span of J is crossed in D. Therefore, v_{i+6} is not in the span of J. Furthermore, if e^w is not in s_{i+1}, then $H \subseteq G - e^w$ and, therefore Lemma 6.6 (10) implies f is not crossed in D, a contradiction. This implies $w = v_{i+1}$. We summarize these remarks as follows.

SUBCLAIM 1.
- $w = v_{i+1}$ and
- there is a 2.5-jump J so that:
  - f is spanned by J;
  - f is in the H-rim branch whose interior contains an end of J; and
  - v_{i+6} is not in the span of J. □

SUBCLAIM 2. Let $j \in \{i+4, i+5, i+6, i+7\}$ so that f is in the H-rim branch r_j. Then no edge of r_j is H-yellow.

PROOF. Suppose some edge e' of r_j is H-yellow. This implies e' is not H-green and, therefore, is not spanned by J. Let C and C' be the witnessing H-yellow and H-green cycles, respectively.

Suppose first that $j \in \{i+4, i+5\}$. Then $r_j = [v_j, r_j, f, r_j, e', r_j, v_{j+1}]$. Because $e \in r_i$ is not H-green, $v_{j+5} \in \{v_{i-1}, v_i\}$ is in the interior of $C' \cap R$. This implies there is an H-yellow cycle containing s_j and the portion of r_j from v_j to e'. By Lemma 11.2 (3), this H-yellow cycle must be C and, therefore, $f \in C$. Now the fact that f is crossed in D contradicts Lemma 12.4. A completely analogous argument holds for $j \in \{i+6, i+7\}$. □

Let $\widehat{w}$ be the vertex in r_{i+5} that is nearest w in P_w. Observe that $\widehat{w}$ is not necessarily in the peak of Δ_e. (See Figure 12.1, where $\widehat{w}$ is the vertex of Δ_e at the top right hand corner of Δ_e.) The following claim will be helpful in completing the proof of Case 1.

SUBCLAIM 3. If $\widehat{w} \neq v_{i+6}$, then $[\widehat{w}, r_{i+5}, v_{i+6}]$ is in an H-green cycle contained in $\mathrm{cl}(Q_i)$.

PROOF. Let P'_w be the $\widehat{w}s_{i+1}$-subpath of P_w. Since $e^w \in s_{i+1}$, $P'_w \subseteq P_w - w$. Let $\widehat{w}^e$ be the end of P'_w in s_{i+1}. Since $\widehat{w} \notin s_{i+1}$ and $\widehat{w}^e \in s_{i+1}$, $\widehat{w} \neq \widehat{w}^e$. By definition of $\widehat{w}$, $P'_w - \widehat{w}$ is disjoint from r_{i+1}. Therefore, $P'_w[\widehat{w}^e, s_{i+1}, v_{i+6}, r_{i+5}, \widehat{w}]$ is an H-green cycle containing $[\widehat{w}, r_{i+5}, v_{i+6}]$, as required. $\square$

The proof of Claim 1 is completed now by treating separately each of the four possibilities for f: $f \in r_{i+4}$, $f \in r_{i+5}$, $f \in r_{i+6}$, and $f \in r_{i+7}$.

Subcase 1: $f \in r_{i+4}$.

In this case, J has an end $x' \in \langle r_{i+4}\rangle$ and the other end of J is v_{i+2}. Lemma 7.2 (3b) implies $f' \in r_i\, r_{i+1}$. We claim that if $f' \in r_{i+1}$, then there is another 1-drawing of $G - e^w$ in which f crosses e.

Since $f \in r_{i+4}$ and $f' \in r_{i+1}$, we see that s_i is exposed in the 1-drawing D of $G - e^w$. Note that $D[Q_{i-1}]$ consists of a simple closed curve crossed by $D[f']$, with $D[r_i]$ on one side (the *inside* of $D[Q_{i-1}]$) and most of $D[H]$ on the other side (this is the *outside* of $D[Q_{i-1}]$).

We claim that we may reroute f inside $D[Q_{i-1}]$ so that it crosses e instead of f'. If this fails, then there is an $(H - \langle s_{i+1}\rangle)$-avoiding path P having one end in the component of $r_{i+1} - f'$ that contains v_{i+1}, and having its other end in $Q_{i-1} \cup [v_i, r_i, u]$.

We note that $D[s_{i+1} - v_{i+1}]$ (which is possibly just v_{i+6}) is completely outside $D[Q_{i-1}]$. Therefore, P is H-avoiding. In $\mathbb{R}P^2$, we conclude that P cannot start inside Q_{i+1}. Thus, P is contained in a global H-bridge. Therefore, P is a global H-bridge; we note that P has one end in the component of $r_{i+1} - f$ containing v_{i+1}. No edge of r_{i+2} can be spanned by P, as such an edge is already spanned by J and therefore would contradict Theorem 6.7. In the other direction, P cannot span e, as e is red and not H-green. This contradiction shows that f may be redrawn as claimed. Consequently, we may assume $f' \in r_i$.

Observe that no global H-bridge can have an end y in $\langle r_i\rangle$, since yv_{i+3} shows e is H-green, a contradiction, and yv_{i-2} shows f is H-yellow and, therefore, by Lemma 12.4 cannot be crossed in D. It follows from this, using Lemmas 12.4 and 12.5 and Theorem 11.3, that f' is red in G.

Suppose first that some edge e' of $[x', r_{i+4}, v_{i+5}]$ is red in G. Then Δ_e and $\Delta_{e'}$ are R-separated in G as witnessed by a subdivision H' of V_8 consisting of R, s_{i-3}, s_{i-2}, and two RR-paths P_1 and P_2, contained in Δ_e and $\Delta_{e'}$, respectively. The paths P_1 and P_2 are disjoint from s_{i+1} except that, possibly P_1 contains v_{i+6}. Thus, H' and Lemma 7.2 show that f cannot be crossed in D, a contradiction. Therefore, there is no red edge in $[x', r_{i+4}, v_{i+5}]$.

Furthermore, no global H-bridge other than J has an end in $[x', r_{i+4}, v_{i+5}\rangle$, as otherwise either e is H-yellow, or f is in two H-green cycles, both contradictions, the latter of Theorem 6.7. We conclude that each edge of $[x', r_{i+4}, v_{i+5}]$ is either H-yellow or contained in an H-green cycle in $\mathrm{cl}(Q_{i-1})$. Subclaim 2 shows the following.

Subcase 1 Observation: *Each edge of* $[x', r_{i+4}, v_{i+5}]$ *is in an* H*-green cycle contained in* $\mathrm{cl}(Q_{i-1})$.

Suppose there is a red edge e' in r_{i+5}. By Lemma 11.7, e' is R-separated from e in G. Therefore, P_u is disjoint from s_i and now we see that $G - e^w$ contains the subdivision H' of V_{10} consisting of $(H - \langle s_{i+1}\rangle) \cup P_u$. But J is in an H'-green cycle C and so, by Lemma 6.6 (10), C, and in particular, f, is not crossed in D, a contradiction.

Thus, no edge of r_{i+5} is red in G. We consider next a 1-drawing D_{i-1} of $G - \langle s_{i-1}\rangle$. By Corollary 12.7 and Lemma 5.9, $\overline{Q}_{i-1}$ is crossed in D_{i-1}. From Lemmas 12.4, 12.5, and 7.2 (1), no edge in $r_{i+2}\,r_{i+3}\,r_{i+4}$ is crossed in D_{i-1}. Therefore, it is some edge f'' in r_{i+5} that is crossed in D_{i-1}. Since no edge of r_{i+5} is red in G, Lemmas 12.4 and 12.5 imply that f'' is spanned by a 2.5-jump $J'' = x''v_{i-2}$, with $x'' \in \langle r_{i+5}\rangle$.

Now consider a 1-drawing D_{i+3} of $G - \langle s_{i+3}\rangle$. As for $\overline{Q}_{i-1}$ in the preceding paragraph, $\overline{Q}_{i+3}$ is crossed in D_{i+3}. In this case, r_{i+1} is contained in the H-yellow cycle Q_{i+1} (with witnessing H-green cycle containing J''). Therefore, r_{i+1} is not crossed in D_{i+3}. Lemma 7.2 (1) implies no edge in the span of J is crossed in D_{i+3}. Subcase 1 Observation combines with Lemma 12.5 to show that no edge in $[x', r_{i+4}, v_{i+5}]$ is crossed in D_{i+3}. But now we see that $\overline{Q}_{i+3}$ cannot be crossed in D_{i+3}, a contradiction that shows Subcase 1 cannot occur.

Subcase 2: $f \in r_{i+5}$.

In this case, J has an end $x' \in \langle r_{i+5}\rangle$. Subclaim 1 implies that v_{i+6} is not spanned by J, so the other end of J is v_{i+3}. Lemma 7.2 implies the edge f' (crossed by f in D) is in r_{i+2}.

We first show that there is no global H-bridge spanning any edge in $r_i\,r_{i+1}\,r_{i+2}$. For if J' is a global H-bridge that spans such an edge, then J' does not span e, while Lemma 10.9 (1) shows it cannot be the 2-jump $v_{i+1}v_{i+3}$. Theorem 6.7 shows J' cannot span any edge in r_{i+3}, so no edge of $r_{i+1}\,r_{i+2}$ is spanned by a global H-bridge. On the other side, J' would have to span $r_{i-2}\,r_{i-1}$. In that case, J and J' contradict Lemma 10.9 (4).

We also conclude that no edge of $r_{i+5}\,r_{i+6}\,r_{i+7}$ is H-yellow.

Our next principal aim is to show that each edge of $[x', r_{i+5}, v_{i+6}]$ is H-green, witnessed by a cycle in $\mathrm{cl}(Q_i)$. We have already seen that none of the edges in $[x', r_{i+5}, v_{i+6}]$ is H-yellow; to see they are H-green, it suffices by Theorem 11.3 to show none is red.

If e' is one of these edges that is red, then Lemma 11.7 implies it is R-separated from e. We note that Δ_e and $\Delta_{e'}$ are disjoint, both are in $\mathrm{cl}(Q_i)$, and $w = v_{i+1}$. Therefore, e' is in r_{i+5}, between x' and the peak of Δ_e. However, this shows e' and e are R-separated in $G - e^w$ and, therefore, f and r_{i+2} are R-separated in $G - e'$, showing that f cannot cross anything in D, a contradiction. Therefore, no edge of $[x', r_{i+5}, v_{i+6}]$ is red, and so they are all H-green.

We next show they are not spanned by a global H-bridge. Recall that $\widehat{w}$ is the vertex in r_{i+5} that is nearest w in P_w.

If $\widehat{w} \neq v_{i+6}$, then $(P_w - e^w) \cup (s_{i+1} - e^w) \cup [\widehat{w}, r_{i+5}, v_{i+6}]$ contains an H-green cycle that contains $[\widehat{w}, r_{i+5}, v_{i+6}]$ and is contained in $\mathrm{cl}(Q_i)$. Theorem 6.7 shows no edge of $[\widehat{w}, r_{i+5}, v_{i+6}]$ is spanned by a global H-bridge, so no edge of $[x', r_{i+5}, v_{i+6}]$ is H-green by a global H-bridge. In this case, every edge of $[x', r_{i+5}, v_{i+6}]$ is H-green by a local cycle.

So suppose $\widehat{w} = v_{i+6}$. By way of contradiction, we suppose there is a global H-bridge J'' spanning the edge of r_{i+5} incident with v_{i+6}. Then J'' must be $x''v_{i+8}$, for some $x'' \in [x', r_{i+5}, v_{i+6}]$. All edges in $[x', r_{i+5}, x'']$ are H-green by local cycles. For $j \in \{i+3, i+8\}$, let e_j be the edge of s_{i+3} incident with v_j and let D_j be a

1-drawing of $G - e_j$. Corollary 12.7 implies $\overline{Q}_{i+3}$ has BOD and Lemma 5.9 implies $\overline{Q}_{i+3}$ is crossed in D_j. Lemma 7.2 (3a) implies neither $r_{i+6}\, r_{i+7}$ nor $r_{i+3}\, r_{i+4}$ is crossed in D_j, while (2) of the same lemma implies neither r_{i+9} nor r_{i+1} is crossed in D_j. Therefore, r_{i+8} crosses r_{i+2}.

If the edge e'_{i+8} of r_{i+8} that is crossed in D_{i+3} is H-green because of some 2.5-jump, then Lemma 7.2 implies e'_{i+8} can cross only r_{i+1} in D_{i+3}. Therefore, Theorem 11.3 and Lemmas 12.4 and (because no H-green cycle containing e'_{i+8} can contain e_{i+3}) 12.5 imply e'_{i+8} is red in G. Likewise the edge e'_{i+2} of r_{i+2} that is crossed in D_{i+8} is red in G.

By Lemma 11.7, e'_{i+2} and e'_{i+8} are R-separated in G. Moreover, the nearer of the $(r_{i+7}\, r_{i+8})(r_{i+2}\, r_{i+3})$-paths P_2 in $\Delta_{e'_{i+2}}$ and P_8 in $\Delta_{e'_{i+8}}$, along with s_i and s_{i+1} witness their R-separation. We now show that P_8 is contained in $\mathrm{cl}(Q_{i+8})$ and must be disjoint from s_{i+4}.

If P_8 intersects s_{i+4} at a vertex other than v_{i+4}, then $P_8 \cup s_{i+4} \cup r_{i+8}$ contains an H-green cycle that includes e'_{i+8}. Otherwise, P_8 and s_{i+4} intersect just at v_{i+4}, in which case $P_8 \cup s_{i+4} \cup r_{i+8}$ contains a cycle C that includes e'_{i+8}. The H-green cycle containing J shows C is H-yellow. Both possibilities contradict the fact that e'_{i+8} is red.

Symmetrically, we use J'' to show that P_2 is disjoint from s_{i+2}. Thus, G contains a subdivison of V_{12} consisting of R, P_2, P_8, s_{i-1}, s_i, s_{i+1} and s_{i+2}. But then $G - e^w$ contains a subdivision of V_{10}, yielding the contradiction that f cannot be crossed in D. Therefore, there is no global H-bridge J'' spanning the edge of r_{i+5} incident with v_{i+6}.

We conclude that every edge of $[x', r_{i+5}, v_{i+6}]$ is in an H-green cycle contained in $\mathrm{cl}(Q_i)$.

We are now in a position to show that r_{i+6} has a red edge. By way of contradiction, we suppose r_{i+6} has no red edge. If there were a global H-bridge having an end in $\langle r_{i+6}\rangle$, then r_{i+2} is H-yellow; Lemma 12.4 shows r_{i+2} is not crossed in D, a contradiction. Thus, no global H-bridge has an end in $\langle r_{i+6}\rangle$.

Let D_i be a 1-drawing of $G - \langle s_i\rangle$. Then Corollary 12.7 and Lemma 5.9 imply $\overline{Q}_i$ is crossed in D_i. However, Lemma 7.2 shows none of $r_{i+3}\, r_{i+4}\, r_{i+5}\, r_{i+6}$ can be crossed in D_i, a contradiction.

Thus, r_{i+6} has a red edge e'. Then e is R-separated from e' in G. If e is R-separated from e' in $G - e^w$, then f is R-separated from r_{i+2} in $G - e^w$ and so f cannot be crossed in D, a contradiction. Therefore, e is not R-separated from e' in $G - e^w$, so Lemma 13.3 implies there is w-consecutive red edge for e, completing the proof in Subcase 2.

Subcase 3: $f \in r_{i+6}$.

In this case, J has an end $x' \in \langle r_{i+6}\rangle$ and the other end is v_{i+9}. Also, Lemma 7.2 implies f' (crossed by f in D) is in r_{i+9}.

Suppose by way of contradiction that no edge of r_{i+6} is red in G. We show that no edge of r_{i+6} is H-yellow. As every edge in $[x', r_{i+6}, v_{i+7}]$ is H-green (because of J), we assume by way of contradiction that there is an H-yellow edge in $[v_{i+6}, r_{i+6}, x']$. Let C and C' be the witnessing H-yellow and H-green cycles, respectively. Lemma 11.2 (1) implies there is a global H-bridge B contained in C', while (4) shows $C \subseteq \mathrm{cl}(Q_{i+1})$.

The edges of the span P_B of B are all H-green, so P_B does not contain the red edge e. One end of B is in $[w, r_i, v_{i+1}, r_{i+1}, v_{i+2}]$ and the other end is in r_{i+3}. Furthermore, Lemma 10.9 (4) and the presence of J shows v_{i+4} is not the other end of B.

Write $C = P_1P_2P_3P_4$ as in Definition 11.1 (H-yellow). Because C bounds a face $\Pi[G]$, $C \subseteq \mathrm{cl}(Q_{i+1})$, so that $P_1 = r_{i+1} \cap C$. In particular, $e^w \notin C$.

Choose the labelling of P_2 and P_4 so that the end of P_2 in r_{i+6} is nearer to v_{i+6} than is the corresponding end of P_4. Since there is an H-yellow cycle containing P_2 and s_{i+2}, Lemma 11.2 (3) shows this must be C. It follows that $P_4 = s_{i+2}$.

Consider the subdivision H' of V_6 whose rim consists of $(R - \langle P_B \rangle) - \langle x', r_{i+6}, v_{i+6} \rangle$, B, $C - \langle r_{i+6} \cap C \rangle$, and whose spokes are s_{i-1}, s_i, and s_{i+3}, $[v_{i+3}, r_{i+3}, z]$. Then H' does not contain e^w and so must contain the unique crossing in D. Since f is not in H', this is a contradiction, showing that no edge of $[v_{i+6}, r_{i+6}, x']$ is H-yellow.

Because of J, a global H-bridge spanning an edge in $[v_{i+6}, r_{i+6}, x']$ would have to be a 2.5-jump having v_{i+4} as an end. But then e is in an H-yellow cycle, which is impossible. Thus, for each edge $\bar{e}$ of $[v_{i+6}, r_{i+6}, x']$, $\bar{e}$ is in an H-green cycle $C_{\bar{e}}$ contained in $\mathrm{cl}(Q_{i+1})$. Theorem 6.7 implies $C_{\bar{e}}$ is disjoint from s_{i+2}.

Let D_{i+2} be a 1-drawing of $G - \langle s_{i+2} \rangle$. We know that $\overline{Q}_{i+2}$ is crossed in D_{i+2} (Corollary 12.7 and Lemma 5.9). Lemma 12.5 shows no edge in $[v_{i+6}, r_{i+6}, x']$ is crossed in D_{i+2}, while J and Lemma 7.2 show no edge in $[x', r_{i+6}, v_{i+7}]\, r_{i+7}\, r_{i+8}$ is crossed in D_{i+2}. Therefore, the crossing in D_{i+2} must be of an edge f'' in r_{i+5} crossing $r_{i+1}\, r_{i+2}$.

If f'' is red in G, then Lemma 11.7 implies f'' and e are R-separated in G. Since $e^w \in s_{i+1}$, f'' is between (in r_{i+5}) v_{i+5} and the peak of Δ_e. Thus, f'' and e are R-separated in $G - \langle s_{i+2} \rangle$ (using s_{i+3} and s_{i+4} as two of the four spokes). In turn, this implies f'' cannot cross $r_{i+1}\, r_{i+2}$ in D_{i+2}, a contradiction that shows f'' is not red. Therefore, Lemmas 12.4 and 12.5 imply f'' is spanned by a 2.5-jump $v_{i+3}x''$, with $x'' \in \langle r_{i+5} \rangle$.

Now let D_{i+3} be a 1-drawing of $G - \langle s_{i+3} \rangle$. We know that $\overline{Q}_{i+3}$ is crossed in D_{i+3}. However:

- Lemma 12.5 implies $[v_{i+6}, r_{i+6}, x']$ is not crossed in D_{i+3};
- Lemma 7.2 (1) implies $[x', r_{i+6}, v_{i+7}]\, r_{i+7}\, r_{i+8}$ is not crossed in D_{i+3}; and
- Lemma 12.4 implies r_{i+9} is not crossed in D_{i+3}.

These three observations imply the contradiction that $\overline{Q}_{i+3}$ cannot be crossed in D_{i+3}, showing that some edge e' in r_{i+6} is red in G.

Obviously, $e' \in [v_{i+6}, r_{i+6}, x']$. By way of contradiction, suppose e and e' are R-separated in $G - e^w$. Because $e \in r_i$ and $e' \in r_{i+6}$, Lemmas 12.12 and 12.13 imply that there is a a witnessing subdivision H' of V_8 with two H'-spokes in $\mathrm{cl}(Q_i) \cup \mathrm{cl}(Q_{i+1})$ and the other two H'-spokes are s_{i+3} and s_{i+4}. Furthermore, six of the eight ends of the H'-spokes are in the component R' of $R - \{e, e'\}$ containing $r_{i+1}\, r_{i+2}\, r_{i+3}\, r_{i+4}$.

Let y be the end of e' in R'. Because $w = v_{i+1}$ and $x' \in [v_{i+6}, r_{i+6}, x\rangle$, R' is contained in

$$r_{i+1}\, r_{i+2}\, r_{i+3}\, r_{i+4}\, r_{i+5}\, [v_{i+6}, r_{i+6}, x\rangle\ .$$

In particular, J is not an H'-spoke and at most two of the H'-spokes have ends in the span of J. Lemma 7.2 (1) implies the contradiction that the span of J, which includes f, cannot be crossed in D. We conclude that e and e' are not R-separated in $G - e^w$. Lemma 13.3 implies that e has a w-consecutive edge, as required.

Subcase 4: $f \in r_{i+7}$.

In this case, J has an end $x' \in \langle r_{i+7} \rangle$. If the other end of J is v_{i+5}, then Lemma 7.2 (3b) implies f' is in r_{i+3}. The contradiction is that $\overline{Q}_{i+1}$ is not crossed in D. Therefore, the other end of J is v_i. Lemma 7.2 (3b) implies f' is in $r_i\, r_{i+1}$.

Suppose there is no red edge in $r_{i+6}\, r_{i+7}$. Let e_{i+8} be the edge of s_{i+3} incident with v_{i+8} and let D_{i+8} be a 1-drawing of $G - e_{i+8}$. Corollary 12.7 and Lemma 5.9 imply $\overline{Q}_{i+3}$ is crossed in D_{i+8}. No edge in r_{i+6} is spanned by a 2.5-jump having an end in $\langle r_{i+6} \rangle$, as otherwise e is H-yellow. Therefore, Lemmas 12.4 and 12.5 imply no edge of r_{i+6} is crossed in D_{i+8}. Lemma 7.2 (1) shows that no edge of $[x', r_{i+7}, v_{i+8}] r_{i+8}\, r_{i+9}$ is crossed in D_{i+8}. We conclude that some edge $\hat{f}$ of $[v_{i+7}, r_{i+7}, x']$ is crossed in D_{i+8}.

Lemmas 12.4 and 12.5 imply that there is a 2.5-jump $v_{i+5}x''$, with $x'' \in \langle v_{i+7}, r_{i+7}, x'\rangle$, and, furthermore, that $\hat{f} \in [v_{i+7}, r_{i+7}, x'']$. Lemma 7.2 (3b) implies $\hat{f}$ crosses an edge e' in r_{i+4}. Lemmas 12.4 and 12.5 imply e' is red in G.

Let y be the end of e' nearest v_{i+5} in r_{i+4}. The $r_i r_{i+5}$-path P_0 contained in the uu^e-subpath of $\Delta_e - e$ must have v_{i+5} as an end, since otherwise e is either H-green or H-yellow. Symmetrically, the $r_{i+4}r_{i+9}$-path P_4 contained in the $yy^{e'}$-subpath of $\Delta_{e'} - e'$ has v_i as an end.

Lemma 11.7 implies e' is R-separated from e in G. Therefore, P_0 and P_4 are disjoint. This implies that $R \cup P_0 \cup P_4 \cup s_{i+2} \cup s_{i+3} \cup s_{i+4}$ is a subdivision V_{10} in $G - e^w$, showing that f cannot be crossed in D, a contradiction that proves there is a red edge e'' in $r_{i+6}\, r_{i+7}$.

Suppose e and e'' are R-separated in $G - e^w$. Lemma 12.12 implies that a witnessing subdivision H' of V_8 is such that the component R' of $R - \{e, f\}$ containing six of the eight ends of H'-spokes contains $r_{i+1}\, r_{i+2}\, r_{i+3}\, r_{i+4}\, r_{i+5}$.

However, J spans $[x', r_{i+7}, v_{i+8}]\, r_{i+8}\, r_{i+9}$, so at most two H'-spokes have ends that are in the span of J. Lemma 7.2 (1) combines with H' to yield the contradiction that the span of J, including f, cannot be crossed in D. It follows that e and e'' are not R-separated in $G - e^w$, and now Lemma 13.3 implies e has a w-consecutive red edge, completing the proof of Claim 1. $\square$

With Claim 1 in hand, we may assume f is red. Recall that f and f' are the edges crossed in D, with $f \in r_{i+4}\, r_{i+5}\, r_{i+6}\, r_{i+7}$ and $f' \in r_{i-1}\, r_i\, r_{i+1}\, r_{i+2}$. The proof in Case 1 is completed by finding a w-consecutive red edge for e. We proceed in four cases, basically depending on which side of Δ_e each of f and f' is on.

Subcase 1: *f is in $r_{i+4}[v_{i+5}, r_{i+5}, u^e]$ and f' is in $r_{i-1}\, [v_i, r_i, u]$.*

Since f and f' are not R-separated in $G - e^w$ and, therefore, not R-separated in G, f' cannot be red (Lemma 11.7). If f' is H-yellow in G, then Lemma 12.4 shows it is not crossed in D. Therefore, Theorem 11.3 implies f' is H-green in G. Lemma 12.5 says there is a 2.5-jump J spanning f' so that f' is in the partial H-rim branch spanned by J. As J cannot span e (e is not H-green), Lemma 7.2 (3b) and our current context (f in $r_{i+4}\, r_{i+5}$ and f' in $r_{i-1}\, r_i$) implies this is possible only if $f' \in r_{i-1}$ and $f \in r_{i+5}$. However, the red edges f and e are R-separated in G, implying that $G - e^w$ still has five spokes (we may replace s_{i+1} with the $r_i r_{i+5}$ subpath of P_u). Thus, f' is H'-green in $G - e^w$, for some $H' \cong V_{10}$. This is impossible, as f' is crossed in D (Lemma 6.6 (10)).

Subcase 2: $f \in r_{i+4}[v_{i+5}, r_{i+5}, u^e]$ *and* $f' \in [u, r_i, v_{i+1}]\, r_{i+1}\, r_{i+2}$.

In this subcase, f is R-separated in G from e. The witnessing subdivision H' of V_8 can be chosen to contain the "nearer" $(r_{i-1}\, r_i)(r_{i+4}\, r_{i+5})$-paths, one from each of Δ_f and Δ_e, together with the H-spokes s_{i+2} and s_{i+3} to construct H'.

We claim that this H' also shows that f is R-separated from f' in $G - e^w$. If $f \in r_{i+4}$, then, since $\overline{Q}_{i+1}$ is crossed in D, $f' \in r_i\, r_{i+1}$. In this case, H' contains the spokes s_{i+2} and s_{i+3}, so indeed f and f' are in disjoint H'-quads, as required. If $f \in r_{i+5}$, then $f' \in r_{i+2}$ by Lemma 7.2 (3b), and again f and f' are in disjoint H'-quads, showing f and f' are R-separated in $G - e^w$. Observation 11.6 (1) yields the contradiction that f and f' do not cross each other in D.

Subcase 3: $f \in [w^e, r_{i+5}, v_{i+6}] r_{i+6}\, r_{i+7}$ *and* $f' \in r_{i-1}\, [v_i, r_i, u]$.

If f is R-separated from e in $G - e^w$, then it cannot cross f' in D, a contradiction. Otherwise, Lemma 13.3 implies there is a w-consecutive red edge for e.

Subcase 4: $f \in [w^e, r_{i+5}, v_{i+6}] r_{i+6}\, r_{i+7}$ *and* $f' \in [u, r_i, v_{i+1}] r_{i+1}\, r_{i+2}$.

If $f' = e$, then we are done: Lemma 13.3 implies e has a w-consecutive edge.

So we assume $f' \neq e$. If f' is red in G, then Lemma 11.7 implies it is R-separated from f in G. Therefore, f' is R-separated from f in $G - e^w$, a contradiction; so f' is not red in G.

Suppose by way of contradiction that f' is H-yellow, with witnessing H-yellow and H-green cycles C and C', respectively. If e^w is not in C, then Lemma 12.4 yields the contradiction that f' is not crossed in D.

If e^w is in C, then let P_2 be the RR-subpath of C containing e^w, let P' be the RR-subpath of $\Delta_e - e$ that contains e^w, and let J be the global H-bridge contained in C'. The end of P' in r_{i+5} cannot be in the interior of the span of J, as then either the peak of Δ_e is a vertex, in which case we have that Δ_e is H-yellow, yielding the contradiction that e is H-yellow, or the peak of Δ_e consists of parallel edges, both in the span of J, contradicting Theorem 6.7.

It follows that P' has its end in r_{i+5}, but not in the interior of the span of J. On the other hand, P_2 has, by Definition 11.1, one end in the interior of the span of J. But now $(P_2 \cup P') - e^w$ contains an R-avoiding subpath that intersects at most the one spoke s_{i+1}. Therefore, this subpath is in an H-green cycle and contains an edge spanned by J, contradicting Theorem 6.7. It follows that f' is H-green.

Theorem 12.6 implies that H has no 3-jumps. If f' is H-green by a 2.5-jump J, then, because J cannot span e, Lemma 7.2 (3b) implies $f \in [w^e, r_{i+5}, v_{i+6}] r_{i+6}$ and $f' \in r_{i+2}$. Let x be the end of f closest to w^e in $r_{i+5}\, r_{i+6}$. Let H' be the subdivision of V_8 obtained from $H - \langle s_{i+1} \rangle$ by replacing s_{i+2} with P_x (recall this is defined in Theorem 12.1 (3)). Now f and f' violate Lemma 7.2 (3b) relative to H'. Therefore, f' is not H-green by a 2.5-jump.

Lemma 7.2 implies f' is not H-green by a 2-jump, as then it is not crossed in D. Thus, f' is H-green by a local H-green cycle C. Lemma 12.5 implies e^w is in C. Since f cannot be R-separated from f' in $G - e^w$, we see that f is not R-separated from e in $G - e^w$. Now Lemma 13.3 implies there is a w-consecutive red edge for e, concluding the proof for Case 1.

Case 2: e^w *not incident with* w.

By Theorem 12.1 (3), w is incident with a global H-bridge J_w. Since w is not incident with e^w, $w \neq v_{i+1}$, and therefore J_w is the 2.5-jump wv_{i+3}.

We observe that, since e_w is not incident with w, its incident vertex in r_i is in the interior of the span of J_w. Moreover, e_w is the first edge of an R-avoiding $r_i r_{i+5}$-path P in $\Delta_e - e$, which, together with a subpath of $r_i\, r_{i+1}$, s_{i+2}, and a subpath of $r_{i+5}\, r_{i+6}$ makes an H-yellow cycle C. By Lemma 11.2 (3), there is only one C-bridge in G and, therefore, $P = s_{i+1}$. In particular, $e_w \in s_{i+1}$.

Claim 2. No edge in $r_{i+7}\, r_{i+8}$ is H-yellow.

Proof. Suppose some edge e' in $r_{i+7}\, r_{i+8}$ is H-yellow. Let C and C' be the witnessing H-yellow and H-green cycles, respectively. By Lemma 11.2 (1), C' contains a global H-bridge J'.

In the case e' is in r_{i+7}, the span of J' contains a vertex of r_{i+2} in its interior. Theorem 6.7 implies $J' = J_w$. But now $C \cup Q_{i+1}$ contains an H-yellow cycle C'' for which there is a C''-interior C''-bridge containing an edge of s_{i+2}, contradicting Lemma 11.2 (3). Therefore, no edge in r_{i+7} is H-yellow.

Now we suppose e' is in r_{i+8}. Lemma 10.9 (1) shows J' does not have v_{i+3} as an end, so J' has one end $x \in \langle r_{i+3}\rangle$ and its other end is v_{i+6}. But now $C \cup Q_{i+4}$ contains an H-yellow cycle C'' having a C''-interior C''-bridge containing an edge of s_{i+4}, contradicting Lemma 11.2 (3). $\square$

Claim 3. Some edge of r_{i+7} is red.

Proof. Suppose no edge of r_{i+7} is red. By Theorem 11.3 and Claim 2, every edge in r_{i+7} is H-green.

Subclaim 1. If there is a red edge in either $r_{i+3}\, r_{i+4}$ or $r_{i+8}\, r_{i+9}$, then there is a red edge in $r_{i+8}\, r_{i+9}$. Furthermore, among all such red edges, the one e'' with an end x'' nearest v_{i+8} in $r_{i+8}\, r_{i+9}$ is such that $(e'')^{x''}$ is not incident with x'' (that is, Case 1 does not apply to e'' and x'').

Proof. We first suppose no edge of $r_{i+3}\, r_{i+4}$ is red. Then there is a red edge in $r_{i+8}\, r_{i+9}$. For any such red edge e'', if the end x'' of e'' nearest to v_{i+8} is incident with $(e'')^{x''}$, then Case 1 shows there is an x''-consecutive red edge $\hat{e}$ for e''. By Definition 13.1 (1), $\hat{e} \in r_{i+1}\, r_{i+2}\, r_{i+3}\, r_{i+4}$. Since the edges in $r_{i+1}\, r_{i+2}$ are H-green, $\hat{e} \notin r_{i+1}\, r_{i+2}$. But then $\hat{e}$ is a red edge in $r_{i+3}\, r_{i+4}$, a contradiction. Therefore, x'' is not incident with $(e'')^{x''}$, as required.

The alternative is that there is a red edge in $r_{i+3}\, r_{i+4}$. Among all such edges, let e' be the one having an incident vertex x' nearest v_{i+3} in $r_{i+3}\, r_{i+4}$. Because of Theorem 6.7 and J_w, x' is not incident with a 2.5-jump $x'v_{i+1}$ or $x'v_{i+2}$. Therefore, x' is incident with $(e')^{x'}$, and we conclude from Case 1 that there is an x'-consecutive red edge e'' for e'. Because of J_w, every edge in r_{i+6} is either H-yellow or H-green and so, in particular, is not red. By assumption, no edge of r_{i+7} is red. By Definition 13.1 (1), $e'' \in r_{i+8}\, r_{i+9}$. Also, $\Delta_{e''}$ separates s_{i+3} from $\Delta_{e'}$ in $\mathrm{cl}(Q_{i+3}) \cup \mathrm{cl}(Q_{i+4})$.

Let x'' be the end of e'' nearest v_{i+8} in $r_{i+8}\, r_{i+9}$. By way of contradiction, suppose x'' is incident with $(e'')^{x''}$. Then Case 1 shows there is an x''-consecutive red edge $\hat{e}$ for e''. But $\hat{e}$ is not in $r_{i+1}\, r_{i+2}$ because J_w makes every one of those edges H-green. Therefore, $\hat{e}$ is in $r_{i+3}\, r_{i+4}$. Since $\Delta_{\hat{e}}$ separates s_{i+3} from $\Delta_{e''}$ in $\mathrm{cl}(Q_{i+3}) \cup \mathrm{cl}(Q_{i+4})$, we see that $\hat{e}$ is nearer to v_{i+3} than e' is, contradicting the choice of e'. Therefore x'' is not incident with $(e'')^{x''}$, as required. $\square$

Subclaim 2. No edge in either $r_{i+3}\, r_{i+4}$ or $r_{i+8}\, r_{i+9}$ is red.

PROOF. Suppose by way of contradiction that there is a red edge in either $r_{i+3}\,r_{i+4}$ or $r_{i+8}\,r_{i+9}$. By Subclaim 1, there is a red edge e'' in $r_{i+8}\,r_{i+9}$ so that the end x'' of e'' nearest v_{i+8} in $r_{i+8}\,r_{i+9}$ is not incident with $(e'')^{x''}$. Therefore, Theorem 12.1 (3) implies x'' is incident with a 2.5-jump that is either $x''v_{i+6}$ or $x''v_{i+7}$. It cannot be the former, as the 2.5-jumps $x''v_{i+6}$ and J_w contradict Lemma 10.9 (4). Therefore, x'' is in the interior of r_{i+9} and the 2.5-jump is $x''v_{i+7}$. The contradiction is obtained by showing that $\operatorname{cr}(G) \le 1$.

Let D be a 1-drawing of $G - \langle r_{i+7}\rangle$. There is still a subdivision H' of V_8 in $G - \langle r_{i+7}\rangle$ consisting of the rim $(R - \langle r_{i+7}\rangle) \cup x''v_{i+7}$ and the four spokes s_i, s_{i+1}, s_{i+2} and $s_{i+3}\,r_{i+8}[v_{i+9}, r_{i+9}, x'']$. We note that $x''v_{i+7}$ is an H'-rim branch, contained in an H'-quad Q consisting of s_{i+2}, r_{i+2}, $s_{i+3}\,r_{i+8}[v_{i+9}, r_{i+9}, x'']$, and $x''v_{i+7}$.

We aim to show $D[Q]$ is clean, so by way of contradiction, we assume $D[Q]$ is not clean. The H'-rim branches of Q are r_{i+2} and $x''v_{i+7}$. Since $r_{i+1}\,r_{i+2}$ is not crossed in D (Lemma 7.2 (3a)), we deduce that $x''v_{i+7}$ is crossed in D. Furthermore, the cycle $r_{i+3}\,s_{i+4}\,r_{i+8}\,s_{i+3}$ (which is Q_{i+3} in G) is H'-close and, therefore Lemmas 5.3 and 5.4 imply Q_{i+3} is not crossed in D. It follows that $x''v_{i+7}$ crosses r_{i+4} in D, so $s_{i+3}\,r_{i+8}[v_{i+9}, r_{i+9}, x'']$ is exposed in D, from which $D[H']$ is completely determined. (See Figure 13.1.)

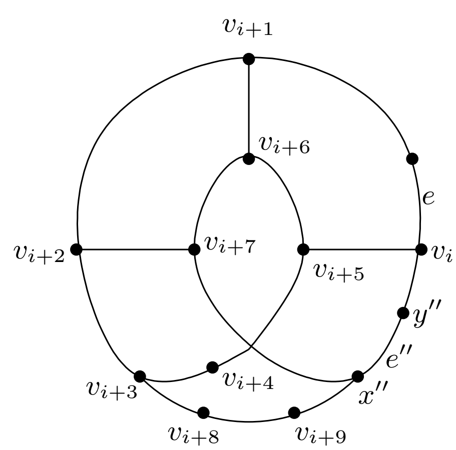

FIGURE 13.1. $D[H']$

Our contradiction is obtained from a detailed consideration of $\Delta_{e''}$. We first show that v_{i+4} is in the peak of $\Delta_{e''}$. To see this, we note that the $r_{i+9}r_{i+4}$-subpath of $\Delta_{e''} - e''$ that starts nearest x'' is simply s_{i+4}, as otherwise there is an H-yellow cycle C with more than one C-bridge. Theorem 12.1 (3) implies the subpath of $\Delta_{e''} - e''$ from x'' to the peak of $\Delta_{e''}$ has at most one edge in R; therefore, there is no edge of r_{i+4} between v_{i+4} and the peak of $\Delta_{e''}$. That is, v_{i+4} is in the peak of $\Delta_{e''}$.

Let y'' be the end of e'' different from x''. Because y'' is too close to J_w, it is not incident with a global H-bridge. Thus, the edge of $\Delta_{e''} - e''$ incident with y'' is not in R and, therefore, is the first edge of an $r_{i+9}r_{i+4}$-subpath P of $\Delta_{e''} - e''$. Let z'' be the other end of P.

We note that $z'' \neq v_{i+4}$, as $D[P]$ cannot cross $D[H']$. Therefore, $z'' \in \langle v_{i+4}, r_{i+4}, v_{i+5}]$. If z'' is in the peak of $\Delta_{e''}$, then z'' and v_{i+4} are joined by parallel edges, one of which is not in H'. That one must cross $D[H']$, which is a contradiction. Therefore, z'' is not in the peak of $\Delta_{e''}$. But now Theorem 12.1 (3) implies z'' is in the interior of the span of a global H-bridge J'' that has an end in the peak of $\Delta_{e''}$; therefore, this end of J'' is in r_{i+4}.

The end of J'' in r_{i+4} must be v_{i+4}, as otherwise J'' is a 2.5-jump with one end being v_{i+7}, which, together with $x''v_{i+7}$, contradicts Lemma 10.9 (1). Therefore, J'' is either $v_{i+4}v_{i+6}$ or $v_{i+4}u''$, with $u'' \in \langle r_{i+6} \rangle$. However, Lemma 7.2 (1) or (3a) and J'' show that r_{i+4} cannot be crossed in D, a contradiction that finally shows $D[Q]$ is clean.

We can now obtain the claimed 1-drawing of G. Observe that $x''v_{i+7}$ is in an H-green cycle that, by Lemma 6.6 (8), has only one bridge. Also, if there is a Q_{i+2}-bridge other than $M_{Q_{i+2}}$, then $\mathrm{cl}(Q_{i+2})$ has an edge f not in Q_{i+2}. But Theorem 5.23 and Lemma 5.9 imply Q_{i+2} would be crossed in any 1-drawing of $G - f$; however, both r_{i+2} and r_{i+7} are H-green courtesy of J_w and $x''v_{i+7}$. Therefore, Q_{i+2} has only one bridge. It follows that there are only two Q-bridges in G, one of which is r_{i+7}. Since $D[Q]$ is clean, it bounds a face of $D[G - \langle r_{i+7} \rangle]$ and it is easy to put r_{i+7} into this face so as to obtain a 1-drawing of G. That is, $\mathrm{cr}(G) \leq 1$, a contradiction completing the proof of the subclaim. □

We are now in a position to finish the proof of Claim 3. Let e_3 be the edge of s_{i+3} incident with v_{i+3} and let D be a 1-drawing of $G - e_3$. Corollary 12.7 and Lemma 5.9 imply $\overline{Q}_3$ is crossed in D. It follows that there is an edge $\hat{e}$ in $r_{i+6}\, r_{i+7}\, r_{i+8}\, r_{i+9}$ that is crossed in D.

The H-yellow cycle Q_{i+1} contains r_{i+6}, so Lemma 12.4 implies r_{i+6} is not crossed in D. By assumption for r_{i+7} and by Subclaim 2 for $r_{i+8}\, r_{i+9}$, no edge of $r_{i+7}\, r_{i+8}\, r_{i+9}$ is red. Lemmas 12.4 and 12.5 imply that $\hat{e}$ is spanned by some 2.5-jump J', and, moreover, $\hat{e}$ is in the H-rim branch whose interior contains the end x' of J'.

If $\hat{e} \in r_{i+7}$, then J' is either $x'v_{i+5}$ or $x'v_i$. Suppose first that $J' = x'v_{i+5}$. Lemma 7.2 (3b) implies $\hat{e}$ crosses an edge in r_{i+4}. But Theorem 6.7 shows r_{i+4} cannot be in the span of a 2.5-jump, so Lemmas 12.4 and 12.5 imply no edge of r_{i+4} is crossed in D. Thus, $J' \neq x'v_{i+5}$.

Now we suppose $J' = x'v_i$. In this case, Lemma 7.2 (3b) implies $\hat{e}$ crosses an edge in r_{i+1}, while (1) of the same lemma implies no edge in the span of J, which includes r_{i+1}, is crossed in D. We conclude that $\hat{e} \notin r_{i+7}$.

If $\hat{e} \in r_{i+8}$, then J' is either $x'v_{i+6}$ or $x'v_{i+1}$. Theorem 6.7 shows the latter does not happen. Lemma 10.9 (4) shows the former does not happen. Therefore, $\hat{e} \notin r_{i+8}$.

The last possibility is that $\hat{e} \in r_{i+9}$. In this instance, J' is either $x'v_{i+7}$ or $x'v_{i+2}$. Theorem 6.7 precludes the latter possibility, so we assume $J' = x'v_{i+7}$. However, in this case, Lemma 7.2 (3b) implies $\hat{e}$ crosses an edge $\tilde{e}$ in r_{i+5}, in which case neither $\hat{e}$ nor $\tilde{e}$ is in $\overline{Q}_{i+3}$, contradicting the fact that $\overline{Q}_{i+3}$ is crossed in D. □

We now finish the proof of Case 2 and, therefore, Theorem 13.2. By Claim 3, we may let $e' = xy$ be the red edge in r_{i+7} that is nearest v_{i+7} in r_{i+7}, labelled so that x is nearer v_{i+7} in r_{i+7} than y is. We look for the x-consecutive red edge for e'. As the edges spanned by J_w are H-green, e is the only possibility for the x-consecutive red edge for e'.

Suppose first that e' and x satisfy the condition for Case 1. We have proved there is an x-consecutive red edge for e' and, as just mentioned, this can only be e. This implies that $x = v_{i+7}$. To see that e' is the w-consecutive red edge for e, it remains to show that e and e' can be crossed in $G - e^w$. (This is the only asymmetric condition in the definition of consecutive.)

The H-quad Q_{i+1} is also an H-yellow cycle and so (Lemma 11.2 (3)) bounds a face of G. It follows that e and e' are not R-separated in $G - e^w$ and, therefore Lemma 11.7 implies there is a 1-drawing of $G - e^w$ in which e and e' are crossed, as required.

The alternative is that e' and x do not satisfy the condition for Case 1. Then, just as for w above, there is a 2.5-jump $J_x = xv_{i+5}$ incident with x. Also, the edge e^x of $\Delta_{e'} - e'$ that is nearest x and not in R is in s_{i+7}. Since Q_{i+1} bounds a face of G, e and e' are not R-separated in $G - e^w$ and, therefore, Lemma 11.7 implies there is a 1-drawing of $G - e^w$ in which they are crossed. ∎

The following is a consequence of Definition 13.1 and Theorem 13.2.

LEMMA 13.4. *Let $G \in \mathcal{M}_2^3$ and $V_{10} \cong H \subseteq G$, with H tidy. With the labelling of $e = uw$ and e_w as in Definition 13.1, if x is the end of e_w nearest w^e in $[w^e, r_{i+5}, v_{i+6}]\, r_{i+6}\, r_{i+7}$, then e is x-consecutive for e_w.*

Proof. By Theorem 13.2, there is an x-consecutive red edge e'' for e_w. Conditions (2) and (3) of Definition 13.1 applied to e_w being w-consecutive for e and the same conditions applied to e'' being x-consecutive for e_w imply that $e = e''$. ∎

The main goal of this work is to prove Theorem 2.14. The following lemma will be very helpful.

LEMMA 13.5. *Let $G \in \mathcal{M}_2^3$, $V_{10} \cong H \subseteq G$, and let Π be an embedding of G in $\mathbb{R}P^2$ so that H is Π-tidy. Let C be a contractible cycle contained in $\mathfrak{M}$ so that C is the union of a 3-rim path $C \cap R$ (recall Definition 11.1 (1)) and an R-avoiding path P. Then, for every edge e of $C \cap R$, there is an H-green cycle containing e and contained in $H \cup P$.*

Proof. The graph $H \cup P$ is 2-connected and not planar, so every face of $\Pi[H \cup P]$ is bounded by a cycle. There is a face F of $H \cup P$ contained in $\mathfrak{M}$ and incident with e; by the preceding remark, F is bounded by a cycle C'.

Let j be the index so that $e \in r_j$; thus, F is Q_j-interior. Since F is also C-interior, $C' \cap H \subseteq \langle s_j\, r_j\, s_{j+1} \rangle$. In particular, there is at least one edge of C' that is in P but not in H.

Observe that $\langle s_j\, r_j\, s_{j+1} \rangle - e$ has two components K_1 and K_2. Since C' contains a vertex in each of K_1 and K_2 (namely the ends of e), C' contains an $\langle s_j\, r_j\, s_{j+1} \rangle$-avoiding K_1K_2-path P'. Thus, $P' \subseteq P$.

Let C'' be the cycle in $\langle s_j\, r_j\, s_{j+1} \rangle \cup P'$. Then C'' is evidently an H-green cycle containing e, as required. ∎

Now for the main result.

Theorem 2.14 *If G is a 3-connected, 2-crossing-critical graph containing a subdivision of V_{10}, then $G \in \mathcal{T}(S)$.*

Proof. By Theorem 10.4, G contains a tidy subdivision H of V_{10}; let Π be an embedding of G in $\mathbb{R}P^2$ so that H is Π-tidy. The strategy is to show that, between every red edge $e = uw$ and its w-consecutive red edge e_w, there is one of the thirteen pictures (as defined just before Lemma 2.11). This is accomplished by showing that e produces "one side" of the picture and e_w produces the other. Let $i \in \{0, 1, 2, \dots, 9\}$ be such that $e \in r_i$; we choose the labelling so that $r_i = [v_i, r_i, u, e, w, r_i, v_{i+1}]$. Thus, $e_w \in r_{i+5}\, r_{i+6}\, r_{i+7}$.

Let x be the end of e_w so that e is the x-consecutive red edge for e_w. Let P_1 be the $w^e x$-subpath of R that is a 3-rim path (Definition 11.1 (1)); likewise P_2 is the $x^{e_w} w$-subpath of R that is a 3-rim path.

Claim 1. Let B be a global H-bridge spanning an edge of P_1. Then:

(a) B has ends w^e and x;
(b) $w^e = v_{i+5}$; and
(c) $e^w \in s_{i+1}$ and $(e_w)^x \in s_{i+2}$.

The analogous claims holds for P_2.

Proof. We remark that the span of B does not include in its interior a peak vertex of Δ_e, and does not include e_w. Therefore, B has both its attachments in P_1.

Consequently, the attachments of B are contained in $r_{i+5}\, r_{i+6} [v_{i+7}, r_{i+7}, v_{i+8}\rangle$. Theorem 10.6 implies one end of B is v_{i+5} and the other end is in $[v_{i+7}, r_{i+7}, v_{i+8}\rangle$.

It follows that $w^e = v_{i+5}$. At the other end, we claim x is in B. We note that e_w is in r_{i+7}, so that $H - \langle s_{i+4}\rangle$ shows that e and e_w are R-separated. Let x' be the end of B in r_{i+7}.

If $(e_w)^x$ is not in s_{i+2}, then let e_{i+7} be the edge of s_{i+2} incident with v_{i+7} and let D be a 1-drawing of $G - e_{i+7}$. Corollary 12.7 and Lemma 5.9 imply $\overline{Q}_{i+2}$ is crossed in D. The presence of J and B combine with Lemma 7.2 (1) to show that neither $[w, r_i, v_{i+1}]\, r_{i+1}\, r_{i+2}$ nor $r_{i+6}\, [v_{i+7}, r_{i+7}, x']$, respectively, is crossed in D.

It follows that some edge e' in $[x', r_{i+7}, v_{i+8}]$ is crossed in D. Let P_w be the path in Δ_e described in Theorem 12.1 (3). Since P_w does not have v_{i+1} as one end, and its other end is v_{i+5}, its only intersection with s_{i+1} can be in $\langle s_{i+1}\rangle$. Such an intersection produces an H-green cycle that shows the edge of r_i incident with v_{i+1} is in two H-green cycles, contradicting Theorem 6.7. Therefore, P_w is disjoint from s_{i+1}.

Using P_w, s_{i+1}, s_{i+3} and s_{i+4} as spokes and R as the rim yields a V_8 that shows r_{i+7} is R-separated in $G - e_{i+7}$ from $[v_i, r_i, w]$; thus, Observation 11.6 (1) shows e' crosses an edge e'' of r_{i+3} in D. Lemmas 12.4 and 12.5 shows e' and e'' are red in G. Lemma 11.7 shows that e' and e'' are R-separated in G. Lemma 12.13 shows that a witnessing V_8 can be chosen to avoid e_{i+7}. But now D contradicts Observation 11.6 (1). Therefore, $(e_w)^x$ is in s_{i+2} and incident with v_{i+7}.

If B has an end in $\langle r_{i+7}\rangle$, then Theorem 12.1 (3) implies $P_x \cap r_{i+7}$ has just one edge, namely xv_{i+7} and, consequently, x is in B.

If, on the other hand, v_{i+7} is an end of B, then Theorem 12.1 (3) implies x must be incident with e^x and, therefore $x = v_{i+7}$. Again, we see that x is in B.

Observe that J_w and B are now seen to be completely symmetric with respect to (e, w) and (x, e_w); in particular, we conclude that $e^w \in s_{i+1}$. $\square$

If there is a global H-bridge B spanning an edge of P_1, then we let $P_1' = B$. Otherwise, we let $P_1' = P_1$. A completely analogous discussion holds for P_2 to yield the wx^{w_e}-path P_2'.

Our next claim identifies the cycle that is the boundary of our picture.

CLAIM 2. The closed walk $P_w P_1' P_x P_2'$ is a cycle.

PROOF. If the edge of P_w incident with w is in R, then Theorem 12.1 (3) shows that w is incident with a global H-bridge B. Claim 1 implies $B = P_2'$. Thus, w has degree 2 in the closed walk $P_w P_1' P_x P_2'$. Otherwise, $P_2' = P_2$, w is incident with e^w, and again w has degree 2 in $P_w P_1' P_x P_2'$.

The other "corners" w^e, x, and x^{e_w} are treated similarly. $\square$

DEFINITION 13.6. Let e and e' be red edges and let w and x be the ends of e and e', respectively, so that e' is the w-consecutive red edge for e and e is the x-consecutive red edge for e'. Let P_1 be the $x\Delta_e$-path in R that is a 3-rim path and let P_2 be the $w\Delta_{e'}$-path in R that is a 3-rim path. Let P_w be the ww^e-path in $\Delta_e - e$ and let P_x be the $xx^{e'}$-path in $\Delta_{e'} - e'$. For $i = 1, 2$, let P_i' be P_i unless there is a global H-bridge B_i spanning an edge of P_i, in which case $P_i' = B_i$.

The cycle C_e is the composition $P_w P_1' P_x P_2'$.

We will see that C_e is the outer boundary of the one of the thirteen pictures that occurs. We observe that C_e is in the boundary of the closed disc in $\mathbb{R}P^2$ consisting of the union of the closed discs bounded by $r_i\, r_{i+1}\, r_{i+2}\, s_{i+3}\, r_{i+7}\, r_{i+6}\, r_{i+5}\, s_i$, $P_1' P_1$ (if $P_1' \neq P_1$), and $P_2' P_2$ (if $P_2' \neq P_2$). Therefore, C_e is the boundary of a closed disc $\mathfrak{D}_e$ in $\mathbb{R}P^2$.

We now prove three claims that will be useful for finding the various parts of the picture.

CLAIM 3. Let C be a cycle contained in $\mathfrak{D}_e$. If either $C \cap P_1'$ or $C \cap P_2'$ is empty, then C bounds a face of $\Pi[G]$.

PROOF. By symmetry, we may suppose $C \cap P_1'$ is empty. Let M be the C-bridge containing s_{i+4}.

SUBCLAIM 1. If B is a C-bridge different from M, then $\Pi[C \cup B]$ is contractible in $\mathbb{R}P^2$.

PROOF. We start by noting that $\Pi[B] \subseteq \mathfrak{M}$, since P_2' is either just an edge that is a global H-bridge (and so in $\mathfrak{D}$ and forcing B to be in $\mathfrak{M}$) or $P_2' = P_2$ and there is no global H-bridge having an attachment in $\langle P_2 \rangle$. In the latter case, any global H-bridge having an attachment at an end of P_2 (say w), has its other attachment in the H-rim $R - \langle P_2 \rangle$. Such an attachment is in $\mathrm{Nuc}(M)$, contradicting the assumption that $B \neq M$.

It follows that $\Pi[C \cup B]$ is contained in $\mathfrak{M}$ and totally disjoint from s_{i+4}. Therefore, $\Pi[C \cup B]$ is contractible, as claimed. $\square$

Let H' be the subgraph of $H \cup P_1' \cup P_2'$ consisting of $(R - (\langle P_1 \rangle \cup \langle P_2 \rangle)) \cup (P_1' \cup P_2')$ and the three H-spokes s_{i+3}, s_{i+4}, and s_i. The following claim shows that H' is a subdivision of V_6. (The notation $\|y\|$ is in Definition 4.1 (1).)

SUBCLAIM 2. $C_e \cap s_i \subseteq \|v_{i+5}\|$ and $C_e \cap s_{i+3} \subseteq \|v_{i+3}\|$.

Proof. Recall that P_w is contained in Δ_e. Theorem 12.1 (the existence of A_u and A_w, together with (3)) implies P_w is internally disjoint from P_u and, therefore, cannot intersect s_i, except possibly at their common end point v_{i+5}. The analogous argument using Δ_{e_w} applies for s_{i+3}. □

If C does not bound a face of $\Pi[G]$, then let e' be any edge of any C-interior C-bridge and let D be a 1-drawing of $G-e$. Subclaim 2 implies that C is H'-close (Definition 5.2). Lemmas 5.3 and 5.4 imply C is clean in D. Therefore, D contains a 1-drawing of $C \cup M$ in which C is clean and Lemma 5.6 implies C has BOD. It now follows from Corollary 4.7 that $\mathrm{cr}(G) \leq 1$, the final contradiction. □

We find structures in the C_e-interior that lead to the pictures. Our discussion will be w-centric; there is a completely analogous discussion for x.

A useful observation is the following. Recall that P_w is the ww^e-path in $\Delta_e - e$ (Theorem 12.1 (3)) and P_x is the analogous xx^{e_w}-path in Δ_{e_w}.

Claim 4. (1) No C_e-interior C_e-bridge has an attachment in each of the components of $(C_e - P_x) - e^w$.
(2) No C_e-interior C_e-bridge has an attachment in each of the components of $(C_e - P_w) - e^x$.

Proof. Let H' be a subdivision of V_8 witnessing the R-separation of e and e_w. As e and e_w are R-separated in neither $G - e^w$ nor $G - e^x$, e^w and e^x are both in H'. Since e and e_w are in disjoint H'-quads, e^w and e^x are in disjoint H'-spokes, which we denote as P^w and P^x, respectively; P^w and P^x are contained in the closed disc bounded by $\Pi[C_e]$.

Subclaim 1. There is such an H' so that $P^x = P_x$.

Proof. As a first case, suppose $C_e \cap s_i = \varnothing$. Then we may choose H' to be R, s_i, s_{i+4}, P_w, and P_x, and we are done. In the second case, $C_e \cap s_{i+3} = \varnothing$; replace s_i with s_{i+3}.

In the final case, $C_e \cap s_i$ and $C_e \cap s_{i+3}$ are not empty. In this instance, $e_w \in r_{i+7}$. We may choose H' to consist of R, s_{i+4}, s_i, s_{i+1}, and P_x, the latter being contained in $\mathrm{cl}(Q_{i+2})$. □

By symmetry, it suffices to prove (1). Suppose by way of contradiction that there is a C_e-interior C_e-bridge B having an attachment in each component of $(C_e - P_x) - e^w$. Subclaim 1 implies there is a subdivision H' witnessing the R-separation of e and e_w so that $P_x \subseteq H'$. Let P^w be the other H'-spoke contained in the interior of C_e.

Let C' be the cycle bounding the C_e-interior face of $C_e \cup P^w$ that is incident with e^w. The C_e-bridge B contains a subpath P' joining the two components of $(C' - P_x) - e^w$. Now $((C' - P_x) - e^w) \cup P'$ contains an R-avoiding path P'' that can replace P^w in H' to get another subdivision of V_8 that witnesses the R-separation of e and e_w in $G - e^w$. However, this contradicts the fact that e and e_w are not R-separated in $G - e^w$. □

Here is our final preliminary claim.

Claim 5. Let B be a C_e-interior C_e-bridge. Then B is just an edge and its ends.

PROOF. Suppose to the contrary that B is a C_e-interior C_e-bridge with at least three attachments.

SUBCLAIM 1. B has at most two attachments in each of $C_e - P_1'$ and $C_e - P_2'$.

PROOF. By symmetry, it suffices to prove the first of these. Suppose B has at least two attachments in $C_e - P_1'$. Let y and z be the ones nearest the two ends of $C_e - P_1'$. There is a cycle in $B \cup C_e$ consisting of a C_e-avoiding yz-path in B and the yz-subpath of $C_e - P_1'$. Claim 3 implies this cycle bounds a face of $\Pi[G]$ and, therefore, B can have no other attachment in $C_e - P_1'$. $\square$

SUBCLAIM 2. $\mathrm{att}(B) \cap P_1' \subseteq \{x, w^e\}$ and $\mathrm{att}(B) \cap P_2' \subseteq \{w, x^{e_w}\}$.

PROOF. By symmetry, it suffices to prove the first of these. By way of contradiction, suppose B has an attachment y in $\langle P_1' \rangle$. Because B has at least three attachments, Subclaim 1 implies B has an attachment z in P_2'. Any C_e-avoiding yz-path in B contradicts Claim 4. $\square$

From these two subclaims, we easily deduce that:

- B has at most four attachments;
- one of w and x^{e_w} is an attachment of B; and
- one of x and w^e is an attachment of B.

Observe that Claim 4 (1) implies that not both w and w^e are attachments of B, while (2) implies that not both x and x^{e_w} are attachments of B. Therefore, $\mathrm{att}(B) \cap (P_1' \cup P_2')$ is either $\{w, x\}$ or $\{w^e, x^{e_w}\}$.

SUBCLAIM 3. $\mathrm{att}(B) \cap (P_1' \cup P_2') = \{w^e, x^{e_w}\}$.

PROOF. Suppose by way of contradiction that $\mathrm{att}(B) \cap (P_1' \cup P_2') = \{w, x\}$. As B has at least three attachments, there is an attachment y in $\langle P_w \rangle \cup \langle P_x \rangle$. By symmetry, we may assume $y \in \langle P_w \rangle$. Let P^{yw} be a C_e-avoiding yw-path in B. Then the union of P^{yw} and the yw-subpath of P_w is a cycle C^{yw} in $\mathfrak{D}_e$.

Since y and w are in $P_w - w^e$, C^{yw} is disjoint from P_1'. Claim 3 implies C^{yw} bounds a face of $\Pi[G]$. On the other hand, P_w is contained in the boundary of the face bounded by Δ_e and, therefore, $C^{yw} \cap P_w$ is in the boundary of two faces of $\Pi[G]$. We deduce that $C^{yw} \cap P_w$ is just the edge wy.

Furthermore, Claim 4 implies w and y are in the same component of $P_w - e^w$. Therefore, the definition of e^w implies wy is in R, and consequently P_2' is a global H-bridge spanning wy. However, any edge of B incident with w — and there is at least one such — must be in the interior of the face of $\Pi[G]$ bounded by the H-green cycle containing P_2' (Lemma 6.6 (8)). This contradiction proves the subclaim. $\square$

We are now ready to complete the proof of the claim. Any vertex in $\mathrm{att}(B) \setminus \{w^e, x^{e_w}\}$ is in $\langle P_w \rangle \cup \langle P_x \rangle$. Subclaim 1 implies there is at most one of these. Since B has at least three attachments, there is at least one of these. We conclude there is exactly one such attachment y. We may choose the labelling so that $y \in \langle P_w \rangle$. Lemma 5.19 implies B is isomorphic to $K_{1,3}$.

The vertex y is in the interior of P_w. Thus, both edges of P_w incident with y are in the boundary of the face bounded by $\Pi[\Delta_e]$. Consequently, any edge of G incident with y is in $\mathfrak{D}_e$.

Let c be the vertex of degree 3 in B. Claim 3 implies that the cycles $[y, c, w^e, y]$ and $[y, c, x^{e_w}, P_2', w, P_w, y]$ both bound faces in $\mathfrak{D}_e$. Therefore, y has degree 3 in G.

Let e' be the edge cw^e of B and let D' be a 1-drawing of $G - e'$. Consider the subdivision H' of V_6 consisting of $(R - (\langle P_1 \rangle \cup \langle P_2 \rangle)) \cup (P_1' \cup P_2')$, P_x, s_i, and s_{i+4}. Then H' shows that $P_w \cup (B - e')$ is not crossed in D'.

The path $P' = [c, cy, y, P_w, w^e]$ is not crossed in D'. Since y has degree 3 in D', we may add the edge $w^e c$ to D' alongside P' without crossing to obtain a 1-drawing of G. This is the final contradiction that shows B has only two attachments. Lemma 5.19 shows B is just an edge and its ends. $\square$

We now have our preliminary lemmas in hand and proceed to complete the proof of Theorem 2.14.

DEFINITION 13.7. Let C_e be decomposed as $P_w P_1' P_x P_2'$ as in Definition 13.6.

(1) If f is an edge not in C_e with ends w and x_{e_w} and P_2' has length 1, then f is a *w-chord*.
(2) If f is an edge not in C_e joining w to a vertex $y \in \langle P_x \rangle$ and the yx^{e_w}-subpath of P_x has length 1, then f is a *w-slope*.
(3) If f and f' are edges not in C_e, with f joining w with $z \in \langle P_2' \rangle$ and f' joining z to $z' \in \langle P_x \rangle$, and if P_2' has length 2, while the $z'x^{e_w}$-subpath of P_x has length 1, then $\{f, f'\}$ is a *w-chord+w-slope*.
(4) If f is an edge not in C_e joining x^{e_w} to a vertex y in $\langle P_w \rangle$, and both P_2' and the yw-subpath of P_w have length 1, then f is a *w-backslope*.
(5) If f is an edge not in C_e joining $y \in \langle P_w \rangle$ and $z \in \langle P_x \rangle$, and the paths P_w and P_x have length 2, while P_1' and P_2' have length 1, then f is a *crossbar*.

The five situations in Definition 13.7 are illustrated in Figure 13.2.

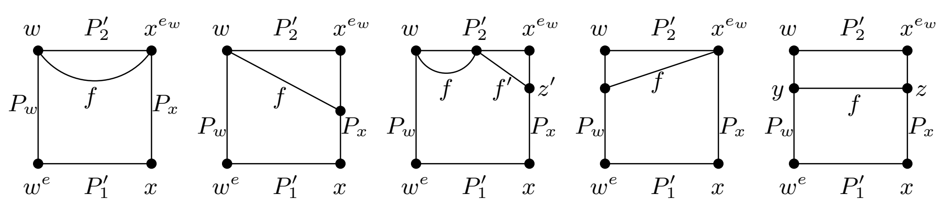

FIGURE 13.2. Definition 13.7.

CLAIM 6. If e^w is in neither an H-yellow nor an H-green cycle, then every edge of P_2 is H-green. If $\mathcal{C}$ is the set of H-green cycles containing edges of P_2, then $C_e \cup (\bigcup_{C \in \mathcal{C}}) C$ contains either:

(a) C_e plus a w-chord;
(b) C_e plus a w-slope; or
(c) C_e plus a w-chord+w-slope.

PROOF. Because e^w is not in an H-yellow cycle, Theorem 12.1 (3) implies w is incident with e^w.

Case 1: *some edge of P_2 is spanned by a global H-bridge.*

Let B be a global H-bridge spanning an edge of P_2. Claim 1 implies B has ends x^{e_w} and w, $x^{e_w} = v_{i+3}$, $e^w \in s_{i+1}$, and $e^x \in s_{i+2}$. Since w is incident with e^w, we have $w = v_{i+1}$.

We show (b) occurs by proving that r_{i+1} is a w-slope. We show that r_{i+1} and r_{i+2} are both paths of length 1, starting with the latter.

We note that P_x is equal to $s_{i+2}\,r_{i+2}$. Moreover, r_{i+2} has the face of $\Pi[G]$ bounded by the H-green cycle C_g containing B on one side and the face bounded by Δ_{e_w} on the other. Thus, r_{i+2} is just a single edge.

Claim 5 shows that the C_e-bridge B' containing r_{i+1} is just an edge and its ends. Thus, r_{i+1} is B' and so has length 1, as required, completing the proof in Case 1.

Case 2: *no edge of P_2 is spanned by a global H-bridge.*

In this case, $P_2' = P_2$. We start by showing that every edge of P_2 is H-green.

Because e_w is w-consecutive for e, Definition 13.1 implies no edge of P_2 is red. By Theorem 11.3, we need only show that none is H-yellow. Suppose to the contrary that there is an H-yellow edge f in P_2, as witnessed by the H-yellow cycle C_y and the H-green cycle C_g. Lemma 11.2 implies there is a global H-bridge B contained in C_g.

The face of $\Pi[G]$ bounded by C_y (Lemma 11.2 (3)) is in $\mathfrak{M}$. Now the faces of $\Pi[G]$ bounded by Δ_e and Δ_{e_w} separate $\mathfrak{M}$ into two parts, one of which contains f, and therefore P_2. It follows that P_1 is also in this part and C_y has at least a vertex in P_1. We conclude that B spans an edge of P_1. Claim 1 implies $B = P_1'$, $w^e = v_{i+5}$, $e^w \in s_{i+1}$, and $e^x \in s_{i+2}$. Because $P_2' = P_2$, and $e^w \in s_{i+1}$, we deduce that $w = v_{i+1}$.

Since e^w is not in an H-yellow cycle, we conclude that Q_{i+1} is not an H-yellow cycle. The other attachment of B, namely x, which is in $[v_{i+7}, r_{i+7}, v_{i+8}\rangle$, must therefore be v_{i+7}.

If the H-yellow edge f is in r_{i+1}, then $C_y \cap P_1$ is contained in the interior of the span of B. This implies that s_{i+1} is in an H-yellow cycle and, therefore, e^w is in an H-yellow cycle, contrary to the hypothesis.

We have noted that $x = v_{i+7}$ is an end of B. Consequently, no edge of $[v_{i+2}, r_{i+2}, x^{e_w}]$ can be H-yellow. That is, every edge of P_2 is H-green.

We now complete the proof in Case 2. Let C be the H-green cycle containing the edge of P_2 that is incident with w. Because e^w is not in any H-green cycle, w is incident with an edge e' in C that is not in C_e. Let B be the C_e-bridge containing e'.

Claim 5 implies B is just an edge with the two ends w and a second vertex z. The path $C \cap P_2$ is in the boundary of the face of $\Pi[G]$ bounded by C (Lemma 6.6 (8)). Also, there is no global H-bridge spanning an edge of P_2 (we are in Case 2). These two facts imply $C \cap P_2$ is just an edge.

Suppose first that $z \in P_x - x^{e_w}$. Because C is H-green, it is disjoint from P_1. Thus, Claim 3 implies that the cycle C' that is the union of the wz-subpath of P_2P_x and B bounds a face of $\Pi[G]$. This face is contained in $\mathfrak{M}$, as is the face bounded by C. Both are incident with the edge of P_2 incident with w and so they are the same face. We conclude that $C = C'$.

Now $C \cap P_x$ is in the boundary of a face inside the disc bounded by $\Delta_{e'}$ on one side and the face bounded by C on the other. Because G is 3-connected, this subpath has length 1. In this case, we have (b).

The other possibility is that z is in P_2. We have already shown that w and z are the ends of a digon. If $z = x^{e_w}$, then we have (a). Therefore, we may suppose $z \neq x^{e_w}$.

Since G is 3-connected, z has a neighbour y distinct from its neighbours in P_2. Let B' be the C_e-bridge containing zy. Claim 5 implies B' is just an edge joining z and y.

The choice of y shows $y \neq w$. Claim 4 (1) and (2) imply, respectively, that $y \notin \langle P_w P_1' \rangle$ and that $y \neq x$. If $y \in P_2$, then (just as for w and z) z and y are the ends of a digon, so y is a neighbour of z in P_2, contradicting the choice of y. Therefore, $y \in \langle P_x \rangle$.

Let C' be the cycle consisting of zy and the zy-subpath of $P_2 P_x$. Claim 3 implies C' bounds a face of $\Pi[G]$.

To see that (c) holds, notice that $C' \cap P_x$ is in the boundary of the faces bounded by C' and Δ_{e_w}. Again, the 3-connection of G shows $C' \cap P_x$ is a path of length 1. Likewise, $C' \cap P_2$ is in the boundary of the face bounded by C'. There is no global H-bridge spanning any edge of P_2, so $C' \cap P_2$ is also a path of length 1, completing the proof that (c) occurs and the proof of Claim 6. □

It remains to consider the possibilities that e^w is in either an H-yellow or an H-green cycle. We do the latter first.

Claim 7. If e^w is in an H-green cycle C, then either

(d) $C_e \cup C$ contains C_e plus a back-slope or
(e) $C_e \cup C$ is C_e plus a crossbar.

Proof. Let F be the face bounded by C (Lemma 6.6 (8)). Obviously F is not inside the face bounded by Δ_e, and, since F is contained in $\mathfrak{M}$, F is C_e-interior. Let y be the end of e^w nearer w in P_2; then $y \in r_i$. From the definition of H-green cycle (Definition 6.2), the edge of the yx^{e_w}-subpath of P_2 incident with y is in C.

If w is an attachment of a global H-bridge, then every edge of $C \cap R$ is in two H-green cycles, which is impossible by Theorem 6.7. Therefore, $P_2 = P_2'$, $y = w$, and C is the union of the wz-path $C \cap P_2$ (this defines z) and an R-avoiding wz-path P.

The path P contains a subpath P' joining a vertex of the zx-subpath of $P_2 P_x$ to a vertex of the component of $P_w - e^w$ containing w^e; we may assume P' is C_e-avoiding. Claim 3 implies that the cycle contained in $P' \cup P_w P_2 P_x$ bounds a face of $\Pi[G]$. As z is in this cycle, it must be that z is an end of P' and, moreover, this cycle is C. In particular, P is just P' plus a subpath of P_w. We know that $C \cap P_2$ is just an edge. Since the path $C \cap P_w$ is in the boundary of the faces bounded by both C and Δ_e, it is also just the edge e_w.

If $z \neq x^{e_w}$, then $P' = P$ and the zw^e-path contained in $P \cup P_w$ contradicts Claim 4 (1). Therefore, $z = x^{e_w}$.

Let B be the C_e-bridge containing P'. Claim 5 implies B has precisely two attachments $w' \in P_w$ and $x' \in P_x$: therefore, B is just the edge $w'x'$ (this is also P'). If x' is x^{e_w}, then B is a w-backslope.

Finally, suppose x' is in $P_x - x^{e_w}$. Then C bounds a face incident with $C \cap P_x$. Since $C \cap P_x$ is also in the boundary of the face bounded by Δ_{e_w}, it has length 1.

On the P_1' side, B together with the $w'x'$-subpath of $\langle P_w P_1' P_x \rangle$ is a cycle C' disjoint from P_2'. By Claim 3, C' bounds a face of $\Pi[G]$. As above, each of $C' \cap P_w$, $C' \cap P_x$, and P_1' all have length 1. Therefore, B is a crossbar. □

Our final case is that e^w is in an H-yellow cycle.

Claim 8. If e^w is in an H-yellow cycle C, then either

(d) $C_e \cup C$ contains C_e plus a back-slope or
(e) $C_e \cup C$ is C_e plus a crossbar.

PROOF. Let C' be the H-green cycle witnessing that the cycle C containing e^w is H-yellow. Then C' contains an H-jump J and either both ends of J are in P_1 or both ends of J are in P_2. In either case, Claim 1 implies the span of J is all of P_1 or P_2. We treat these two possibilities separately.

SUBCLAIM 1. If both ends of J are in P_2, then (e) occurs.

PROOF. In this case, Claim 1 implies J has ends w and x^{e_w}, $x^{e_w} = v_{i+3}$, $e^w \in s_{i+1}$, and $e^x \in s_{i+2}$.

Because e_w is both incident with v_{i+1} and in an H-yellow cycle as witnessed by the H-green cycle C' containing J, v_{i+1} is in the interior of the span of J; consequently, $w \in \langle r_i \rangle$. Therefore, the edge of r_i incident with v_{i+1} is H-green.

We observe that J witnesses that Q_{i+1} is an H-yellow cycle. It follows from Lemma 11.2 (3) that $C = Q_{i+1}$. The same part of the same lemma combines with the fact that e is not H-green to show that P_w consists of $[w, r_i, v_{i+1}, s_{i+1}, v_{i+6}]$ and that P_w has length precisely two. Symmetrically, P_x consists of $s_{i+2}\, r_{i+2}$ and has length 2. Therefore, we have (e), as required. $\square$

It remains to consider the possibility that both ends of J are in P_1. Claim 1 implies $J = w^e x$, $w^e = v_{i+5}$, $e^w \in s_{i+1}$, and $e^x \in s_{i+2}$. Also, r_{i+5} is in the H-green cycle C' containing J, and so P_w contains $r_{i+5}\, s_{i+1}$. Since Theorem 12.1 (3) implies P_w has at most one H-rim edge, we conclude that $w = v_{i+1}$. Recall that P_w is in the boundary of the face of $\Pi[G]$ bounded by Δ_e. The path r_{i+5} is also in the boundary of the face bounded by C' and so is just an edge. The path s_{i+1} is also in the boundary of the face bounded by C, so it too is just an edge.

If J is not incident with v_{i+7}, then the situation is precisely that Subclaim 1 with the roles of (e, w) and (e_w, x) interchanged. Therefore, $C_e \cup C$ is (e), as required.

Therefore, we may assume J is incident with v_{i+7}. At this point, we know that $s_{i+1}\, r_{i+5}, J$ and at least the edge e^x of s_{i+2} are contained in C_e. There is a C_e-bridge containing r_{i+6}; Claim 5 implies this C_e-bridge is precisely r_{i+6} and this is just an edge.

The cycle C has a second edge e' incident with v_{i+6}. There is a C_e-bridge B containing e'. Claim 5 implies B has precisely two attachments, namely v_{i+6} and some other vertex y.

If $y \in P_x - x^{e_w}$, then B together with the yv_{i+6}-subpath of $C_e - P_2'$ contains a cycle disjoint from P_2' and yet does not bound a face (it contains r_{i+6}). We know that $r_{i+5}\, r_{i+6}\, J$ bounds a face of $\Pi[G]$, so y is not in $J\, r_{i+5}$. Claim 4 implies $y \notin P_2' - x^{e_w}$. Thus, $y = x^{e_w}$.

To finish the proof that (d) occurs, note first that s_{i+1} and B are both edges; thus, it suffices to prove that P_2' is just an edge. In fact, Claim 3 implies $P_2'\, B\, s_{i+1}$ bounds a face of $\Pi[G]$. In particular, P_2 is not inside this face; therefore, $P_2' = P_2$. Consequently, $P_2' = P_2$ is just an edge. $\square$

In order to determine the 13 pictures, we remark that, from the perspective of both e and e_w, any of (1)-(5) in Definition 13.7 can occur. However, if (5) occurs for either, then Claim 3 implies C_e and this crossbar is all that is in $\mathfrak{D}_e$. In the cases (2)(4) and (3)(4), there are two possibilities, as the slope and the back-slope can have either distinct or common ends in the spoke; the latter is denoted by a $^+$ in the listing below. There is no third possibility, since the slope and back-slope do not cross in $\mathfrak{D}_e$. Thus, there are the 13 pictures (1)(1), (1)(2), (1)(3), (1)(4), (2)(2), (2)(3), (2)(4), (2)(4)$^+$, (3)(3), (3)(4), (3)(4)$^+$, (4)(4), and (5)(5).

Label the red edges in G as $e_0, e_1, \dots, e_{k-1}$ so that, for $i = 0, 1, \dots, k-1$, e_i has ends u_i and v_i and so that, reading indices modulo k, e_{i+1} is the v_i-consecutive red edge for e_i. This implies that e_i is the u_{i+1}-consecutive red edge for e_{i+1}.

Since there are no red edges between e_{i-1} and e_{i+1} on the "peak of Δ_{e_i}" portion of R, defining adjacency to mean "consecutive" shows the set of red edges make a cycle. Furthermore, v_i and u_{i+1} are both in the cycle C_{e_i} that determines the picture $\mathfrak{P}_i$ between e_i and e_{i+1}. Taking any $v_i u_{i+1}$-path P_i in $\mathfrak{P}_i$, we see that P_i together with either of the $v_i u_{i+1}$-subpaths of R makes a non-contractible cycle in $\mathbb{R}P^2$. In this sense, e_i and e_{i+1} are on opposite sides of R.

If we think of e_0 as being on "top" and e_1 on the "bottom", then e_2, e_4, ... are all on top and e_3, e_5 ..., are on the bottom. When we get back to e_0 from e_{k-1}, we have gone once around the Möbius strip, so e_0 is now on the bottom. It follows that e_{k-1} is on top and, therefore, $k-1$ is even, showing k is odd.

It follows that G contains a subgraph H that is in $\mathcal{T}(S)$. (There may be edges in the interior of C_e "between" the structures we identified "near" P_1' and P_2'.) However, Theorem 5.5 implies $H \in \mathcal{M}_2^3$, so we conclude $G = H$. That is, $G \in \mathcal{T}(S)$. ■

CHAPTER 14

Graphs that are not 3-connected

The rest of this work is devoted to: describing all the 2-crossing-critical graphs that are not 3-connected, discussed in this section; finding all 3-connected 2-crossing-critical graphs that do not contain a subdivision of V_8, treated in Section 15; and showing that the number of 3-connected 2-crossing-critical graphs that do not contain a subdivision of V_{2n} is finite, which is Section 16. These last two combine with the preceding work to show that there are only finitely many 3-connected 2-crossing-critical graphs to be determined, namely those that have a subdivision of V_8 but no subdivision of V_{10}.

In this section we show that every 2-crossing-critical graph that is not 3-connected is either one of a few known examples or is obtained from a graph in $\mathcal{M}_2^3$ by replacing 2-parallel edges with a "digonal" path (that is, a path in which every edge is duplicated). We remark that we continue assuming that the minimum degree is at least 3, as subdividing edges does not affect crossing number. We first determine all the 2-crossing-critical graphs that are not 2-connected.

14.1. 2-critical graphs that are not 2-connected

Since the crossing number is additive over components, any 2-crossing-critical graph can have at most two components, each of them equal to either $K_{3,3}$ or K_5. Thus, there are only three different such graphs: two disjoint copies of K_5, two disjoint copies of $K_{3,3}$, and disjoint copies of each.

Similarly, the crossing number is easily seen to be additive over blocks. Thus, the blocks of a connected, but not 2-connected, 2-crossing-critical graph must be 1-critical graphs, and therefore all such graphs can be obtained from the aforementioned disconnected 2-crossing-critical graphs by identifying two vertices from distinct components. The identified vertex may be a new vertex that subdivides some edge. For example, there are three possibilities in which both blocks are K_5: the identified vertex is a node in both, or only in one, or in neither. Likewise for $K_{3,3}$. There are four 2-crossing-critical graphs in which one block is a subdivision of K_5 and the other is a subdivision of $K_{3,3}$.

Proposition 14.1. *The thirteen graphs in Figure 14.1 are precisely those 2-crossing-critical graphs that are not 2-connected.*

14.2. 2-connected 2-critical graphs that are not 3-connected

In this subsection, we treat 2-crossing-critical graphs that are 2-connected, but not 3-connected. With 36 exceptions, these all arise from 3-connected 2-crossing-critical graphs that have *digons* (i.e., two edges with the same two ends). The digons may be replaced with arbitrarily long "digonal paths" — these are simply paths in which every edge is converted into a digon.

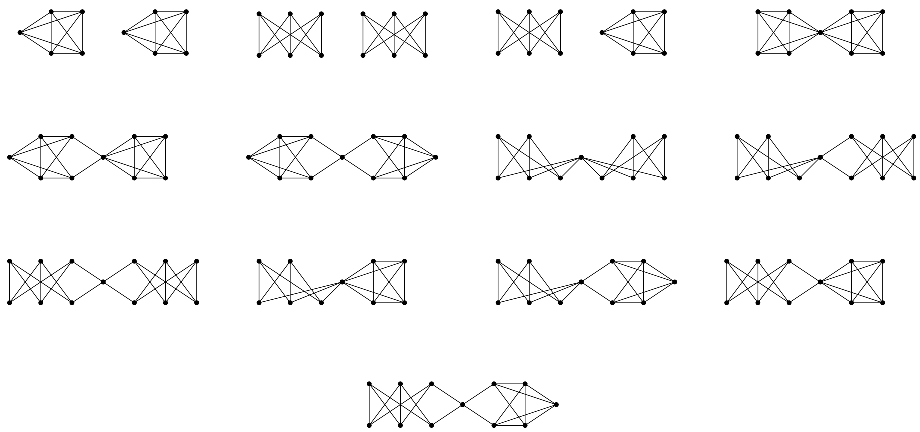

FIGURE 14.1. The 2-crossing-critical graphs that are not 2-connected.

Tutte [**34, 35**] developed a decomposition theory of a 2-connected graph into its *cleavage units*, which are either 3-connected graphs, cycles of length at least 4, or for $k \geq 4$, k-bonds (a *k-bond* is a graph with k edges, all having the same two ends). We provide here a brief review of this theory. A *2-separation* of a 2-connected graph G is a pair (H, K) of edge-disjoint subgraphs of G, each having at least two edges, so that $H \cup K = G$ and $H \cap K = \|\{u, v\}\|$ (recall $\|\{u, v\}\|$ is the graph with just the vertices u and v and no edges.). Notice that a 3-cycle and a 3-bond have no 2-separations and, therefore, are to be understood in this context to be 3-connected graphs.

The 2-separation (H, K) with $H \cap K = \|\{u, v\}\|$ is a *hinge-separation* if at least one of H and K is a $\|\{u, v\}\|$-bridge and at least one of them is 2-connected. Another way to say the same thing, but in terms of $H \cap K$: $\|\{u, v\}\|$ is a *hinge* if either there are at least three $\|\{u, v\}\|$-bridges, not all just edges, or there are exactly two $\|\{u, v\}\|$-bridges, at least one of which is 2-connected.

The theory of cleavage units develops as follows. Let G be a 2-connected graph.

(1) If $\|\{u, v\}\|$ is a hinge and (H, K) is a hinge-separation (possibly of another hinge), then there is some $\|\{u, v\}\|$-bridge containing either H or K.
(2) G has no hinge if and only if G is 3-connected, a cycle of length at least 4, or a k-bond, for some $k \geq 4$. (Recall that a 3-cycle and a 3-bond are 3-connected.) In each of these cases, G is its own cleavage unit.
(3) If (H, K) is a hinge-separation and $H \cap K = \|\{u, v\}\|$, then the cleavage units of G are the cleavage units of the two graphs $H + uv$ and $K + uv$ obtained from H and K by adding a *virtual edge* between u and v, respectively. This inductively determines the cleavage units.
(4) There is a decomposition tree T whose vertices are the cleavage units of G and whose edges are the virtual edges. A virtual edge joins in T the two cleavage units of G containing it.
(5) G contains a subdivision of each of its cleavage units.
(6) If G contains a subdivision of some 3-connected graph H, then some cleavage unit of G contains a subdivision of H.

In attempting to reconstruct G from its decomposition tree and its cleavage units, each time we combine two graphs along a virtual edge, there are two possibilities for how to identify the vertices of the corresponding hinge. This ambiguity will play a small role in constructing the 2-crossing-critical graphs that are 2- but not 3-connected.

It is easy to see that G is planar if and only if every cleavage unit is planar. (We could apply Kuratowski's Theorem and Item 6 or prove it more directly.) Since we are interested in non-planar graphs, there are two relevant possibilities: one or more than one of the cleavage units of G is not planar. We start by treating the latter case. We remark that the following discussion makes clear that the crossing number is not additive over cleavage units. Related discussions can be found in Širáň [**32**], Chimani, Gutwenger, and Mutzel [**10**] (but see [**5**] for significant comments about the latter), Beaudou and Bokal [**5**], and Leaños and Salazar [**21**].

LEMMA 14.2. *Let G be a 2-connected graph. If two cleavage units of G are not planar, then* $\mathrm{cr}(G) \geq 2$.

It is an important consequence that, if G is 2-crossing-critical, 2-connected, and has 2 non-planar cleavage units, then G is simple, i.e., has no digons.

Proof of Lemma 14.2. Among all 2-separations (H, K) of G, we choose the one that has K minimal so that both $H + uv$ and $K + uv$ are not planar, where $H \cap K = \|\{u, v\}\|$. If the crossing number of G is not at least 2, then $\mathrm{cr}(G) \leq 1$, so, by way of contradiction, suppose D is a 1-drawing of G.

Let P_K and P_H be uv-paths in K and H respectively. Since G contains the subdivision $H \cup P_K$ of $H + uv$, G is not planar. Therefore, D has a crossing. Evidently, $D(H \cup P_K)$ and $D(K \cup P_H)$ both contain the crossing. We conclude that the crossing in D is of an edge of P_H with an edge of P_K. It follows that there are not edge-disjoint uv-paths in either H or K and that the crossed edges are cut-edges in their respective subgraphs.

Let w and x be the ends of the edge in K that is crossed, labelled so that w is nearer to u in P_K than x is. Let K_u and K_v be the two components of $K - wx$, with the former containing u. Since $K + uv$ is not planar, either $K_u + uw$ or $K_v + vx$ is not planar. We may assume it is the former. Notice that $(H \cup K_v) + xu$ contains a subdivision of $H + uv$ and, therefore, is not planar. But then $((H \cup K_v), K_u)$ is a 2-separation contradicting the minimality of K. ∎

We are now in a position to determine the 2-connected, 2-crossing-critical graphs having two non-planar cleavage units.

THEOREM 14.3. *Let G be a 2-connected, 2-crossing-critical graph having two non-planar cleavage units. Then G is one of the 36 graphs in Figures 14.2 and 14.3.*

Proof. Let C_1 and C_2 be non-planar cleavage units of G.

CLAIM 1. G has at most three cleavage units: C_1, C_2 and possibly a 3- or 4-cycle; if there are three, then the 3- or 4-cycle is the internal vertex in the decomposition tree.

PROOF. For $i = 1, 2$, let $\{u_i, v_i\}$ be the hinge of G contained in C_i such that C_1 and C_2 are contained in different $\|\{u_i, v_i\}\|$-bridges. For any other virtual edge xy

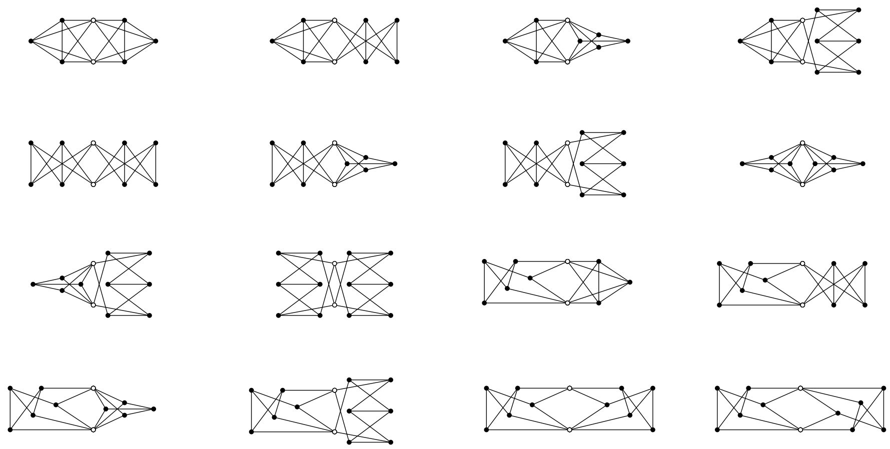

FIGURE 14.2. 2-connected, not 3-connected, 2-crossing-critical graphs, 2 non-planar cleavage units

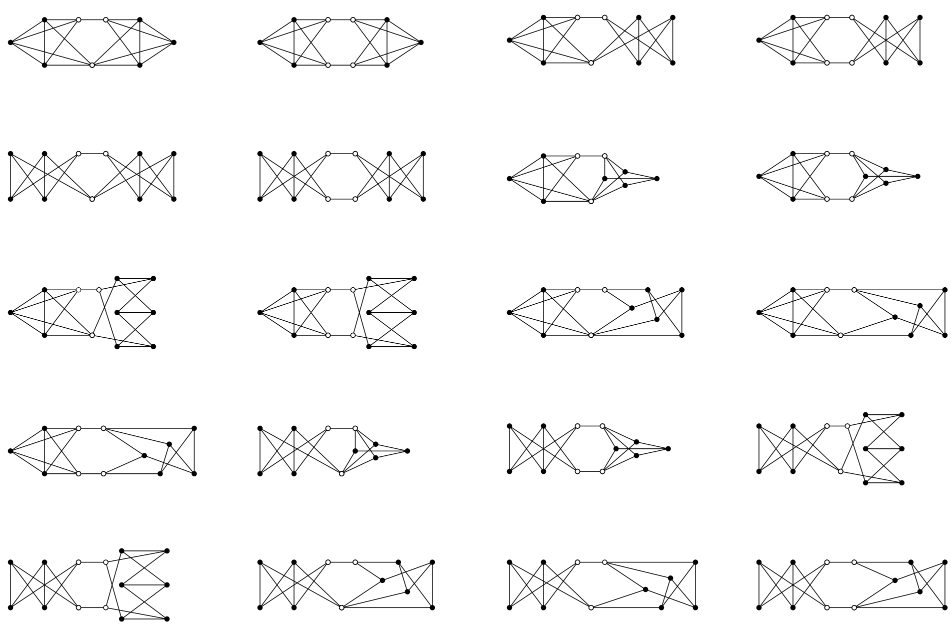

FIGURE 14.3. 2-connected, not 3-connected, 2-crossing-critical graphs, 3 cleavage units, 2 of which are non-planar.

in C_i, there is a path P_{xy} in G that is $C_1 \cup C_2$-avoiding. Let $\widetilde{C}_i$ be $C_i \cap G$ (i.e., C_i with none of its virtual edges) together with all these P_{xy}. Let H be the subgraph of G consisting of $\widetilde{C}_1 \cup \widetilde{C}_2 \cup Q$, where Q consists of two disjoint $\{u_1, v_1\}\{u_2, v_2\}$-paths in G. Evidently, H is 2-connected and C_1 and C_2 are cleavage units of H.

Lemma 14.2 implies $\mathrm{cr}(H) \geq 2$. Since $H \subseteq G$ and G is 2-crossing-critical, $H = G$. Since G has no vertices of degree 2, G consists of either two or three cleavage units, namely C_1, C_2, and possibly a 3- or 4-cycle between them. $\square$

We next determine the possibilities for C_1 and C_2.

CLAIM 2. For each $i = 1, 2$, one of the following occurs:
(1) C_i is K_5;
(2) C_i is $K_{3,3}$;
(3) $C_i - u_iv_i$ is a subdivision of $K_{3,3}$.

PROOF. Hall proved that every 3-connected non-planar graph is either K_5 or contains a subdivision of $K_{3,3}$ [**16**]. Since G is simple and C_i is 3-connected, we deduce that C_i is either K_5 or contains a subdivision of $K_{3,3}$. So suppose C_i contains a subdivision K of $K_{3,3}$.

Suppose $C_i - u_iv_i$ has an edge e for which $C_i - e$ is not planar. Since $C_i - e$ is 2-connected, $G - e$ is 2-connected and has at least two non-planar cleavage units (C_{3-i} and another contained in $C_i - e$). By Lemma 14.2, $\mathrm{cr}(G - e) \geq 2$, contradicting 2-criticality of G. So $C_i - u_iv_i \subseteq K$. Thus, either $C_i = K$ or $C_i - u_iv_i = K$, as claimed. □

CLAIM 3. There are five possibilities for C_i, namely:
(1) C_i is K_5;
(2) C_i is $K_{3,3}$;
(3) $C_i - u_iv_i$ is $K_{3,3}$ and u_iv_i joins two non-adjacent nodes of $K_{3,3}$;
(4) $C_i - u_iv_i$ is $K_{3,3}$ with one edge subdivided once and u_iv_i joins the degree 2 vertex to a node of $K_{3,3}$ that is not incident with the subdivided edge; and
(5) $C_i - u_iv_i$ is $K_{3,3}$ with two non-adjacent edges both subdivided once and u_iv_i joins the two degree 2 vertices.

PROOF. If C_i is neither K_5 nor $K_{3,3}$, then it must be a subdivision K of $K_{3,3}$ with the additional edge u_iv_i. Clearly K has at most two vertices of degree 2. If K has no vertices of degree 2, then, since C_i is simple, we have (3). Likewise, if K has only one vertex of degree 2, that vertex (one of u_i and v_i) cannot be in a branch incident with the other one of u_i and v_i, which is (4). Finally, suppose u_i and v_i are both of degree 2 in K. Then their containing branches cannot be incident with a common vertex w, as otherwise, we could delete the edge u_iw and still have two non-planar cleavage units, contradicting 2-criticality. This proves (5). □

Note that in all five cases of Claim 3, there is only one possibility for C_i, up to isomorphism. Only (4) has non-isomorphic labellings of u_i and v_i.

CLAIM 4. If G has just two cleavage units, then G is one of the 16 graphs in Figure 14.2.

PROOF. If neither C_1 nor C_2 is (4) from Claim 3, then, with repetition allowed, there are 10 possible unordered pairs for C_1 and C_2. Each of the pairs uniquely produces the graph G. There are four graphs having C_1 but not C_2 satisfying Claim 3 (4), and there are two graphs having both C_1 and C_2 satisfying Claim 3 (4). □

CLAIM 5. If G has three cleavage units, then at least one of C_1 and C_2 is either K_5 or $K_{3,3}$.

PROOF. Let e be an edge of G in the third cleavage unit of G; recall that this cleavage unit is either a 3- or a 4-cycle. The blocks of $G - e$ include $C_1 - u_1v_1$ and $C_2 - u_2v_2$; if these were both non-planar, then $\mathrm{cr}(G - e) \geq 2$, contradicting

2-criticality of G. Hence, at least one of $C_1 - u_1v_1$ and $C_2 - u_2v_2$ is planar. By Claim 3, such a one must be either K_5 or $K_{3,3}$. □

CLAIM 6. If G has three cleavage units, then G is one of the 20 graphs in Figure 14.3.

PROOF. There are three pairs in which both C_1 and C_2 are one of K_5 and $K_{3,3}$ and two possibilities for the third cleavage unit, yielding six graphs. Now suppose C_1 is one of K_5 and $K_{3,3}$ and C_2 is not. There are three possibilities for C_2 and two possibilities for the third bridge. However, when the third bridge is a 3-cycle, there are two ways to attach C_2 when it is of Type (4) from Claim 3. Thus, there are 6 graphs with the third cleavage unit a 4-cycle and 8 when it is a 3-cycle. □

From the claims, we see that the 36 graphs shown in Figures 14.2 and 14.3 are all the cases in which G is 2-connected, but not 3-connected, and has two non-planar cleavage units. ■

In the remaining cases of 2-connected, but not 3-connected, 2-crossing-critical graphs, there is only one non-planar cleavage unit C. The graph C is simple. The following result shows how to obtain G from a 3-connected 2-crossing-critical graph. It requires the following definition.

DEFINITION 14.4. A *digonal path* is a graph obtained from a path P by adding, for every edge e of P, an edge parallel to e.

THEOREM 14.5. *Let G be a 2-crossing-critical graph with minimum degree at least 3. Suppose that G is 2-connected but not 3-connected and has only exactly one non-planar cleavage unit, C. The graph $\widetilde{C}$ obtained from C by replacing each of its virtual edges with a digon is 2-crossing-critical and 3-connected. The graph G is recovered from $\widetilde{C}$ by replacing these virtual edge pairs by digonal paths.*

Proof. That $\widetilde{C}$ is 3-connected is a trivial consequence of the fact that C is 3-connected.

As for the rest, let uv be a virtual edge in C. Then $\|\{u,v\}\|$ is a hinge of G. We consider the $\|\{u,v\}\|$-bridges in G; let B_{uv} be the one that contains $C \cap G$, and let $B^{\#}_{uv}$ be the union of the remaining $\|\{u,v\}\|$-bridges. We have two objectives: to show that $\widetilde{C}$ is 2-crossing-critical and that, for each uv, $B^{\#}_{uv}$ is a digonal uv-path.

For the former, we first show $\operatorname{cr}(\widetilde{C}) \geq 2$. Otherwise $\widetilde{C}$ has a 1-drawing D. Obviously no edge in a digon of $\widetilde{C}$ is crossed in D. For each virtual edge uv of C, $B^{\#}_{uv} + uv$ is planar, so it may be inserted into D in place of the uv-digon in D to obtain a 1-drawing of G, which is a contradiction. Therefore, $\operatorname{cr}(\widetilde{C}) \geq 2$.

We next claim that each $B^{\#}_{uv}$ consists of digonal uv-paths. Assume first that $B^{\#}_{uv}$ has a cut-edge e separating u and v. Since G has no vertices of degree 2 and $B^{\#}_{uv}$ is not just a single edge, $B^{\#}_{uv}$ contains some edge e' so that $B^{\#}_{uv} - e'$ still contains a uv-path.

If no edge of $B^{\#}_{uv}$ is crossed in a 1-drawing $D_{e'}$ of $G - e'$, then, since $B^{\#}_{uv} - e'$ contains a uv-path, $B^{\#}_{uv}$ may be substituted for $B^{\#}_{uv} - e'$ in $D_{e'}$ to obtain a 1-drawing of G, which is impossible. So some edge of $B^{\#}_{uv}$ is crossed in $D_{e'}$. Deleting edges from $B^{\#}_{uv} - e'$ to leave only a uv path shows that $D_{e'}$ restricts to a 1-drawing of $B_{uv} + uv$ in which there is at most one crossing; if there is a crossing, then uv is crossed. Since every planar embedding of $B^{\#}_{uv} + uv$ has uv and e on the same face, the 1-drawing of $B_{uv} + uv$ and a planar embedding of $B^{\#}_{uv} + uv$ may be merged to produce a 1-drawing of G in which e is crossed. This contradiction that shows $B^{\#}_{uv}$ contains edge-disjoint uv-paths.

Let e be an edge of $B^{\#}_{uv}$. Then a 1-drawing D_e of $G - e$ must have a crossing of some edge e' of $B^{\#}_{uv}$. If $B^{\#}_{uv} - \{e, e'\}$ has a uv-path P, then D_e restricts to a planar embedding of C by using P to represent uv. But C is non-planar, so every edge of $B^{\#} - uv$ is in an edge-cut of size at most 2 separating u and v. Combining this with the preceding paragraph shows that every edge of $B^{\#}_{uv}$ is in an edge-cut of size exactly 2. It is an easy exercise to see that this implies $B^{\#}_{uv}$ is a pair of digonal paths.

We conclude by showing that, for every edge e of $\widetilde{C}$, $\operatorname{cr}(\widetilde{C} - e) \le 1$. Suppose first that e is not in a digon. Each $B^{\#}_{uv}$ has a uv-path P_{uv} that is clean in D_e. Thus, $D_e[G - e]$ contains a subdivision of $C - e$ in which no virtual edge (represented in the subdivision by P_{uv}) is crossed. Therefore, the virtual edges may be replaced with digons to give a 1-drawing of $\widetilde{C} - e$, as claimed.

Now suppose e is in the uv-digon. Let e' be any edge of $B^{\#}_{uv}$. Then $D_{e'}$ contains a 1-drawing of C, in which every other virtual edge wx is represented by a wx-path P_{wx} in $B^{\#}_{wx}$ that is clean in $D_{e'}$. All these other virtual edges may be replaced with digons to give a 1-drawing of $\widetilde{C} - e$, as required. ■

CHAPTER 15

On 3-connected graphs that are not peripherally-4-connected

In this chapter, we reduce the problem of finding all 3-connected 2-crossing-critical graphs to the consideration of non-planar, peripherally-4-connected graphs. Our motivation for doing this is to use a known characterization of internally-4-connected graphs (a concept intimately related to peripherally-4-connected graphs) with no subdivision of V_8 to find all the 3-connected, 2-crossing-critical graphs with no subdivision of V_8.

DEFINITION 15.1. A graph G is *peripherally-4-connected* if G is 3-connected and, for any 3-cut S of G and any partition of the components of $G - S$ into two non-null subgraphs H and K, at least one of H and K has just one vertex.

We begin this section by finding the four 3-connected, not peripherally-4-connected, 2-crossing-critical graphs that are not obtained from planar substitutions into a peripherally-4-connected graph. The bulk of the section is devoted to explaining in detail how to obtain the remaining 3-connected 2-crossing-critical graphs from peripherally-4-connected graphs. Finally, this theory is used to explain how to find all the 3-connected 2-crossing-critical graphs that do not contain a subdivision of V_8.

15.1. A 3-cut with two non-planar sides

In this section we find the four 3-connected, not peripherally-4-connected, 2-crossing-critical graphs that are not obtained by substituting planar pieces into degree-3 vertices in a peripherally-4-connected graph (this substitution process being the remainder of the section). We start by describing the four graphs and showing that they are 2-crossing-critical.

DEFINITION 15.2. The graph $K^*_{3,4}$ is obtained from disjoint copies of $K_{2,3}$ by joining the parts of the bipartition having three vertices in each of the copies by a perfect matching M.

Observe that $K_{3,4}$ is obtained from $K^*_{3,4}$ by contracting all the edges of the matching M. The following generalizes the well-known fact that $K_{3,4}$ is 2-crossing-critical.

LEMMA 15.3. *If H is obtained from $K^*_{3,4}$ by contracting some subset of M, then H is 2-crossing-critical.*

Proof. Suppose e is an edge of $K^*_{3,4}$ not in M. Then there is a 1-drawing of $K^*_{3,4} - e$ in which no edge of M is crossed. Thus, $\mathrm{cr}(H - e) \leq 1$. If $e \in M$, then $H - e$ is planar. It remains to show $\mathrm{cr}(H) \geq 2$.

Suppose to the contrary that H has a 1-drawing D. Let H_1 and H_2 be the $K_{2,3}$ subgraphs of H contained in $K^*_{3,4} - M$. For each vertex v of degree 3 in H_2, there are three disjoint vH_1-paths in H; adding v and these paths to H_1 yields a subdivision H_v of $K_{3,3}$ in H. Thus, $D[H_v]$ has a crossing, and, since there are two choices for v, this crossing involves only edges of H_1 and M.

Interchanging the roles of H_1 and H_2 shows the crossing in D involves only edges of M. But then $D[H_v]$ has its only crossing on branches incident with v, which is impossible. ■

We remark that there are splits of $K_{3,4}$ that have crossing number 1 — split two of the degree 4 vertices so that the two partitions of the four neighbors are different. Fortunately, they do not occur in our context.

In order to show that these are the only four graphs with “non-planar 3-cuts”, we need to understand just what “non-planar 3-cuts” are.

Definition 15.4. Let S be a 3-cut in a 2-connected graph, so there are subgraphs H and K of G such that $G = H \cup K$ and $H \cap K = \|S\|$. For $L \in \{H, K\}$, L^+ denotes the graph obtained from L by the addition of a new vertex adjacent to precisely the vertices in S.

We will see that, in the case G is 2-crossing-critical, with the exception of $K_{3,4}$, there are at most three non-trivial S-bridges, and so at least one of H and K is an S-bridge. Our next goal is to show that the four graphs in Lemma 15.3 are the only four that have both H^+ and K^+ non-planar. We start with the following, which is likely well-known; however, we could not find a reference. It extends Hall's Theorem [**16**] that there is a subdivision of $K_{3,3}$.

Lemma 15.5. *Let G be a 3-connected non-planar graph different from K_5 and let v be a vertex of G. Then G has a subdivision H of $K_{3,3}$ in which v is an H-node.*

Proof. Here is an outline of the easy, but tedious, proof. As a first step, we show that there is a subdivision of $K_{3,3}$ containing v. By Hall's Theorem [**16**], G contains a subdivision L of $K_{3,3}$. If $v \notin L$, then there are three disjoint vL-paths. There are three possibilities for the ends of these paths in L: two are in the same closed L-branch; two are in L-branches incident with a common L-node; and the L-branches containing the ends of the paths are pairwise disjoint. In the first case, v is incorporated into the interior of a branch of a new subdivision of $K_{3,3}$, while in the other cases, v is incorporated as a node of the new subidivision of $K_{3,3}$.

So now assume that v is in L, but not as a node. Then v is interior to some L-branch b with ends u and w. Let $L' = L - \langle b \rangle$ — this is a subdivision of $K_{3,3}$ less an edge. Because there are, in G, disjoint L'-avoiding $v\{u, w\}$-paths, standard proofs of Menger's Theorem imply that there are three disjoint L'-avoiding vL'-paths, having u and w among their three L'-ends. Therefore, we may assume not only is v in the interior of b, but there is an L-avoiding vx-path from v to some other vertex x of L'.

Up to symmetry, there are three possibilities for x: it is a node of L other than u and w; it is interior to an L-branch incident with u but not with w; and it is interior to an L-branch not incident with either u or w. Let y and z be nodes of L' (note that u and w are not actually nodes of L'). We can assume x is either y, or in the L-branch $[w, y]$, or in the L-branch $[y, z]$. Let Y be a $\{u, w, x\}$-claw with centre v, so that $Y \cap L'$ is just u, w, and x.

If x is either z or in $\langle y, z\rangle$, then $(L' \cup Y) - \langle w, x\rangle$ is a subdivision of $K_{3,3}$ with v as a node. If x is in $\langle y, z\rangle$, then $(L' \cup Y) - \langle w, y\rangle$ is a subdivision of $K_{3,3}$ having v as a node. ∎

We are now ready for the classification of the 3-connected 2-crossing-critical graphs with two non-planar sides to a 3-cut.

THEOREM 15.6. *Let $G \in \mathcal{M}_2^3$ have subgraphs H and K of G and a set S of three vertices of G such that:*

(1) $G = H \cup K$;
(2) $H \cap K = \|S\|$;
(3) H and K both have an $\|S\|$-bridge having all of S as attachments; and the two graphs H^+ and K^+ are both non-planar.

Then G is one of the four graphs obtained from $K_{3,4}^*$ by contracting some subset of M.

Proof. Let u, v, and w be the vertices in S. For $L \in \{H, K\}$, let v_L^+ denote the vertex in L^+, but not in L. The graph L^+ is a subdivision of a 3-connected graph (the only possible vertices of degree 2 are u, v, and w). Since L^+ is not planar and has a vertex of degree 3, it is not a subdivision of K_5 and, therefore, by Lemma 15.5 contains a subdivision L' of $K_{3,3}$ in which v_L^+ is a node. Now $G' = (H' - v_H^+) \cup (K' - v_K^+)$ is a subdivision of $K_{3,4}^*$, with some subset of M contracted. By Lemma 15.3, $\operatorname{cr}(G') = 2$, so $G' = G$, as required. ∎

15.2. 3-reducing to peripherally-4-connected graphs

In this section, we discuss the general details of reducing a 3-connected graph to a peripherally-4-connected graph. These results apply in some generality and not just in the context of 2-crossing-critical graphs. These are the first of several steps toward finding all the 3-connected 2-crossing-critical graphs that do not contain a subdivision of V_8.

These results are fairly technical but essential to this part of the theory.

DEFINITION 15.7. (1) A 3-cut S in a 3-connected graph is *reducible* if $G - S$ has at most 3 components and they partition into two subgraphs each having at least two vertices.
(2) The set $\mathfrak{K}$ consists of those 3-connected graphs that do not contain a subdivision of $K_{3,4}$.

The following result is obvious from the definitions and begins to explain the appearance of $K_{3,4}$ in Definition 15.7 (2).

LEMMA 15.8. *Let G be a 3-connected graph that is not peripherally-4-connected. Then either G has a reducible 3-cut or G has $K_{3,4}$ as a subgraph.*

The next result sets up the basic scenario that we will use throughout our reduction to peripherally-4-connected graphs.

LEMMA 15.9. *Let $G \in \mathfrak{K}$. Then there is a sequence $G_0, G_1, \ldots, G_k$ of 3-connected graphs in $\mathfrak{K}$ so that: $G_0 = G$; G_k is peripherally-4-connected; and, for each $i = 1, 2, \ldots, k$, there is a 3-cut S_i in G_{i-1} and a non-trivial, planar S_i-bridge B_i so that $\operatorname{Nuc}(B_i)$ has at least two vertices and G_i is obtained from G_{i-1} by contracting the nucleus of B_i.*

Proof. Suppose G_{i-1} is 3-connected. Among all the choices of S_i and S_i-bridges B_i so that $\mathrm{Nuc}(B_i)$ has at least two vertices, choose B_i to be inclusion-wise maximal. We claim that the graph G_i obtained from G_{i-1} by contracting $\mathrm{Nuc}(B_i)$ to a vertex is 3-connected.

Otherwise, there is some pair $\{u,v\}$ of vertices so that $G_i - \{u,v\}$ is not connected. If the vertex of contraction of $\mathrm{Nuc}(B_i)$ is neither u nor v, then $\{u,v\}$ is a 2-cut in G_{i-1}, a contradiction. Therefore, we can assume u is the contraction of $\mathrm{Nuc}(B_i)$.

Let H and K be components of $G_i - \{u,v\}$, with the labelling chosen so that $|S_i \cap V(H)| \geq |S_i \cap V(K)|$; in particular, $|S_i \cap V(K)| \leq 1$. Let $h \in V(H)$; if there is a vertex $k \in V(K) \setminus S_i$, then $\{v\} \cup (S_i \cap V(K))$ separates k from h in G_{i-1}, which contradicts the assumption that G_{i-1} is 3-connected.

Therefore $V(K) \subseteq S_i$, so there is a single vertex s in K, and $s \in S_i$. It follows that s is adjacent to only vertices in $\mathrm{Nuc}(B_i)$ and possibly to v. But this contradicts the maximality of B_i: let $S' = (S \setminus \{s\}) \cup \{v\}$. Observe that $B_i + s$ is a planar S'-bridge, contradicting maximality of B_i.

Lastly, we show that if G_{i-1} does not have a subdivision of $K_{3,4}$, then neither does G_i. Any subdivision of $K_{3,4}$ in G_i must contain the vertex v_i of contraction. Since v_i has degree 3 in G_i and B_{i-1} is an S-bridge, we can reroute the subdivision of $K_{3,4}$ in G_i into B_{i-1} to obtain a subdivision of $K_{3,4}$ in G_{i-1}. ■

Definition 15.10. Let $G \in \mathfrak{K}$.

(1) Then G *reduces to* G' *by 3-reductions* if there is a sequence $G_0, G_1, \ldots, G_k$ of 3-connected graphs so that $G_0 = G$; $G_k = G'$; and, for each $i = 1, 2, \ldots, k$, there is a 3-cut S_i in G_{i-1} and an S_i-bridge B_i, whose nucleus at least two vertices, so that G_i is obtained from G_{i-1} by contracting the nucleus of B_i.
(2) For each vertex v of G' and each $i = 0, 1, 2\ldots, k$, K_v^i denotes the connected subgraph of G_i that contracts to v. We also set $K_v = K_v^0$.
(3) If v has just three neighbours x, y, and z in G', then G_v is the graph obtained from K_v by adding x, y, and z, and, for each $t \in \{x,y,z\}$ and each edge $v't'$ of G with $v' \in K_v$ and $t' \in K_t$, adding the edge $v't$.

We now commence a lengthy series of technical lemmas that all play vital roles in usefully reducing the 3-connected graph 2-crossing-critical graph G to a smaller 3-connected 2-crossing-critical graph $G_{\mathrm{rep}(v)}$. The culmination of this part of the work is Theorem 15.25 in the next section, showing that $G_{\mathrm{rep}(v)}$ is 2-crossing-critical. This will lead to a program for determining all the 3-connected 2-crossing-critical graphs that reduce to a particular peripherally-4-connected graph.

Lemma 15.11. *Let $G \in \mathfrak{K}$ and suppose G reduces by 3-reductions to the peripherally-4-connected graph G^{p4c}. For any two vertices u, v of G^{p4c}, there is a single vertex in G incident with all edges having one end in K_u and one end in K_v.*

Proof. Let $G = G_0, G_1, \ldots, G_k = G^{\mathrm{p4c}}$ be a sequence of 3-reductions. Choose i to be largest so that there are disjoint $K_u^{i-1}K_v^{i-1}$-edges ab and cd with $a, c \in K_u^{i-1}$ and $b, d \in K_v^{i-1}$. In G_i, either a and c have been identified or b and d have; by symmetry, we may assume the former.

The vertices b and d are obviously attachments of B_i and so these are in S_i. Let z_i be the third vertex in S_i. Since K_u^{i-1} is connected and since, by Definition

15.10, u has three neighbours in G^{p4c}, $z_i \in K_u$. Continue using the label a for the vertex obtained by contracting $\mathrm{Nuc}(B_i)$.

At some point in the later 3-reductions, a and z_i are identified and at another point b and d are identified. We show that neither can be done before the other, which is impossible.

Suppose z_i and a are identified first. When this identification occurs, a 3-cut S_j and an S_j-bridge B_j so that z_i and a are in $\mathrm{Nuc}(B_j)$. The vertices b and d are again attachments of B_j and so are in S_j; let z_j be the third vertex in S_j.

Because i is largest so there are disjoint $K_u^{i-1}K_v^{i-1}$-edges, all edges between K_u^j and K_v^j at this moment are incident with a. It follows that $\{a, z_j\}$ is a 2-cut in the current graph, separating z_i from b. But this contradicts the fact that G_{j-1} is 3-connected. Therefore, z_i and a are not identified before b and d.

On the other hand, suppose b and d are identified first, by the contraction of $\mathrm{Nuc}(B_j)$. When b and d are identified, the only neighbours of a are b, d, and z_i. Following the identification of b and d, the only neighbours of a are z_i and the vertex of identification, again contradicting 3-connection of G_j. ■

We need a slight variation on a standard definition.

Definition 15.12. Let G be a connected graph.

(1) An *isthmus* is a set I of parallel edges so that $G - I$ is not connected.
(2) A *cut-edge* is an edge e so that $G - e$ is not connected.

Obviously, e is a cut-edge of G if and only if $\{e\}$ is an isthmus, but an isthmus may have more than one edge. The distinction comes into play because at various points we will consider edge-disjoint paths in certain subgraphs of our 2-crossing-critical graph; if there are not two edge-disjoint uv-paths, then there is a cut-edge separating u and v. On the other hand, the 3-connection of G does not preclude the possibility of parallel edges; at several points we will be able to identify that two vertices u and v have the property that they must be adjacent, but be unable to distinguish whether they are joined by 1 or 2 edges. A common scenario will have the set of edges between them making an isthmus in some subgraph.

In particular, the case that K_v has an isthmus is a central one in reducing 2-crossing-critical graphs.

Lemma 15.13. *Let $G \in \mathfrak{K}$ reduce to the peripherally-4-connected graph G^{p4c} by a sequence of 3-reductions. Suppose there is a vertex v of G^{p4c} so that the graph K_v has an isthmus I. Then, for each component K of $K_v - I$, there are at least two neighbours x and y of v in G^{p4c} so that there are KK_x- and KK_y-edges in G.*

Proof. At some moment in the reduction of G, G_{i-1} has a 3-cut S_i and B_i is the planar S_i-bridge in G_{i-1} that contains I. Then $B_i - I$ is not connected; the ends u and w of the edge or edges in I are in different components K and L, respectively, of $B_i - I$.

Let x, y, and z be the neighbours of v in G^{p4c} and let t be any vertex of G_{i-1} not in $K_v^i \cup K_x^i \cup K_y^i \cup K_z^i$. (Since G^{p4c} is not planar, it has at least five vertices.) In G_{i-1} there are three pairwise internally-disjoint ut-paths. These three paths leave B_i through distinct attachments of B_i; these are the vertices in S_i. The same argument applies for wt-paths.

In particular, two of the ut-paths leave K on edges incident with vertices in S_i. Likewise for L. Therefore, K and L are both joined by edges to the same attachment $s \in S_i$. It follows that s is not in K_v^i, so s is in K_x^i, say. Moreover, since the K_v^i-ends of these two edges are not the same, Lemma 15.11 implies all the edges between K_v^i and K_x^i are incident with s.

Since G_{i-1} is 3-connected, $G_{i-1} - (\{s\} \cup I)$ is connected. Therefore, there are edges of G_{i-1} leaving each of K and L; each of these edges is also leaving K_v^i and, therefore, has its other end in one of K_x^i, K_y^i, and K_z^i. However, this other end cannot be s and, consequently, cannot be in K_x^i, as required. ∎

The connectivity of G has further implications about the structure of the K_v.

Lemma 15.14. *Let $G \in \mathfrak{K}$ reduce by 3-reductions to a peripherally-4-connected graph G^{p4c}. Let v be a vertex of G^{p4c} with just the three neighbours x, y, and z and suppose K_v has at least two vertices. For each $t \in \{x, y, z\}$, let t' be any vertex incident with all the K_vK_t-edges. Then x', y', and z' are all distinct.*

Proof. Suppose $x' = y'$. Then x' is in K_v. Observe that no vertex of $K_v - \{x', z'\}$ is adjacent to any vertex of of $G - \{x', z'\}$ not in K_v. Since G is 3-connected, it follows that K_v consists of just x' and z'. In particular, $z' \neq x'$. Also, recall that K_v contracts to a single vertex in the sequence of planar 3-reductions.

At the moment of contraction of K_v, G_{i-1} is 3-connected and $x'z'$ is an isthmus. Therefore, Lemma 15.13 implies that z' is joined to at least one of K_x^i and K_y^i; this contradicts the fact that all edges from K_v^i to $K_x^i \cup K_y^i$ are incident with x'. ∎

The vertices x', y', and z' are not uniquely determined. It is possible that there is only one vertex in each of K_v and K_x incident with all K_vK_x-edges; one obvious instance is if there is only one K_vK_x-edge. We will follow up on this a little later.

Here is a very simple and very useful observation.

Lemma 15.15. *Let H be a simple, non-planar, peripherally-4-connected graph. There is no 3-cycle of H having two vertices with just 3 neighbours.*

Proof. Suppose to the contrary there are three vertices x, y, z making a 3-cycle, with x and y having only three neighbours each. Let v and w be the other neighbours of x and y. Then x and y are the vertices of one component of $H - \{v, w, z\}$.

Observe that H is non-planar, 3-connected, and has a vertex of degree 3. Therefore H is not K_5 and so contains a subdivision of $K_{3,3}$. It follows that H has at least six vertices. Thus, there is another component of $H - \{v, w, z\}$.

Since H is peripherally-4-connected, the only possibility is that there is exactly one other component and it consists of a single vertex u, adjacent to all of v, w, and z. The only other possible edges in H are between v, w, and z. However, the resulting graph is planar, a contradiction. ∎

The following result assures us that useful (and expected) paths exist in each K_v.

Lemma 15.16. *Let:*

(1) $G \in \mathfrak{K}$ reduce by 3-reductions to the peripherally-4-connected graph G^{p4c};
(2) G^{p4c} have at least five vertices;
(3) v be a vertex of G^{p4c} so that K_v has at least two vertices; and
(4) x, y, and z be the neighbours of v in G^{p4c}, with corresponding vertices x', y', and z' in G as in Lemma 15.14.

Then:

a) *for any vertex w in $K_v - \{x', y', z'\}$, there are three $w\{x', y', z'\}$-paths in G_v that are pairwise disjoint except for w; and*
b) *if $x' \in K_v$, then there are $x'y'$- and $x'z'$-paths in $G_v - x$ that are disjoint except for x'.*

Proof. For a), let u be any vertex of G not in $K_v \cup K_x \cup K_y \cup K_{t_3}$. Since G is 3-connected, there are three pairwise internally-disjoint wu-paths in G. The result follows from the observation that w and u are in different components of $G - \{x', y', z'\}$.

If b) fails, then there is a vertex w of $G_v - x$ that separates x' from $\{y', z'\}$. Since K_v is an $\|\{x, y, z\}\|$-bridge in G_v, w is in K_v (possibly $w = y'$ or $w = z'$). Since $\{x', w\}$ is not a 2-cut in G, x' and w are adjacent in K_v. But now they are joined by an isthmus I in K_v. Since x' is a component of $K_v - I$ joined only to K_x, we have a contradiction of Lemma 15.13. ■

15.3. Planar 3-reductions

In this section we now turn our attention to the particular case $G \in \mathcal{M}_2^3$. We want to show that the 3-reductions can be taken to be contractions of planar bridges. So suppose S is a non-peripheral 3-cut in G.

If there are four or more non-trivial S-bridges (that is, having a nucleus), then G has a subdivision of $K_{3,4}$ and so is $K_{3,4}$. In the remaining cases, there are at most three non-trivial S-bridges. If there are three and B is one of them so that B^+ is not planar (as in Subsection 15.1), then the union K of the remaining S-bridges has K^+ not planar. Theorem 15.6 implies that G is one of four 2-crossing-critical graphs. Thus, if there are three non-trivial S-bridges, we may assume that, for each one B, B^+ is planar. Finally, consider the case that there are precisely two non-trivial S-bridges B_1 and B_2. Since S is not peripheral, both B_i have at least two vertices. If both B_i^+ are non-planar, then we are in the case dealt with in Theorem 15.6, so we may assume that one of them is planar. In summary, in every case, we may assume that G^{p4c} is obtained from 3-reductions in G in which the contracting S_i-bridge B_i is always planar.

Definition 15.17. Let G be a 3-connected graph and let G^{p4c} be a peripherally-4-connected graph. Then G *reduces to G^{p4c} by planar 3-reductions* if there is a sequence $G = G_0, G_1, G_2, \dots, G_k = G^{\mathrm{p4c}}$ of 3-reductions so that, for each $i = 1, 2, \dots, k$, G_i is obtained from G_{i-1} by contracting $\mathrm{Nuc}(B_{i-1})$ and B_{i-1}^+ is planar.

We need two results about K_v in the context of planar 3-reductions. This requires further definitions.

Definition 15.18. Let G be a 3-connected graph that reduces by 3-reductions to the peripherally-4-connected graph G^{p4c}. Suppose v is a vertex of G^{p4c} having only the neighbours x, y, and z. For each $t \in \{x, y, z\}$, let m_t denote the number of vertices in K_v adjacent to vertices in K_t and let n_t denote the number of vertices in K_t adjacent to vertices in K_v. (Lemma 15.11 implies that at least one of m_t and n_t is 1.)

(1) The subgraph $K_v^{\max}$ induced by K_v together with, for each $t \in \{x, y, z\}$ with $n_t = 1$, the vertex of K_t adjacent to vertices in K_v.

(2) The subgraph $K_v^{\min}$ induced by K_v together with, for each $t \in \{x, y, z\}$ with $m_t > 1$, the vertex of K_t adjacent to vertices in K_v.

We remark that $K_v \subseteq K_v^{\min} \subseteq K_v^{\max}$, and, for $t \in \{x, y, z\}$, $K_v^{\max}$ has a vertex $t' \in K_t$ that is not in $K_v^{\min}$ precisely when $n_t = m_t = 1$.

LEMMA 15.19. *Let $G \in \mathfrak{K}$ reduce by 3-reductions to a peripherally-4-connected graph G^{p4c}. Let v be a vertex of G^{p4c} with just the three neighbours x, y, and z and suppose K_v has at least two vertices. Then there is a cycle C in $K_v^{\min}$ containing all of x', y' and z'.*

Proof. Suppose w is a cut-vertex of $K_v^{\min}$, so there are subgraphs X and Y of $K_v^{\min}$ with $X \cup Y = K_v^{\min}$, $X \cap Y = \|w\|$, and both $X - w$ and $Y - w$ are not empty. We may choose the labelling so that X has at least the two vertices x' and z' from $\{x', y', z'\}$, while $Y - w$ has at most one; we may further assume $x' \neq w$. If $y' \notin Y - w$, then w is a cut-vertex of G, contradicting the fact that G is 3-connected. Therefore, $y' \in Y - w$.

However, if $y' \in K_v$, then we have a contradiction to Lemma 15.16 (b). Therefore, $y' \notin K_v$. If there is a vertex in Y other than w and y', then we contradict 3-connection of G, so y' is adjacent only to w in G_v. But then $y' \notin K_v^{\min}$.

It follows that there is no cut-vertex in $K_v^{\min}$. Thus, there is a cycle C in $K_v^{\min}$ containing x' and y'. Obviously, we are done if $z' \in C$, so we assume $z' \notin C$.

Since there is no cut-vertex in $K_v^{\min}$, there are two $z'C$-paths P_1 and P_2 that are disjoint except for z'. If the C-ends of P_1 and P_2 are not both on the same $x'y'$-subpath of C, then G_v^+ contains a subdivision of $K_{3,3}$. This contradicts the fact that we are doing planar 3-reductions. Therefore, the C-ends of P_1 and P_2 are on the same $x'y'$-subpath of C and it is easy to find the desired cycle through all of x', y', and z'. ■

The following is the last lemma we need to get the main result of this section.

LEMMA 15.20. *Let $G \in \mathcal{M}_2^3$ and suppose G reduces by planar 3-reductions to the peripherally-4-connected graph G^{p4c}. Let v and x be adjacent vertices in G^{p4c}. Then there are at most two vertices in K_v adjacent to vertices in K_x.*

Proof. This is obvious if K_v has at most one vertex. In the remaining case, v has degree 3 in G^{p4c}; let y and z be its other neighbours.

Suppose by way of contradiction that s, t, and u are distinct vertices in K_v all adjacent to vertices in K_x. By Lemma 15.11, there is a vertex x' incident will all the K_vK_x-edges and, evidently, $x' \in K_x$.

In the planar embedding D_v^+ of G_v^+, letting w denote the new vertex adjacent to each of x, y, and z, we may choose the labelling so that the edges xw, xs, xt, xu occur in this cyclic order around x.

CLAIM 1. There is an su-path in K_v containing t.

PROOF. As K_v is connected, there is an su-path P in K_v. We are obviously done if $t \in P$, so we assume $t \notin P$. Let C be the cycle obtained by adding x' to P and joining it to s and u.

The rotation at x implies that t is on one side of $D_v^+[C]$, while w, y, and, consequently, z, are on the other. Therefore, every $t\{y, z\}$-path in G_v^+ goes through either x or P.

If there is a cut-vertex r in K_v separating t from P, then $\{r, x'\}$ is a 2-cut in the 3-connected graph G, which is impossible. Therefore, there are tP-paths Q and R in K_v that are disjoint except for t. We can now reroute P through t to obtain the desired path. □

Since G is 2-crossing-critical, there is a 1-drawing D of $G - x't$. From Claim 1, there is an su-path P in K_v containing t. Let C be the cycle obtained from P by adding x', $x's$ and $x'u$.

CLAIM 2. All the vertices of $G - (K_v \cup K_x)$ are in the same face of $D[C]$.

PROOF. Suppose by way of contradiction that there are vertices in $G - (K_v \cup K_x)$ that are in different faces of $D[C]$.

Case 1: *there is a vertex p in G^{p4c} so that K_p contains vertices that are in different faces of $D[C]$.*

In this case there is an edge f of K_p that crosses $D[C]$. As D has at most one crossing, f is a cut-edge of K_p. Lemma 15.13 implies each component of $K_p - f$ is adjacent to at least two different K_n's. If one of them is adjacent to both K_x and K_v, then we have a 3-cycle pxv in G^{p4c} in which both p and v have degree 3, contradicting Lemma 15.15.

Therefore, we may assume each is adjacent to one, say K_q and K_r, that is neither K_x nor K_v. However, now $\{v, x, p\}$ is a 3-cut in G^{p4c} separating q and r in G^{p4c}. Therefore one of them — say q — is adjacent to precisely these three vertices in G^{p4c}, producing the 3-cycle $\{q, v, x\}$ in G^{p4c} that contradicts Lemma 15.15.

Case 2: *any two vertices of $G - (K_v \cup K_x)$ in different faces of $D[C]$ are in different K_p's.*

Since $G - (K_v \cup K_x)$ is connected, there is a path in $G - (K_v \cup K_x)$ joining vertices in different faces of $D[C]$. Therefore, there is, for some vertices q and r of G^{p4c}, a K_qK_r-edge f that crosses $D[C]$. It follows that $D[C]$ has no self-crossings, so $D[C]$ has only two faces.

Clearly $G^{\mathrm{p4c}} - \{x, v, f\}$ has K_q and K_r in different components. Since G^{p4c} has at least six vertices, it has a vertex m different from all of v, x, q and r. We may choose the labelling so that $D[K_q]$ is in one face of $D[C]$, while $D[K_r \cup K_m]$ is contained in the other. It follows that $\{v, x, r\}$ is a 3-cut in G^{p4c} separating q from m.

Since G^{p4c} is peripherally-4-connected, one of q and m — say q — is adjacent precisely to v, x, and r, yielding the 3-cycle $\{v, x, q\}$ in G^{p4c} that has two vertices with only three neighbours, contradicting Lemma 15.15. □

We note that the crossing in D cannot involve two edges, each incident with a vertex in K_v, as otherwise G^{p4c} is planar. In particular, $D[C]$ is not self-crossing.

CLAIM 3. $OD_{G_v^+}(C)$ is isomorphic to $OD_G(C)$. In particular, $OD_G(C)$ is bipartite.

PROOF. The main point is that there is a single C-bridge in G containing $G - (K_v + x')$. To prove this, we show that any two vertices in $G - (K_v + x')$ are connected by a C-avoiding path. For vertices not in $K_v \cup K_x$, this is easy: for any two vertices p and q in $G^{\mathrm{p4c}} - \{v, x\}$, there is a pq-path in $G^{\mathrm{p4c}} - \{v, x\}$, showing that any two vertices in $K_p \cup K_q$ are joined by a path in $G - (K_v \cup K_x)$.

If $p \in K_x - x'$, then Lemma 15.14 implies that the three vertices separating K_x from its neighbours are distinct. For one of these vertices w' that is not x', Lemma 15.16 implies there is a pw'-path in $K_x - x'$, completing the proof that there is a single C-bridge B in G containing $G - (K_v + x')$.

Every other C-bridge in G is contained in $K_v + x'$. These are all C-bridges in G_v^+; the only other C-bridge in G_v^+ is the one containing the vertex joined to x, y, and z. This C-bridge has precisely the same attachments as B. This shows that $OD_G(C)$ and $OD_{G_v^+}(C)$ are isomorphic.

Since $G_v^+(C)$ is planar, $OD_{G_v^+}(C)$ is bipartite, yielding the fact that $OD_G(C)$ is bipartite. □

Suppose first that C is clean in D. Since B is the unique non-planar C-bridge in G, D yields a 1-drawing of $C \cup B$ with C clean. Therefore, Corollary 4.7 implies $\mathrm{cr}(G) \le 1$, a contradiction.

If, on the other hand, C is not clean in D, then C is crossed by an edge f. By Claim 2, f is incident with a vertex in $K_v \cup K_x$. If f is incident with a vertex in K_v, then contract K_v (with a vertex inserted at the crossing point, if necessary, to get a 1-drawing of G^{p4c} so that both edges incident with the crossing are incident with v. This implies the contradiction that G^{p4c} is planar.

If f is not incident with x', then $K_x - x'$ has vertices on both sides of $D[C]$. One of these is in a component K_x^1 of $K_x - f$ that is on the side of $D[C]$ that does not contain any vertex of $G - (K_v \cup K_x)$. Lemma 15.13 implies $K_x^1 - x$ is joined to a vertex in some other K_w, $w \neq v$, which cannot happen without crossing $D[C]$ a second time, a contradiction. It follows that f is incident with x'. Furthermore, Lemmas 15.19 and 15.16 (a) imply that f is in a cycle C_f in $G - K_v$. The ends of the edge e_v of K_v crossed in D are separated by $D[C_f]$, so e_v is a cut-edge of K_v. Moreover, e_v is in C.

We now see that the C-bridges are B, those contained in one component of $K_v - e_v$, and those contained in the other component of $K_v - e_v$. Notice that B is a cut-vertex of $OD_G(C)$, and so it overlaps C-bridges of both the other types.

Since $OD_G(C)$ is connected and bipartite, it follows that the C-bridges in either of the components of $K_v - e_v$ occur on the same side of $D[C]$ that they do in D_v^+. In particular, $x't$ may be reintroduced to D to obtain a 1-drawing of G, which is impossible. ■

Strategy. *The strategy now is to show that if we replace any K_v with a smallest possible representative subject to the preceding observations, then we produce a 2-crossing-critical graph. This is the last part of this section. This implies that G^{p4c} turns into a 2-crossing-critical graph by choosing these smallest possible representatives. From this, it is then possible to determine (although not in a theoretical sense, but rather in a definite, finite — really manageable — way that we shall describe) all the 3-connected 2-crossing-critical graphs that have these configurations and reduce to G^{p4c} by planar 3-reductions.*

There will remain the issue of determining all the possible G^{p4c}. Of course, one can list them all, but it is not clear at what point to stop. Fortunately, Theorem 2.14 shows that we do not need to do this when G contains a subdivision of V_{10}, as we already know what G looks like. When G does not contain a subdivision of V_8, a theorem of Robertson plus some analysis implies that G^{p4c} has at most 9 vertices. We are left with the open question of finding the graphs in $\mathcal{M}_2^3$ that *contain a subdivision of* V_8 *but do not contain a subdivision of* V_{10}*. In Section 16, we show that any such graph has at most about 4 million vertices.*

We next characterize certain properties of the graphs G_v; our goal is to show that these (more or less) determine the crossing number of G.

DEFINITION 15.21. Let x, y, and z be vertices in a graph H so that H is an $\|\{x,y,z\}\|$-bridge. Then:

- T is the set of vertices $w \in \{x,y,z\}$ so that there are edge-disjoint $w(\{x,y,z\} \setminus \{w\})$-paths in H; and
- U is the set of vertices $w \in \{x,y,z\}$ for which there are edge-disjoint paths in $H - w$ joining the two vertices in $\{x,y,z\} \setminus \{w\}$.
- $(H, \{x,y,z\})$ is a (T,U)-*configuration* if the graph H^+ obtained from H by adding a new vertex adjacent just to x, y, and z is planar.

Our entire argument depends on the fact, to be proved in the next section, that the pairs (T,U) effectively characterize 2-criticality. Theorem 15.24, the main point of this section, shows that substituting one (T,U)-configuration for another retains the fact that the crossing number is at least 2.

For a (T,U)-configuration, obviously there are only four possibilities for $|T|$. It is a routine analysis of cut-edges to see that, if $|T| \le 1$, then U is empty, while if, for example, $T = \{x,y\}$, then $U = \{z\}$. Thus, for $|T| \le 2$, U is determined by T. This is not the case for $|T| = 3$. In this instance, if $z \notin U$, then there is a cut-edge in $G_v - z$ separating x and y. From here and the fact that $T = \{x,y,z\}$, one easily sees that $x, y \in U$. Thus, if $T = \{x,y,z\}$, then $|U|$ can be either 2 or 3. Therefore, there are in total five possibilities for the pair $(|T|, |U|)$.

We first show that replacing a (T,U)-configuration with another (T,U)-configuration does not lower the crossing number below 2. First the definition of substitution.

DEFINITION 15.22. Let G reduce by planar 3-reductions to the peripherally-4-connected graph G^{p4c}. Suppose v is a vertex of G^{p4c} with neighbours x, y, and z so that $(G_v, \{x,y,z\})$ is, for some subsets T and U of $\{x,y,z\}$, a (T,U)-configuration. Let N be the set of vertices t in $\{x,y,z\}$ for which $K_v^{\max} \cap K_t$ is null. (See Definition 15.18 for $K_v^{\max}$.) Let $\bar{N}_v$ denote the attachments of $K_v^{\max}$: these are the vertices that are of the form t', $t \in \{x,y,z\}$, chosen to be in K_t whenever possible.

(1) A (T,U)-configuration $(H, \{x,y,z\})$ is (G, K_v)-*compatible* if:
   (a) for each $t \in N$, then there is only one neighbour of t in H;
   (b) the degrees of each $t \in \{x,y,z\}$ are the same in both G_v and H; and
   (c) setting $\bar{N}_H$ to consist of the union of the set of vertices of H in $\{x,y,z\} \setminus N$ together with the neighbours in H of the vertices in N, $H - N$ either has a single vertex or contains a cycle through all the vertices in $\bar{N}_H$.

(2) The *substitution of the* K_v*-compatible* (T,U)*-configuration* $(H, \{x,y,z\})$ *for* K_v *in* G is the graph G_v^H obtained from G by adding $H - N$ by identifying the vertices in $\bar{N}_v$ with those in $\bar{N}_H$ in the natural way, and then deleting all vertices in $K_v^{\max} - \bar{N}_v$.

We are almost ready for a major plank in the theory.

Our plan is to show that we can replace a "large" (T, U)-configuration by a "small" (T, U)-configuration and still be 3-connected and 2-crossing-critical. There is one special case that requires particular attention.

DEFINITION 15.23. A (T, U)-configuration $(H, \{x, y, z\})$ is *doglike with nose* n if $|T| = 3$ and $|U| = 2$ and n is the vertex in $T \setminus U$.

THEOREM 15.24. *Let G reduce by planar 3-reductions to the peripherally-4-connected graph G^{p4c}. Suppose v is a vertex of G^{p4c} with precisely the neighbours x, y, and z so that K_v has at least two vertices so that $(G_v, \{x, y, z\})$ is, for some subsets T and U of $\{x, y, z\}$, a (T, U)-configuration. Let $(H, \{x, y, z\})$ be a (G, K_v)-compatible (T, U)-configuration. If $\mathrm{cr}(G) \geq 2$, then $\mathrm{cr}(G_v^H) \geq 2$.*

Proof. We remark that the non-planarity of G and the fact that we are doing *planar* 3-reductions implies G^{p4c} is not planar. This fact will be used throughout the proof.

Let $H' = H - \{x, y, z\}$ and let N be the set of vertices t in $\{x, y, z\}$ so that $K_v^{\max} \cap K_t$ is null. By way of contradiction, we suppose G_v^H has a 1-drawing D.

We start with two simple observations.

CLAIM 1. Some edge of H' is crossed in D.

PROOF. If no edge of H' is crossed in D, then Definition 15.22 (1b) implies we may resubstitute K_v for H' to obtain a 1-drawing of G, a contradiction. □

CLAIM 2. There is no drawing D' of G_v^H in which each crossed edge is incident with a vertex in H'.

PROOF. Otherwise, insert a vertex at each crossing point, and add this vertex to H'. Then contract every edge in the new graph that has both ends in H', and also contract all the K_u to single vertices. The result is a planar embedding of G^{p4c}, a contradiction. □

Therefore, we may assume the crossing edges are $e_v \in H'$ with some other edge f not incident with any vertex in H'. Observe that H' cannot be a single vertex.

CLAIM 3. f is not a cut-edge of $G_v^H - H'$.

PROOF. Suppose f is a cut-edge of $G_v^H - H'$. Since $D[G_v^H - H']$ has no crossing, it is planar. Therefore, the faces on each side of f in $D[G_v^H - H']$ are the same. Thus, the ends of e_v are in the same face of $D[G_v^H - H']$.

Consider now the planar embedding $D[G_v^H - e_v]$. The two ends of e_v are in the same face of the subembedding $D[G_v^H - H']$ and so may be joined by an arc that is disjoint from $D[G_v^H - H']$. This produces a drawing of G_v^H in which all the crossings involve e_v and edges incident with at least one vertex in H'. This contradicts Claim 2. □

Since f is not a cut-edge of $G_v^H - H'$, there is a cycle C_f of $G_v^H - H'$ containing f. Moreover, $D[C_f]$ separates the two ends of e_v, so e_v is an cut-edge of H'. Let H^1 and H^2 be the two components of $H' - e_v$.

The next claim is central to the remainder of the argument.

CLAIM 4. Let $t \in \{x, y, z\}$ be a common neighbour of H^1 and H^2. Then f is incident with $t' \in K_t$ and one of the faces of $H' + t'$ incident with both t' and e_v is empty except for the segment of f from t' to the crossing with e_v.

PROOF. Let C be any cycle in $H' + t'$ containing e_v. Since e_v is a cut-edge of H', $t' \in C$. Since $G^{\mathrm{p4c}} - \{v, t\}$ is connected, $G - (K_v \cup K_t)$ is connected.

Suppose by way of contradiction that there are vertices u and w of $G - (K_v \cup K_t)$ on both sides of $D[C]$. By the preceding paragraph, there is a uw-path P in $G - (K_v \cup K_t)$. Since P is graph-theoretically disjoint from C, but $D[u]$ and $D[w]$ are on different sides of $D[C]$, $D[P]$ crosses $D[C]$; this must be at the unique crossing of D, so $f \in P$ and the crossing of $D[P]$ with $D[C]$ is the crossing of f with e_v.

Moreover, $D[C_f]$ crosses $D[C]$ at the crossing of D and so they must cross somewhere else. As C_f and H' are disjoint, the second crossing is at the vertex t'. Since this is true of any cycle C_f in $G - K_v$, f is a cut-edge of $(G - K_v) - t'$.

We now consider two cases.

Case 1: *there are distinct vertices t_1 and t_2 of $G^{\mathrm{p4c}} - \{t, v\}$ so that $D[K_{t_1}]$ and $D[K_{t_2}]$ are on different sides of $D[C]$.*

In this case, either (i) for some vertex s of G^{p4c}, $f \in K_s$, in which case t_1 and t_2 are in different components of $G^{\mathrm{p4c}} - \{t, v, s\}$, or (ii) since G^{p4c} is non-planar and so has at least five vertices, for some vertex s of G^{p4c} that is an end of f, we may choose t_1 and t_2 to again be in different components of $G^{\mathrm{p4c}} - \{t, v, s\}$.

In either case, the internal 4-connection of G^{p4c} implies that there is an $i \in \{1, 2\}$ so that t_i is the only vertex in its component of $G^{\mathrm{p4c}} - \{t, v, s\}$. But then tvt_i is a 3-cycle in G^{p4c} having v and t_i as degree 3 vertices, contradicting Lemma 15.15.

Case 2: *there are not distinct vertices t_1 and t_2 of $G^{\mathrm{p4c}} - \{t, v\}$ so that $D[K_{t_1}]$ and $D[K_{t_2}]$ are on different sides of $D[C]$.*

In this case, there is a vertex s of $G^{\mathrm{p4c}} - \{t, v\}$ so that $f \in K_s$ and all the vertices of $G - (K_v \cup K_x)$ on one side of $D[C]$ are in one component K_s^1 of $K_s - f$, while all the other vertices of $G - (K_v \cup K_x)$, including the other component K_s^2 of $K_s - f$, are on the other side of $D[C]$.

Lemma 15.13 implies that K_s^1 has neighbours in two K_r's. According to D, these can only be K_v and K_t. But now the 3-cycle tvs has the two degree 3 vertices v and s, contradicting Lemma 15.15.

Since f is on both sides of $D[C]$, but one side has no vertex, it must be that the end of f on that side is in C. But f is disjoint from H', and so this end can only be t'. □

Our proof proceeds by considering how many common neighbours among K_x, K_y, and K_z there are for H^1 and H^2. We start by noting that there cannot be three, since then the graph H^+ is not planar, contradicting Definition 15.21.

CLAIM 5. H^1 and H^2 have exactly one common neighbour.

PROOF. We have already ruled out the possibility that H^1 and H^2 have three common neighbours.

To rule out two common neighbours, suppose by way of contradiction that H^1 and H^2 have the two common neighbours K_x and K_y. By the preceding remark, at least one of H^1 and H^2 does not have a neighbour in K_z. Since H' does have a neighbour in K_z, we may choose the labelling so that H^1 has a neighbour in K_z and H^2 does not.

Claim 4 implies f is incident with both x' and y'. But now $D[f]$ can be rerouted along the other side of the $x'H^2$-edges, around H^2, and on to y' so that G_v^H has no crossings. This implies the contradiction that G^{p4c} is planar. We conclude that H^1 and H^2 have at most one common neighbour.

If they have no common neighbours, then H^1 has neighbours just in K_x, while H^2 has neighbours in K_y and K_z, but not in K_x. In this case, e_v is a cut-edge in H separating x from $\{y,z\}$. It follows that $x \notin T$. Since G_v is also a (T,U)-configuration, there is an cut-edge e'_v of G_v separating x from $\{y,z\}$. Now we can replace H' in T with K_v in such a way that e'_v (in fact the only edge of G_v incident with x) is crossed by f to yield a 1-drawing of G. This contradiction completes the proof of the claim. □

We conclude from Claim 5 that H^1 and H^2 have precisely one common neighbour x'. Claim 4 implies that f is incident with x'.

If, for some $i \in \{1,2\}$, H^i has no other neighbour, then we may reroute f to go around $D[H^i]$, yielding a planar embedding of G_v^H and, therefore, of the non-planar graph G^{p4c}, a contradiction.

Thus, we may choose the labelling so that H^1 has at least one neighbour in K_y, while H^2 has at least one neighbour in K_z. If, say, H^1 is joined to K_y by only one edge, then $y \notin T$; therefore, y is incident with a unique edge in G_v and we can replace $D[H]$ with the planar embedding of K_v so that it is the yK_v-edge that is crossed by f. This yields that contradiction that G has a 1-drawing.

Thus, we may assume that $T = \{x,y,z\}$. However, there are not edge-disjoint yz-paths in $H - x$ (e_v is a cut-edge separating y and z). Therefore, $U = \{y,z\}$, showing G_v is doglike. It follows that $G_v - x$ has a cut-edge e'_v separating y and z. We may substitute the planar embedding of K_v for $D[H]$ so that e'_v crosses f, yielding the final contradiction that G has a 1-drawing. ■

15.4. Reducing to a basic 2-crossing-critical example

In this section, we show that if G is a 3-connected 2-crossing-critical graph that reduces by planar 3-reductions to a peripherally-4-connected graph, then there is a "basic" 3-connected 2-crossing-critical graph from which G is obtained by the regrowth mechanism of the preceding section.

THEOREM 15.25. *Let $G \in \mathcal{M}_2^3$ reduce by planar 3-reductions to a peripherally-4-connected graph G^{p4c}. Let v be a vertex of G^{p4c} with just the three neighbours x, y, and z, so that $(G_v, \{x,y,z\})$ is a (T,U)-configuration and K_v has at least two vertices. Let $G_{\mathrm{rep}(v)}$ be the graph obtained from G by contracting as indicated in the following cases.*

(1) If $(G_v, \{x,y,z\})$ is doglike, then let e be the cut-edge of K_v and contract each component of $K_v - e$ to a vertex.

(2) If $(G_v, \{x,y,z\})$ is not doglike, then we have the following subcases.

(a) If none of G_x, G_y, and G_z is doglike, then contract K_v to a vertex.

(b) If $(|T|,|U|) = (3,3)$, then contract K_v to a vertex.

(c) If G_x is doglike and $y \notin T$, then let C be a cycle in G_v^+ containing x', y', and z', delete everything in $K_v - E(C)$ and contract the edges of C to the 3-cycle $x'y'z'$.

Then $G_{\mathrm{rep}(v)} \in \mathcal{M}_2^3$.

There is one clarification that is required to understand one fine detail of $G_{\mathrm{rep}(v)}$. If, for example, the vertex x' is in K_v, then we proceed precisely as described in the statement. If, however, x' is in K_x and $x \in T$, then in $G_{\text{rep}(v)}$ we retain only two edges between x' and the contracted vertex in $K_{\text{rep}(v)}$ to which it is joined. This especially applies in the case 2c: if $z' \in K_z$, then we keep only the two edges of C incident with z', while if $z' \in K_v$, then we keep all the $z'K_z$-edges.

There is also an important remark to be made. We had long thought that it was possible to reduce each K_v to a single vertex and retain 2-criticality. This might be true in the particular cases of 3-connected 2-crossing-critical graphs with no subdivision of V_8, but it is certainly not true of all 3-connected 2-crossing-critical graphs.

In Definition 2.10 we described the set $\mathcal{S}$ of all graphs that can be obtained from the 13 tiles and the two frames. These graphs are all 3-connected and 2-crossing-critical. Consider any one of these that uses the right-hand frame in Figure 2.1 and uses the second picture in the third row of Figure 2.2. With appropriate choices of the neighbouring pictures, the 3-cycle in the upper half of the picture is part of a doglike G_v that contains the parallel edges in the picture and the parallel edges in the frame: the horizontal edge in the 3-cycle is K_v. The vertical edge in the other 3-cycle in the picture is a K_x. When we do the planar 3-reductions in this case, the contractions of K_x and K_v produce a pair of parallel edges not in the rim. The conclusion is that the resulting peripherally-4-connected graph plus parallel edges is not 2-crossing-critical. Thus, the technicalities we must endure in the statement of Theorem 15.25 seem to be unavoidable.

Proof. We use the notation $K_{\text{rep}(v)}$ for the contraction of K_v in $G_{\text{rep}(v)}$.

Phase 1: *showing $G_{\text{rep}(v)}$ is 3-connected.*

Let t and u be vertices of $G_{\text{rep}(v)}$. We show $G_{\text{rep}(v)} - \{t, u\}$ is connected.

Let w_t and w_u be the vertices of G^{p4c} so that $t \in K_{w_t}$ and $u \in K_{w_u}$ (taking, for example, K_{w_t} to be $K_{\text{rep}(v)}$ if $t \in K_{\text{rep}(v)}$). It follows from Lemma 15.16 that every vertex of every K_s has a path in $G - \{t, u\}$ to at least one neighbour of K_s that is not one of K_{w_t} or K_{w_u}. This is also true of $K_{\text{rep}(v)}$, as may be seen by checking the analogues for $K_{\text{rep}(v)}$ of Lemma 15.16 in the three cases for which $K_{\text{rep}(v)}$ has at least two vertices. (Note there are two possible outcomes for $K_{\text{rep}(v)}$ in Case 2c, depending on whether $z' \in K_v$, in which case $K_{\text{rep}(v)}$ is a 3-cycle, or $z' \in K_z$, in which case K_v is an edge.)

Since each K_s is connected, $G_{\text{rep}(v)} - \{t, u\}$ is connected.

Phase 2: *showing* $\text{cr}(G_{\text{rep}(v)}) \geq 2$.

The graph $\bar{K}_{\text{rep}(v)}$ obtained from $K_{\text{rep}(v)}$ by adding x, y, and z is a (G, K_v)-compatible (T, U)-configuration. Therefore, Phase 2 follows immediately from Theorem 15.24.

Phase 3: *showing that $G_{\text{rep}(v)}$ is 2-crossing-critical.*

Let e be any edge of $G_{\text{rep}(v)}$. Then there is an edge e_G in G naturally corresponding to e (in the sense that precisely the same contractions and deletions of G and $G - e_G$ can be used to obtain both $G_{\text{rep}(v)}$ and $G_{\text{rep}(v)} - e$).

Special situation. *There is one case where the choice of e_G must be made with special care. Suppose K_v contracts down to the single vertex v and e is one of two parallel edges vx. In the case K_v has a cut-edge e', Lemma 15.13 implies each component of $K_v - e'$ is joined to two of the neighbours of v. Suppose that K_x*

is the only common neighbour of these two components. Since G_v is not doglike, some component L of $K_v - e'$ is joined by exactly one edge to its other neighbour; choose e_G to be an xL-edge.

DEFINITION 15.26. For each vertex w of $K_{\mathrm{rep}(v)}$, L_w denotes the subgraph of K_v that contracts to w.

Since G is 2-crossing-critical, there is a 1-drawing D of $G - e_G$. If no edge of any $L_w \subseteq K_v$ is crossed in D, then these may each be contracted to obtain a 1-drawing of $G_{\mathrm{rep}(v)} - e$, and we are done.

CLAIM 1. If there is a drawing of $G - e_G$ in which all the crossings are between edges incident with vertices in L_w, then $G_{\mathrm{rep}(v)} - e$ is planar.

PROOF. Insert vertices at each crossing point and contract every edge in the new graph that has both ends in some L_u. The result is a planar embedding of $G_{\mathrm{rep}(v)} - e$. □

Therefore, we may assume the crossing edges are $e_v \in L_w \subseteq K_v$ with some other edge f not incident with any vertex in L_w.

Case 1: *f is a cut-edge of $(G - e_G) - L_w$.*

In this case, $D[(G - e_G) - L_w]$ has no crossing, so it is planar. Therefore, the faces on each side of f in $D[(G - e_G) - L_w]$ are the same. Thus, the ends of e_v are in the same face of $D[(G - e_G) - L_w]$.

Consider now the planar embedding $D[(G - e_G) - e_v]$. The two ends of e_v are in the same face of the subembedding $D[(G - e_G) - L_w]$ and so may be joined by an arc that is disjoint from $D[(G - e_G) - L_w]$. This produces a drawing of $G - e_G$ in which all the crossings involve e_v and edges incident with at least one vertex in L_w. Claim 1 implies $G_{\mathrm{rep}(v)} - e$ is planar, as required.

Case 2: *f is not a cut-edge of $(G - e_G) - L_w$.*

In this case, f is in a cycle C_f of $(G - e_G) - L_w$. Moreover, $D[C_f]$ separates the two ends of e_v, so e_v is a cut-edge of L_w. Let L_w^1 and L_w^2 be the components of $L_w - e_v$.

We consider separately two cases for G_v.

Subcase 2.1: *G_v is doglike.*

In this subcase, $K_{\mathrm{rep}(v)}$ is two vertices w and $\bar{w}$ joined by a cut-edge e' of $G_v - x$, each joined by an edge to x', w is joined by at least two edges to K_y and $\bar{w}$ is joined by at least two edges to K_z. Lemma 15.20 implies that K_x has at most two neighbours in K_v. We already know there is one in each of L_w and $L_{\bar{w}}$. Lemma 15.11 now implies there is a vertex $x' \in K_x$ incident with all the K_vK_x-edges in G. Thus, we may choose the labelling of L_w^1 and L_w^2 so that the neighbour of x' in L_w is in L_w^1.

We see that x' and the end of e_v in L_w^2 are neighbours of vertices in L_w^1, and neither of these vertices is in L_w^1. The only other possibilities for neighbours of L_w^1 outside of L_w^1 are in K_y and $L_{\bar{w}}$, the latter being the end of e'. A similar remark holds for L_w^2: it has the neighbour (via e_v) in L_w^1, and can have at most neighbours in K_y and $L_{\bar{w}}$ (via e').

Since G is 3-connected, for each $i = 1, 2$, L_w^i has at least two neighbours outside of L_w^i other than x'. From the neighbour analysis of the preceding paragraph, there are at most three in total: two to K_y and one to $L_{\bar{w}}$. There are two ways this can happen.

In the first way, both edges from L_w to K_y have their ends in L_w^2, while e' has an end in L_w^1. But then e_v is a cut-edge of K_v that violates Lemma 15.13: the edge e_G cannot connect L_w^2 to either x' (Lemma 15.20 or K_z (because e' is a cut-edge of $G_v - x$), so the component L_w^2 of $K_v - e_v$ is joined only to K_y.

Therefore, e' has one end in L_w^2 and the two K_vK_y edges have ends in different ones of L_w^1 and L_w^2. It follows that y' is incident with these edges, so Lemma 15.20 implies y' has precisely these neighbours in K_v.

Contract $D[e_v]$ so that L_w^1 is pulled across f and, if necessary, shrink $D[L_w^1]$ so that we obtain a new drawing D^1 of $G - e_G$ in which f crosses the edges from x' and y' to L_w^1.

Claim 2. $f \notin L_{\bar{w}}$.

Proof. If $f \in L_{\bar{w}}$, then exactly the same analysis as for L_w implies that $L_{\bar{w}} - f$ has two components $L_{\bar{w}}^1$, from which there is an edge to x' and an edge to z', and $L_{\bar{w}}^2$, from which there is an edge to z' and L_w^2. But now the graph-theoretically disjoint cycles in $L_w + y'$ containing e_v and $L_{\bar{w}} + z'$ containing f cross exactly once in D, which is impossible. □

It follows from Claim 2 that $f \notin L_{\bar{w}}$. We contract the uncrossed $D^1[L_w]$ and $D^1[L_{\bar{w}}]$ to obtain a drawing D^2 of $G_{\text{rep}(v)} - e$, in which the only crossings are of f with the edges from x' and y' to L_w^1. In D^2, there are parallel edges $y'w$; the one from y' to L_w^2 is not crossed in D^2, so we may make all the others go alongside the uncrossed one. This yields a drawing D^3 of $G_{\text{rep}(v)} - e$ in which the only crossing is $x'w$ with f, so D^3 is a 1-drawing of $G_{\text{rep}(v)} - e$, as required.

Subcase 2: *G_v is not doglike.*

Subcubcase 2.1: *there is a neighbour x of v in G^{p4c} so that G_x is doglike and $x' \in K_v$ is the nose of G_x.*

Let C be the cycle in G_v that we contracted to the 3-cycle $x'y'z'$. We let G^C be the subgraph of G obtained by deleting all edges between the various L_u except the one or three edges in C. Choose the labelling so that y is a neighbour of v in G^{p4c} so that there is exactly one K_vK_y-edge in G; thus $y' \in K_v$.

Let r be that element of $\{x, y, z\}$ so that $r' \in L_w$. There are precisely two edges e_1 and e_2 in G^C coming out of L_w in $G_v - r$.

Let L_w^1 be the component of $L_w - e_v$ containing r' and let L_w^2 be the other. Since C goes through r', at least one of e_1 and e_2 is incident with a vertex in L_w^1. Therefore, at most one of e_1 and e_2 has an end in L_w^2.

We claim that L_w^2 is not joined to any other vertex in G^C. The only possibility is that there is an edge from L_w^2 to $K_x \cup K_y \cup K_z$. Since all the K_vK_x- and K_vK_y-edges in G are incident with x' and y', respectively and x' and y' are not in L_w^2, there are no edges in G from L_w^2 to $K_x \cup K_y$.

As for the possibility of an $L_w^2K_z$-edge, this can only exist if $z' \in K_z$. But z' already has two known neighbours in K_v, namely the K_v-ends of the edges of C incident with z'. Lemma 15.20 implies these are the only vertices of K_v adjacent to vertices in K_z. Therefore these known z'-neighbours are the only ones; in particular, z' has no neighbour in L_w^2, as claimed.

We obtain a 1-drawing of $G_{\mathrm{rep}(v)} - e$ by partially contracting $D[e_v]$ and, if necessary, scaling $D[L_w^2]$ down so that L_w^1 and L_w^2 are now drawn on the same side of f. The only crossing in this new drawing is of the edge of $D[G^C]$, if it exists, that is not e_v and joins L_w^2 to the rest of G^C. Now we may contract all the L_u to single vertices to obtain the required 1-drawing of $G_{\mathrm{rep}(v)} - e$.

Subsubcase 2: *there is no neighbour x of v in G^{p4c} so that G_x is doglike and $x' \in K_v$ is the nose of G_x..*

At this stage, K_v contracts to a single vertex of $G_{\mathrm{rep}(v)}$. In this case, $K_v - e_v$ has two components K_v^1 and K_v^2. Lemma 15.13 implies each of K_v^1 and K_v^2 are connected in G to at least two of K_x, K_y and K_z. Because G_v^+ is planar, at most two of K_x, K_y, and K_z can be adjacent to both K_v^1 and K_v^2.

If both K_x and K_y have neighbours in both K_v^1 and K_v^2, then there is an $i \in \{1, 2\}$ so that K_v^i has adjacencies only in those two. Now pull $D[K_v^i]$ across f and, scaling $D[K_v^i]$ if necessary, to obtain a planar embedding of $G - e_G$. This contracts to a planar embedding of $G_{\mathrm{rep}(v)} - e$, as required.

Thus, we may assume K_v^1 and K_v^2 have precisely one common neighbour in G. Each has its own neighbour. Since G_v is not doglike, one of these, say K_v^1, is joined by a single edge to that unique neighbour and now we can drag K_v^1 across f. This works unless e goes to K_v^2 and K_v^2 is joined to its unique neighbour by two edges. But this is the special situation, and e is joined to K_v^1, not K_v^2. ■

15.5. Growing back from a given peripherally-4-connected graph

The important corollary of Theorem 15.25 is that, if we replace each K_v with its $K_{\mathrm{rep}(v)}$, then we get a 2-crossing-critical model of G^{p4c} with very simple replacements for the vertices of G^{p4c}. In this section, we explain how to obtain all the 3-connected 2-crossing-critical graphs that reduce by planar 3-reductions to a particular peripherally-4-connected graph.

Let L be a non-planar peripherally-4-connected graph. For each vertex v of L having only three neighbours x, y, and z, we decide on the type of v; that is, we choose $T_v \subseteq \{x, y, z\}$ and, in the case $|T_v| = 3$, we decide on U_v: either $U_v = \{x, y, z\}$, or U_v consists of two of $\{x, y, z\}$. For each edge of L joining two vertices of degree at least 4, we decide whether the edge will be a single edge or a parallel pair.

The choices must be made so that $x \in T_v$ if and only if $v \in T_x$. If, for some v, $(|T_v|, |U_v|) = (3, 2)$ (v is chosen to be doglike), then some other implications (as in Theorem 15.25) must be maintained. Choose the labelling so that $x \notin U_v$. Then x is the nose of the dog, v is replaced with K_v, so that K_v is an edge $y'z'$, so that y' incident with two edges going to K_y, and likewise for z' to K_z. Each of y' and z' is also incident with an edge to $x' \in K_x$. Furthermore, K_x can be either a vertex, or, if $|T_x| \neq 3$, an edge, or a 3-cycle.

Once all these choices have been made, the resulting graph is tested for 2-criticality. Thus, for a given peripherally-4-connected graph L, there will be many graphs that require testing. If one of the resulting graphs L' is found to be 2-crossing-critical, then there may be many other 3-connected 2-crossing-critical graphs that arise from L'. Recall that, for each vertex of L that has only three neighbours, we have made a choice as to what type that vertex has. The following lemma explains what may replace the vertex of each type.

LEMMA 15.27. *Suppose the peripherally-4-connected graph L has choices as explained in the preceding paragraphs to produce a 3-connected 2-crossing-critical graph L'. Suppose G is a 3-connected 2-crossing-critical graph that reduces by planar 3-reductions to L so that L' is the graph obtained from G by the replacements described in Theorem 15.25. Then, for each K_v in L', K_v is replaced by one of the possibilities shown in Figures 15.1, depending on (T_v, U_v).*

Proof. We only illustrate the tedious proof in a couple of cases.

Case 1: $(T_v, U_v) = (\{x, y, z\}, \{y, z\})$.

Let e be a cut-edge in $G_v - x$ separating y and z. Let $K_v - e$ have the two components K_v^y, containing the neighbour(s) of y, and K_v^z, containing the neighbour(s) of z. If K_v^y, for examples, is not just either a single vertex or an edge joining the two neighbours of y, then it contains a subdivision of one of these (either pick a path in K_v^y joining the neighbour of y to the K_v^y-end of e or pick a path joining the two neighbours of y). It is easy to see that the subdivision (making a similar choice on the z-side) is also a (T_v, U_v)-configuration. By Theorem 15.24, the subgraph has crossing number 2, and so is all of G. Thus, K_v can be at most one of the three figures in Figure 15.1 corresponding to $(|T|, |U|) = (3, 2)$.

Case 2: $T_v = \{x, y, z\} = U_v$.

In this case, $G_v - x$ contains edge-disjoint yz-paths. Therefore, it contains two such paths P and Q that make a digonal pair. If P and Q are internally disjoint, then there is a $(P - \{y, z\})(Q - \{y, z\})$-path R. If P and Q are not internally disjoint, then set $R = \varnothing$. In either case, set $M = P \cup Q \cup R$. There are two $x(M - \{y, z\})$-paths R_1 and R_2 in G_v.

If the ends of P and Q are in the same digon of $P \cup Q$, then planarity of G_v^+ implies R_1 and R_2 have their ends in the same one of P and Q. It follows that $M \cup R_1 \cup R_2$ is a (T_v, U_v)-configuration, and so is G_v by 2-criticality and Theorem 15.24.

The fact that G is 3-connected implies that there cannot be more than four common internal vertices to P and Q, as if there were six digons, then some two consecutive ones would not contain an end of either R_1 or R_2. This would readily yield a 2-cut in G, which is impossible. This is why the number of possibilities for G_v in this case is finite. ■

In some of the larger (T, U)-configurations, there are edges that are not required to produce the relevant paths between s, t, and u, but, rather, are there to maintain the connectedness of the configuration. These edges might be deletable without reducing the crossing number below 2. Thus, each candidate 3-connected graph produced by the method described needs to have its criticality checked.

15.6. Further reducing to internally-4-connected graphs

In order to find the 2-crossing-critical graphs that do not contain V_8, we wish to use the characterization by Robertson of V_8-free graphs. This characterization, described in the next section, is in terms of *internally-4-connected graphs*. These

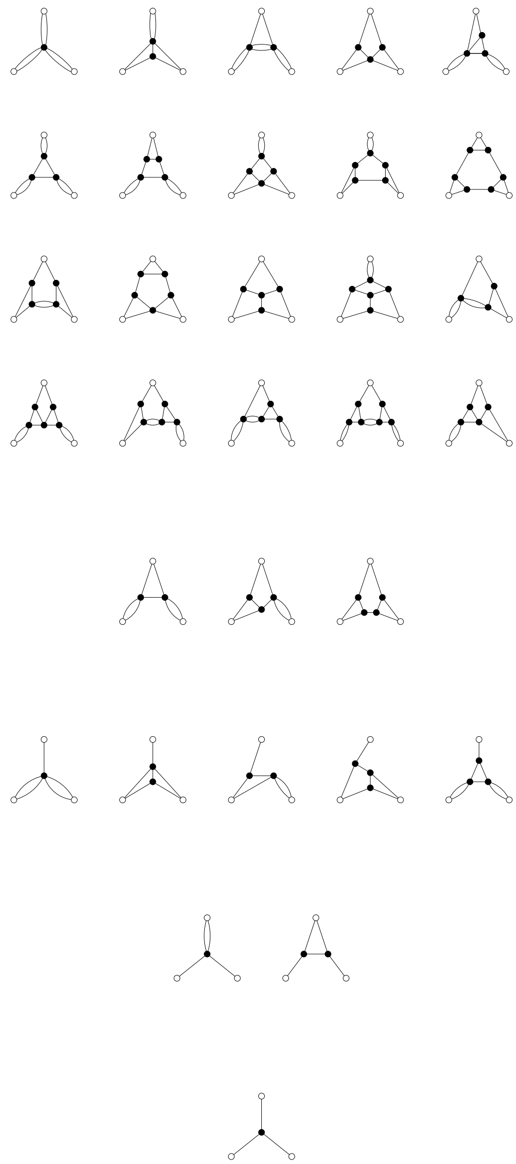

FIGURE 15.1. The possible (T, U)-configurations.

graphs are very closely related to peripherally-4-connected graphs and it is the purpose of this section to describe the reduction of a peripherally-4-connected graph to an internally-4-connected graph, and back again.

DEFINITION 15.28. A *hug* in a graph G is an edge e in a triangle T whose vertex v not incident with e has degree 3. The triangle T is the e-*triangle*, v is the *head* of the hug and the two edges of T other than e are the *arms* of the hug.

DEFINITION 15.29. A G is internally-4-connected if it is peripherally-4-connected and has no hugs.

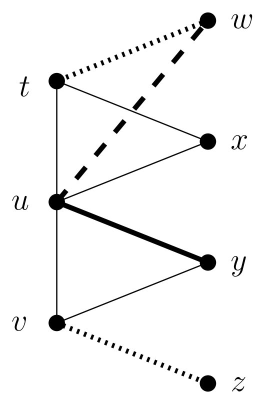

FIGURE 15.2. The thick edge is a bear hug. The dotted edges tw and vz might be subdivided, and the dashed edge uw need not be present. If uw is not present, then $\{ux, uy\}$ is a simultaneously deletable pair of bear hugs.

It is not correct that simply deleting (successively) the hugs from a peripherally-4-connected graph produces an internally-4-connected graph. There is a particular situation that arises that needs special care.

DEFINITION 15.30. (1) A hug e with head v is a *bear hug* if there is an end u of e, incident with a second hug uy whose head t is different from v, and so that, with w the other end of e, the neighbours of u are contained in the union of $\{t, v, w\}$ and the set of neighbours of t. (See Figure 15.2.)
(2) A hug is *deletable* if it is not a bear hug.
(3) A pair of bear hugs having a common end is *simultaneously deletable*.

We are now in a position to reduce a peripherally-4-connected graph to an internally-4-connected graph.

THEOREM 15.31. *Let G be a non-planar peripherally-4-connected graph and let $G = G_0, G_1, \ldots, G_k$ be a sequence of graphs so that, for each $i = 1, 2, \ldots, k$, there is either a hug h_i or a simultaneously deletable pair h_i of bear hugs in G_{i-1} so that $G_i = G_{i-1} - h_i$. Then, for $i = 0, 1, 2, \ldots, k$:*
(1) G_i is a subdivision of a non-planar peripherally-4-connected graph;
(2) if v has degree 2 in G_i but not in G_{i-1}, then h_i is a simultaneously deletable pair of bear hugs in G_{i-1}, both incident with v; and
(3) every degree 2 vertex in G_i has two degree 3 neighbours in G_i.

Furthermore, if the sequence $G_0, G_1, \ldots, G_k$ is maximal, then G_k is a subdivision of an internally-4-connected graph.

We emphasize that, in the reduction process described in the statement, G_i is obtained from G_{i-1} by the deletion of either one or two edges.

Proof. Suppose by way of contradiction that i is least so that G_i is planar. Since G_0 is not planar, $i > 0$, so $G_i = G_{i-1} - h_i$. Each edge in h_i joins two neighbours of a degree 3 vertex in G_i and so may be added to the planar embedding of G_i to produce a planar embedding of G_i together with that edge of h_i. In the case

$|h_i| = 2$, the heads of the hugs are not adjacent. Thus, both hugs may be added simultaneously, while preserving planarity. Thus, G_{i-1} is planar, contradicting the choice of i.

By way of contradiction, we may let i be least so that G_i is not a subdivision of a peripherally-4-connected graph. Thus, $i \geq 1$. Throughout the proof, when we refer to the vertices t, u, v, w, x, y, z, we are always referring to the labelling in Figure 15.2. In each of the three cases, there are two possibilities for h_i to be considered.

It will be helpful to notice that, in the case h_i consists of a simultaneously deletable pair of bear hugs, the vertex u is not a node of G_i and is incident with both deleted edges.

Claim 1. G_i is a subdivision of a 3-connected graph.

Proof. Let a and b be distinct nodes of G_i. Then a and b are distinct nodes of G_{i-1}, so there are three internally disjoint ab-paths P_1, P_2, P_3 in G_{i-1}.

If $e \in h_i$, then the head c of the e-triangle has degree 3. If e is in some P_i and T is the triangle containing e and its head, then we may replace $P_i \cap T$ with the path in T complementary to $P_i \cap T$. The at most two modifications result in three internally disjoint paths that are also paths in G_i. □

Claim 2. If a has degree at least three in G_{i-1} and degree 2 in G_i, then:

(1) $|h_i| = 2$;
(2) a is incident with both edges in h_i; and
(3) both neighbours of a have degree 3 in G_i.

Proof. Let $e \in h_i$. The head b of the e-triangle has degree 3 in G_{i-1} and, since G_{i-1} is a subdivision of a peripherally-4-connected graph, no other vertex of the e-triangle has degree 3, so Lemma 15.15 shows they both have degree at least 4. It follows that if e is the only edge in h_i, then the ends of e have degree at least 3 in G_i and no new vertex of degree 2 is introduced in G_i.

Therefore h_i is a deletable pair. The only new vertex of degree 2 in G_i is u, so $a = u$. Also, the only neighbours of u in G_i have degree 3 in G_i. □

The remaining possibility is that there is a set $\{a, b, c\}$ of nodes of G_i and a 3-separation (H, J) of G_i so that $H \cap J = \|a, b, c\|$ and both $H - \{a, b, c\}$ and $J - \{a, b, c\}$ have at least two nodes of G_i.

Because G_{i-1} is a subdivision of a peripherally-4-connected graph, there is an edge $e \in h_i$ having one end r_H in $H - \{a, b, c\}$ and one end r_J in $J - \{a, b, c\}$.

Suppose for the moment that h_i has a second edge. Since G_{i-1} is a subdivision of a peripherally-4-connected graph, not all the neighbours of u in G_{i-1} can be in the same one of H and J. We may choose the labelling so that $x = r_J$. As t is a common neighbour of $u = r_H$ and $x = r_J$, we conclude that $t \in \{a, b, c\}$, say $t = a$.

It follows that at least one of v and y (the other two neighbours of u) is in $H - \{t, b, c\}$. Since v and y are adjacent, it follows that both are in H and, furthermore, uy is also in H. In particular, there is a unique edge in h_i that has one end in $H - \{a, b, c\}$ and one end in $J - \{a, b, c\}$.

Now the two possibilities for h_i are merged: e is the unique edge in h_i having one end r_H in $H - \{a, b, c\}$ and one end r_J in $J - \{a, b, c\}$. The head q of the e-triangle must be in $\{a, b, c\}$, say $q = a$.

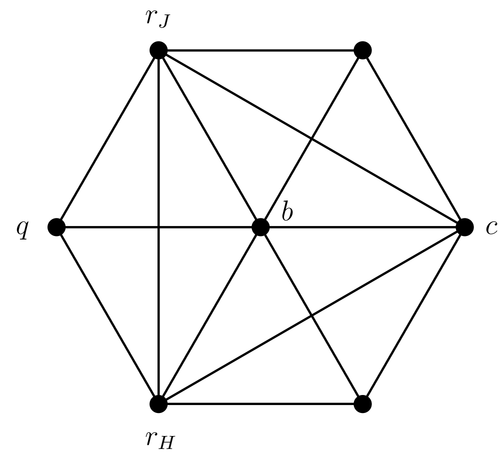

FIGURE 15.3. When $s = b$, G_{i-1} is a subgraph of the illustrated planar graph.

Since q has degree 3, we may choose the labelling so that r_H is the only neighbour of q in $H - \{q, b, c\}$. The neighbour r_J of q is in $J - \{q, b, c\}$. Note that r_H and r_J are both nodes of G_{i-1}.

The third neighbour s of q is in J, so $\{r_H, b, c\}$ is a 3-cut in G_{i-1}. Since G_{i-1} is peripherally-4-connected, there is a unique node p in $H - r_H$, which is joined by branches in G_{i-1} to all of r_H, b, and c.

If $s \in \{b, c\}$, then the discussion in the preceding paragraph applies with r_J and J in place of r_H and H, respectively. The nodes of G_{i-1} are now all known (there are only 7), and the edges are almost completely determined. In particular, G_{i-1} is a subgraph of the planar graph shown in Figure 15.3, contradicting the fact that G_{i-1} is non-planar. Therefore, s is in $J - \{q, b, c\}$.

The vertex r_H is the only candidate for the second branch vertex (after p) of G_i in $H - \{q, b, c\}$, so it must be joined by a G_i-branch to at least one of b and c; choose the labelling so that b is an end of such a G_i-branch.

If b has only one neighbour in $J - \{q, b, c\}$, then p and b are both degree 3 vertices in a triangle in G_{i-1}; since G_{i-1} is a subdivision of a peripherally-4-connected graph, this contradicts Lemma 15.15. The same reasoning implies that both r_H and b have degree at least 4 in G_{i-1}. These imply that $r_H p$, $r_H b$, and pb are all edges of G_{i-1}.

Because $r_H r_J$ is in h_i and q is the head of the $r_H r_J$-triangle, we know that $r_H r_J$, $q r_H$, and $q r_J$ are all edges of G_{i-1}. Furthermore, $r_H s$ is not a G_{i-1}-branch (it would yield a second edge with one end in each of $H - \{a, b, c\}$ and $J - \{a, b, c\}$).

The triangles $p r_H b$ and $q r_H r_J$ show that $r_H r_J$ is a bear hug. Since it was deleted, it must be in a simultaneously deletable pair of bear hugs. This implies that $r_H b$ is the other edge in that pair. Thus, $H - \{a, b, c\}$ has only one node in G_i, a contradiction that completes the proof that each G_i is a subdivision of a peripherally-4-connected graph.

We move on to showing that a maximal sequence ends in a subdivision of an internally-4-connected graph. So suppose G_i is not a subdivision of an internally-4-connected graph. Since it is a subdivision of a peripherally-4-connected graph

H, there is a 3-cut $\{a,b,c\}$ in H so that ab is an edge of G_i. Since H is peripherally-4-connected, there is a vertex p adjacent in H to all of a,b,c and with no other neighbours in H. Lemma 15.15 shows that the triangle p,a,b has at most one vertex of degree 3; since p is such a vertex, a and b have degree at least 4 in H. It follows that pa and pb are edges of G_i and, therefore, ab is a hug in G_i.

It is evident from the definitions that, as soon as G_i has a hug, then either G_i has a hug that is not a bear hug or G_i has a pair of simultaneously deletable bear hugs. In either case, G_i is not the last in a maximal sequence. ■

We conclude this section with a brief discussion of the reverse process: how to generate all the peripherally-4-connected graphs that reduce to a given non-planar internally-4-connected graph G. Every graph created through iterating the following procedure is peripherally-4-connected and non-planar. We choose either two non-adjacent neighbours of a degree 3 vertex and add the edge between them, or we choose an edge e joining degree 3 vertices and a neighbour of each vertex incident with e, subdivide e once, and join both the chosen neighbours to the vertex of subdivision.

Every internally-4-connected graph produces only finitely many peripherally-4-connected graphs through this process, as the number of possible additions is initially finite and strictly decreasing.

15.7. The case of V_8-free 2-crossing-critical graphs

In this section, we complete our analysis of peripherally-4-connected 2-crossing-critical graphs by considering the case of 3-connected 2-crossing-critical graphs that do not contain a subdivision of V_8. This is the whole reason for studying peripherally-4-connected graphs, since there is a characterization of the closely related internally-4-connected graphs that do not contain a subdivision of V_8.

Two important classes of graphs in this context are the following.

DEFINITION 15.32. (1) A *bicycle wheel* is a graph consisting of a *rim*, which is a cycle C, and an *axle*, which is consists of two adjacent vertices x and y not in the rim, together with *spokes*, which are edges from $\{x,y\}$ to C.

(2) A *4-covered graph* is a graph G containing a set W of four vertices so that $G-W$ has no edges.

Maharry and Robertson [**22**] prove Robertson's Theorem that an internally-4-connected graph with no subdivision of V_8 is one of the following:

(1) a planar graph;
(2) a non-planar graph with at most seven vertices;
(3) $C_3 \,\square\, C_3$;
(4) a bicycle wheel; and
(5) a 4-covered graph.

Suppose G is a 3-connected graph that does not contain a subdivision of V_8 and G reduces by planar 3-reductions to the peripherally-4-connected graph G^{p4c}. It follows that G^{p4c} has no V_8. Eliminating hugs as described in Theorem 15.31 produces an internally-4-connected graph G^{i4c}. Deleting hugs does not affect the planarity of the graph; since G^{p4c} is not planar, so is G^{i4c}. By Robertson's Theorem, one of the following happens:

(1) G^{i4c} is not planar and has at most seven vertices;
(2) G^{i4c} is $C_3 \,\square\, C_3$;
(3) G^{i4c} is a bicycle wheel; and
(4) G^{i4c} is a 4-covered graph.

Our ambition in the remainder of this section is to explain how to determine all the peripherally-4-connected graphs G^{p4c} that can be the outcome of a sequence of planar 3-reductions starting from a 3-connected, 2-crossing-critical graph G that has no subdivision of V_8. Any peripherally-4-connected graph with no subdivision of V_8 that either has crossing number exactly 1 or is itself 2-crossing-critical needs to be tested. Those with crossing number 1 might extend to a 2-crossing-critical example by duplication of edges and/or replacing vertices of degree 3 by one of the basic (T, U)-configurations, as explained in the preceding section.

The first two items arising from Robertson's Theorem are easily dealt with. A computer program can easily find all internally-4-connected graphs with at most 7 vertices and determine which ones either have crossing number 1 or are 2-crossing-critical. The graph $C_3 \,\square\, C_3$ is itself 2-crossing-critical, so this is one of the 3-connected, 2-crossing-critical graphs that do not contain a subdivision of V_8.

Definition 15.33. Let G^{p4c} be a peripherally-4-connected graph and let G^{i4c} be the internally 4-connected graph obtained from G^{p4c} by simplifying (that is, leaving only one edge in each parallel class) and eliminating hugs. Then G^{p4c} is a *peripherally-4-connected extension of* G^{i4c}.

We conclude this section by showing how to which bicycle wheels and 4-covered graphs G^{i4c} can have such a 2-crossing-critical G^{p4c} as an extension. In particular, G^{i4c} must either have crossing number 1 or itself be 2-crossing-critical; in the latter case $G^{\mathrm{p4c}} = G^{\mathrm{i4c}}$.

CASE 1: *the bicycle wheels.*

Let x and y be the adjacent vertices making the axle of the bicycle wheel G^{i4c}, and let C be the cycle that is the rim. Our goal is to provide sufficient limitations on C to show that the computation is feasible. Here is our first limitation, which can very likely be improved.

Lemma 15.34. *Suppose $G \in \mathcal{M}_2^3$ reduces by planar 3-reductions to the graph G^{p4c} that is a peripherally-4-connected extension of G^{i4c}. If G^{i4c} is a bicycle wheel with axle xy and rim C, then x is not adjacent in G^{i4c} to six consecutive vertices on C, none of which is adjacent to y.*

Proof. Suppose by way of contradiction that $x_1, x_2, x_3, x_4, x_5, x_6$ are six consecutive (in this order) vertices of C adjacent to x but not y. Lemma 15.15 implies no two consecutive ones of these vertices have only three neighbours in G^{p4c}. By symmetry, we may assume x_3 has a neighbour u that is not adjacent to x_3 in G^{i4c}.

Because G^{p4c} is a peripherally-4-connected extension of G^{i4c}, there are vertices w and z so that x_3, u, and w are the neighbours (in both graphs) of z and no other vertex has just these three neighbours. Since y is not adjacent to x_3 and x has more than 3 neighbours, $z \in C$. If follows that x_3 and u are the C-neighbours of z and w is the neighbour of z that is in $\{x, y\}$. In particular, z, being a neighbour of x_3 is either x_2 or x_4, so $w = x$. In either case, three consecutive vertices from $x_1, x_2, \dots, x_6$ are such that the outer two are adjacent by a chord in G^{p4c}; if necessary, we relabel so these are x_1, x_2, x_3. In particular, x_2 has just three neighbours in G^{p4c}.

Let D be a 1-drawing of $G^{\mathrm{p4c}} - xx_2$ and let K be the subgraph of $G^{\mathrm{p4c}} - xx_2$ induced by x, x_1, x_2, and x_3.

CLAIM 1. K is clean in D.

PROOF. In $G^{\mathrm{p4c}} - xx_2$, x_2 has only two neighbours, so the edge x_1x_3 and the path (x_1, x_2, x_3) make a pair of parallel edges. Therefore, we may assume neither of these is crossed in D.

It suffices to prove that xx_1 is not crossed in D, as the proof for xx_3 is symmetric. Suppose by way of contradiction that xx_1 is crossed in D and consider the planar embedding of $G^{\mathrm{p4c}} - \{xx_1, xx_2\}$ induced by D. Since $G^{\mathrm{i4c}} - \{xx_1, xx_2\}$ is a subgraph, it is also planar, embedded in the plane by D.

Since x_3 has only three neighbours in $G^{\mathrm{i4c}} - \{xx_1, xx_2\}$, we can add the edge xx_2 alongside the path (x, x_3, x_2) to obtain a planar embedding of $G^{\mathrm{i4c}} - xx_1$. Then we may add the edge xx_1 alongside the path (x, x_2, x_1) to get a planar embedding of G^{i4c}. However, this contradicts the fact that G^{i4c} is not planar. □

Now let K be the subgraph of $G^{\mathrm{p4c}} - xx_2$ induced by x, x_1, x_2, and x_3. Because x_1, x_2, and x_3 are consecutive along C, there is a unique K-bridge B in $G^{\mathrm{p4c}} - xx_2$. The claim shows K is clean in D, so $D[B]$ is contained in one face F of $D[K]$.

Adjusting which of $D[x_1x_3]$ and $D[(x_1, x_2, x_3)]$ is which, if necessary, we may arrange D so that both x and x_2 are incident with a face of $D[K]$ that is not F. This permits us to add xx_2 to D without additional crossings, to obtain a 1-drawing of G. This final contradiction yields the result. ■

Along the same lines, we have the following limitation.

LEMMA 15.35. *Suppose $G \in \mathcal{M}_2^3$ reduces by planar 3-reductions to the graph G^{p4c} that is a peripherally-4-connected extension of G^{i4c}. If G^{i4c} is a bicycle wheel with axle xy and rim C, and there are four distinct vertices of C adjacent to both x and y, then these are the only six vertices of G^{i4c}.*

Proof. Suppose to the contrary that u_1, u_2, u_3, and u_4 are distinct vertices of C adjacent to both x and y in G^{p4c} and there is another vertex u_5. We may choose the labelling of x and y so that $xu_5 \in G^{\mathrm{i4c}}$. Let D be a 1-drawing of $G^{\mathrm{p4c}} - xu_5$.

Let K be the subgraph of $G^{\mathrm{p4c}} - xu_5$ consisting of C and all edges between x and vertices of C. (We do not include any chords of C that might exist in G^{p4c}.) If x and y are both in the same face of $D[C]$, then y is in some face F of $D[C]$ and at least two of u_1, u_2, u_3, and u_4 are not incident with F. This implies the contradiction that D has at least two crossings.

We conclude that y is not in the same face of $D[C]$ with x. It follows that xy crosses C in D and this is the only crossing. We claim we can add the edge xu_5 to D to obtain a 1-drawing of G^{p4c}.

Let F be the unique face of $D[K]$ incident with both u_5 and u and let C' be the cycle bounding F. If we cannot add xu_5 in F, then there is an edge e of G^{p4c} that has an end in each of the two components of $C' - \{x, u_5\}$. Since $C' - x \subseteq C$, it follows that both ends w_1 and w_2 of e are in C.

Since e is not an edge of G^{i4c}, there are vertices w_3 and z of G^{p4c} so that z has just the neighbours w_1, w_2, and w_3. Since both x and y have at least four neighbours, $z \notin \{x, y\}$. Since one of x and y is a neighbour of z, $w_3 \in \{x, y\}$. Finally, z has at least two neighbours in C, so these are w_1 and w_2. We conclude that $z = u_5$.

We note that xy cannot cross the 3-cycle $u_5w_1w_2$ in D. Therefore, we can move w_1w_2 to the face of $D[C]$ that contains y; in this new 1-drawing of $G^{\mathrm{p4c}} - xu_5$, x and u_5 are incident with the same face, giving the contradiction that G^{p4c} has a 1-drawing. ■

The final limitation is the following.

LEMMA 15.36. *Suppose $G \in \mathcal{M}_2^3$ reduces by planar 3-reductions to the graph G^{p4c} that is a peripherally-4-connected extension of G^{i4c}. Suppose G^{i4c} is a bicycle wheel with axle xy and rim C, and there are six distinct vertices x_1, y_1, x_2, y_2, x_3, y_3 in this cyclic order on C, so that, for $i = 1, 2, 3,$, x_i is adjacent to x and y_i is adjacent to y. Then these are the only six vertices of C.*

We remark that we allow for the possibility that some (or all) of the x_i are also adjacent to y and, likewise, some of the y_i can be adjacent to x.

Proof. Suppose to the contrary that there is another vertex u in C. If possible, choose the x_i, y_i and u so that u is adjacent to only one of x and y. We may assume that u occurs between x_1 and y_1 in the cyclic order on C. By the choice of the x_i, y_i, and u, if u is adjacent to both x and y, then so are x_1 and y_1 and all vertices between them on C.

Let D be a 1-drawing of $G^{\mathrm{p4c}} - xu$. Let K be the subgraph of $G^{\mathrm{p4c}} - xu$ consisting of C and all edges between x and vertices of C. (We do not include any chords of C that might exist in G^{p4c}.) If x and y are on the same side of $D[C]$, then at most one of the y_i is incident with the face of $D[K]$ containing y, showing D has at least two crossings, a contradiction. Therefore, the crossing of D is of xy with an edge of C.

There is a face of $D[K]$ incident with both x and u; let C' be its bounding cycle. If we cannot add xu to D, it is because there is an edge e of $G^{\mathrm{p4c}} - xu$ with an end in each of the components of $C' - \{x, u\}$. Since $C' - x \subseteq C$, it follows that the ends w_1 and w_2 of e are both in C. Because G^{p4c} is a peripherally-4-connected extension of a bicycle wheel, there are vertices z and w_3 so that z has only the neighbours w_1, w_2, and w_3.

Both x and y have at least four neighbours in G^{i4c}, so $z \notin \{x, y\}$; thus, $z \in C$. Since z has two neighbours in C and at least one in $\{x, y\}$, it follows that $w_3 \in \{x, y\}$, while w_1 and w_2 are the two C-neighbours of z. Therefore, $z = u$. As u is adjacent to x, we conclude that u is not also adjacent to y. But now we can move the edge w_1w_3 to the other side of C so that the resulting 1-drawing of $G^{\mathrm{p4c}} - xu$ extends to a 1-drawing of G, a contradiction. ■

Lemmas 15.34, 15.35, and 15.36 effectively limit the possibilities for G^{i4c}. Each of these must be checked for either having crossing number 1 or being 2-crossing-critical. Those with crossing number 1 must have their peripherally-4-connected extensions tested for 2-criticality. No matter what improvement is made to Lemma 15.34, this will require computer work to complete.

CASE 2: *the 4-covered graphs.*

We begin our analysis by describing three particular internally-4-connected 2-crossing-critical graphs that are 4-covered.

DEFINITION 15.37. (1) The 3-cube Q_3 is the 3-regular, 3-connected, planar, bipartite graph with 8 vertices.

(2) The graph Q_3^v is the bipartite graph obtained from Q_3 by adding one new vertex joined to all four vertices on one side of the bipartition of Q_3.
(3) The graph Q_3^{2e} is the bipartite graph obtained from Q_3 by adding two of the four missing (bipartite-preserving) edges.
(4) The graph Q_3^t is the graph obtained from Q_3 by adding a 3-cycle abc on one side of the bipartition of Q_3 together with one edge joining the fourth vertex d of the same part to the non-adjacent vertex in the other part of the bipartition.

LEMMA 15.38. *The graphs Q_3^v, Q_3^{2e}, and Q_3^t are all 2-crossing-critical.*

Proof. We start with the following observation.

CLAIM 1. If D is a 1-drawing of Q_3, then D is the unique planar embedding of Q_3.

PROOF. If e and f are two non-adjacent edges of Q_3, then it is easy to see that they are in disjoint cycles. Therefore, no two edges of Q_3 cross in D. $\square$

We use Claim 1 to show that $\mathrm{cr}(Q_3^v) \geq 2$, $\mathrm{cr}(Q_e^{2e}) \geq 2$, and $\mathrm{cr}(Q_3^t) \geq 2$.

Adding the one vertex to the planar embedding of the 3-cube yields 2 crossings, since each face of the 3-cube is incident with only 2 of the four vertices joined to the new vertex. This shows $\mathrm{cr}(Q_3^v) \geq 2$.

For Q_3^{2e}, each of the two new edges joins vertices not on the same face of Q_3 and so each has a crossing with Q_3. Thus, $\mathrm{cr}(Q_3^{2e}) \geq 2$.

For Q_3^t, the new edge e incident with d must cross Q_3 in any drawing D of Q_3^t for which $D[Q_3^t]$ has no crossings. If the 3-cycle $D[abc]$ also has a crossing with Q_3, then D has two crossings. Otherwise, $D[abc]$ separates the two ends of $D[e]$, so $D[e]$ crosses $D[abc]$. Thus, $\mathrm{cr}(Q_3^t) \geq 2$.

We now consider 2-criticality in each case.

For Q_3^v, deleting any edge of the 3-cube makes a face incident with 3 of the four vertices and so yields a 1-drawing. Likewise deleting one of the edges incident with the new vertex yields a 1-drawing.

For Q_3^{2e}, obviously deleting either of the edges not in Q_3 yields a 1-drawing. On the other hand, if e is an edge of Q_3 incident with at most one of the vertices of Q_3^{2e} of degree 4, then deleting e makes one of the newly adjacent pairs now lie on the same face, yielding the required 1-drawing. If e is one the remaining two edges of Q_3, there is a 1-drawing of $Q_3 - e$ with one crossing that extends to a 1-drawing of $Q_3^{2e} - e$.

For Q_3^t, criticality of all the edges not incident with d is obvious, as it is the new edge e incident with d. The remaining three edges are symmetric. Deleting any one of these results in a subgraph that has crossing number 1 (we may move the other end of e to the other side of abc to get a 1-drawing). ■

LEMMA 15.39. *Suppose $G \in \mathcal{M}_2^3$ reduces by planar 3-reductions to a peripherally-4-connected G^{p4c} with at least 8 vertices that is an extension of the internally-4-connected 4-covered graph G^{i4c}. Then either G is one of the graphs Q_3^v, Q_3^{2e}, or G^{p4c} has exactly 8 vertices.*

Proof. Let a, b, c, d be the four vertices so that $G^{\mathrm{i4c}} - \{a, b, c, d\}$ is an independent set I. For each $x \in \{a, b, c, d\}$, let X be the set of vertices in I adjacent to everything in $\{a, b, c, d\} \setminus \{x\}$, and let R be the remaining vertices in I; a vertex in R is joined to all of $\{a, b, c, d\}$.

Note that a vertex in R has degree 4 in G^{i4c}, so it is also a vertex of G; it cannot be the outcome of any 3-reductions. If $|R| \geq 3$, then G contains $K_{3,4}$ and so $G = K_{3,4}$, a contradiction. Thus, $|R| \leq 2$.

If, for some $x \in \{a, b, c, d\}$, $|X| \geq 2$, then $\{a, b, c, d\} \setminus \{x\}$ is a 3-cut in G^{i4c} that separates any two vertices v, w in X from all the other vertices in $I \setminus \{v, w\}$, of which there are at least two. This contradicts the fact that G^{i4c} is internally 4-connected. Thus, $|X| \leq 1$.

This implies that G^{p4c} has at most 10 vertices, but we can proceed a little further.

If $R = \varnothing$, then G^{i4c} is planar (adding the K_4 on $\{a, b, c, d\}$ does not affect planarity), which is a contradiction. Thus, $|R| > 0$.

If, for each $x \in \{a, b, c, d\}$, $|X| = 1$, then the bipartite subgraph of G^{i4c} consisting of $\{a, b, c, d\}$ and the four vertices in $A \cup B \cup C \cup D$ is the 3-dimensional cube Q_3. Adding one of the vertices in R to Q_3 produces Q_3^v. That is, if all of A, B, C, and D are not empty, $|R| = 1$ and $G = Q_3^v$.

Thus, we may assume $R \neq \varnothing$ and $D = \varnothing$.

If $|R| = 2$, then for G^{i4c} to have at least 8 vertices, at least two of A, B, and C are not empty. Thus, $Q_3^{2e} \subseteq G^{\mathrm{p4c}}$, so $G^{\mathrm{p4c}} = Q_3^{2e}$.

In the final situation, we have $|R| = 1$ and, because G^{p4c} has at least 8 vertices, all of A, B, and C are not empty. In particular, G^{p4c} has exactly 8 vertices, as required. ∎

A computer search can find all the peripherally-4-connected graphs having 8 vertices. These will include all the examples that are peripherally-4-connected extensions of internally-4-connected, 4-covered graphs having 8 vertices. This completes our analysis of 3-connected, 2-crossing-critical graphs with no subdivision of V_8.

CHAPTER 16

Finiteness of 3-connected 2-crossing-critical graphs with no V_{2n}

This section is devoted to showing that, for each $n \geq 3$, there are only finitely many 3-connected 2-crossing-critical graphs that do not contain a subdivision of V_{2n}. In particular, Theorem 16.14 asserts that if G has a subdivision of V_{2n} but no subdivision of V_{2n+2}, then $|V(G)| = O(n^3)$.

The finiteness has been proved previously by completely different methods in [**13**]. In our particular context, this shows that there are only finitely many 3-connected 2-crossing-critical graphs that have a subdivision of V_8 but do not have a subdivision of V_{10}; these are the only ones missing from a complete determination of the 2-crossing-critical graphs.

The first subsection shows that, if G is a 3-connected 2-crossing-critical graph that does not contain a subdivision of V_{2n+2}, then, for any $V_{2n} \cong H \subseteq G$, each H-bridge in G has at most 88 vertices. The second subsection shows that, for a particular subdivision H of V_{2n}, there are only $O(n^3)$ H-bridges having a vertex that is not an H-node. These easily combine to give the $O(n^3)$ bound of Theorem 16.14.

16.1. V_{2n}-bridges are small

The main result of this subsection is to show that if $G \in \mathcal{M}_2^3$ and $V_{2n} \cong H \subseteq G$, then any H-bridge B is a tree with a bounded number of leaves, so that $|V(B)| \leq 88$. In the next subsection, we show that there are only $O(n^3)$ non-trivial H-bridges.

The next lemma will have as a corollary the first main result of this subsection.

LEMMA 16.1. *Let $G \in \mathcal{M}_2^3$, $V_{2n} \cong H \subseteq G$, $n \geq 3$, and B an H-bridge. Then* $|\mathrm{att}(B)| \leq 11n + 12$.

Proof. Let e be an edge of B incident with $x \in \mathrm{att}(B)$ and $y \in \mathrm{Nuc}(B)$. Then $D_e[B - e]$ is contained in a face F of $D_e[H]$. Because we know the 1-drawings of V_{2n}, we know that each face of $D_e[H]$ is incident with at most $n + 1$ H-branches. Moreover, $B - e$ is an H-bridge in $G - e$ and $\mathrm{att}_{G-e}(B - e)$ is either $\mathrm{att}_G(B)$ or $\mathrm{att}_G(B) \setminus \{x\}$.

If B has at least $11(n + 1) + 2$ attachments, then some H-branch b contains at least 12 attachments of $B - e$. Let $a_1 \dots a_{12}$ be any 12 distinct attachments of $B - e$ occurring in this order in b. Let $T \subseteq B$ be a minimal tree that meets $\mathrm{att}(B)$ at a_1, a_3, a_4, a_6, a_7, a_9, a_{10}, and a_{12}, so that these a_i are the leaves of T, and let $Q = [a_1, b, a_{12}]$. Set $Y = T \cup Q$.

For $i = 1, 4, 7, 10$, there is a unique cycle $C_i \subseteq Y$ that meets b precisely in a_iQa_{i+2}. Let $I \subseteq \{1, 4, 7, 10\}$ be the subset such that, for $i \in I$, $x \notin C_i$; clearly $|I| \geq 3$.

For each $i \in I$, let M_i be the C_i-bridge in $G-e$ with $H \subseteq M_i \cup C_i$. As $x \notin C_i$, $x \in \mathrm{Nuc}(M_i)$. Let B_i be the C_i-bridge in $G-e$ containing y or $B_i = y$ if $y \in C_i$. Let P_i be a minimal subpath of C_i containing $B_i \cap C_i$, so that $a_i Q a_{i+2} \not\subseteq P_i$.

Claim 1. Let $i, j, k \in I$ be distinct. If $y \notin M_i \cup M_j$, then:

- $B_i = B_j$;
- $P_i = P_j \subseteq C_i \cap C_j$; and
- $y \in M_k$.

Proof. If u and v are vertices in $C_i \cap C_j$, then u and v are not in b and there is a unique uv-path P in T. We note that $P \subseteq C_i \cap C_j$. Thus, $C_i \cap C_j$ is a path.

If there were a yC_i-path disjoint from C_j, then $y \in M_i$, a contradiction. Therefore, every yC_i-path meets C_j and, symmetrically, every yC_j-path meets C_i. Thus, every $y(C_i \cup C_j)$-path has one end in $C_i \cap C_j$. It follows that if $y \in C_i \cup C_j$, then $y \in C_i \cap C_j$, so in this case $B_i = B_j = \|y\|$.

In the case $y \notin C_i \cup C_j$, let B be the $(C_i \cup C_j)$-bridge containing y. The preceding paragraphs show that $\mathrm{att}(B) \subseteq C_i \cap C_j$, so that in fact B is also both a C_i- and a C_j-bridge. In particular, $B_i = B_j = B$.

For the last part, we assume $y \notin M_k$ and note that $B = B_i = B_j = B_k$ and $C_i \cap C_j \cap C_k$ is a non-null path P'. If P' has length at least one, then $P' \cup C_i \cup C_j \cup C_k$ contains a subdivision of $K_{2,3}$ and yet has all three of the vertices on one side incident with a common face, which is impossible. Therefore, P' consists of a single vertex z.

If z is not y, B has only z as an attachment in $G-e$. It follows that either z or $\{z, x\}$ is a cut-set of G, contradicting the fact that G is 3-connected. Thus, $z = y$, and so, for some $t \in \{i, j, k\}$, y is an attachment of M_t; in particular, $y \in M_t$, a contradiction. □

By Claim 1, there is an $i \in I$ such that $y \in M_i$. For such an i, set $C = C_i$ and note that $x \in M_i - \mathrm{att}(M_i)$, so that $M = M_i + e$ is a C-bridge in G. Furthermore, $\mathrm{att}_G(M) = \mathrm{att}_{G-e}(M-e)$.

Notice that $D_e[C]$ is clean, since the crossing of D_e is between disjoint H-branches. Thus, C has BOD in $G-e$. Also, any C-bridge $B' \neq M$ has $C \cup B'$ planar. As $\mathrm{att}_G(M) = \mathrm{att}_{G-e}(M-e)$, C has BOD in G.

Recall that the H-bridge B has a_i, a_{i+1}, and a_{i+2} as attachments. For any vertex u of B not in b, there is an H-avoiding ua_{i+2}-path, whose edge e' incident with u is in some C-bridge B'. Since x and y are on the same side of $D_e[C]$, M is contained on that side of $D_e[C]$ and e' is on the other side. Therefore, $B' \neq M$.

In $D_{e'}$, the crossing is in H and $D_{e'}[C]$ is clean. That is, $D_{e'}[C \cup M]$ is a 1-drawing with C clean. Corollary 4.7 shows $\mathrm{cr}(G) \leq 1$, the final contradiction. ■

The following corollary is the first main result of this section.

Corollary 16.2. *Let $G \in \mathcal{M}_2^3$, $V_{2n} \cong H \subseteq G$, $n \geq 3$, B an H-bridge. Then $|\mathrm{att}(B)| \leq 45$.*

Proof. If $n = 3$, then the result is an immediate consequence of Lemma 16.1. Thus, we may assume $n \geq 4$. If B has attachments in the interiors of non-consecutive spokes, then G is the Petersen graph and the result clearly holds.

Otherwise, B has attachments in at most two consecutive spokes. Thus, there is a subdivision H' of V_6 contained in H that contains all the attachments of B. Applying Lemma 16.1 to H', we again see that $|\mathrm{att}(B)| \leq 45$. ■

We now turn to the other half of the argument that bounds the number of vertices in an H-bridge, namely, that the bridge is a tree. We need a new notion.

DEFINITION 16.3. Let T^* be a graph consisting of subdivision of a $K_{2,3}$ together with three pendant edges, one incident with each of the three degree 2 vertices in the $K_{2,3}$. A *tripod* is any graph T obtained from T^* by contracting any subset of the pendant edges; if all three pendant edges are contracted, then an edge is added between the two copies of $K_{1,3}$, but not having a vertex of contraction as an end — this may be done in any of three essentially different ways. The *attachments* of the tripod are the degree 1 and 2 vertices in T.

We are now ready for the second half of the main result of this section.

LEMMA 16.4. *Suppose $G \in \mathcal{M}_2^3$, $V_{2n} \cong H \subseteq G$, $n \geq 3$, G has no subdivision of $V_{2(n+1)}$, and B is an H-bridge. Then either B is a tree or B has a tripod, $n = 3$ and $|V(G)| \leq 10$.*

Proof. By way of contradiction, suppose B has a cycle C. If $\mathrm{att}(B) \cap C \neq \emptyset$, let e be an edge of C incident with $u \in \mathrm{att}(B)$. If $C \cap \mathrm{att}(B) = \emptyset$, then let e be any edge of C. The choice of e shows that $B - e$ is an H-bridge in $G - e$ and that $\mathrm{att}_{G-e}(B-e) = \mathrm{att}_G(B)$. Since $D_e[H]$ contains the crossing in $D_e[G-e]$, $D_e[B-e]$ is contained in a face F of $D_e[H]$.

Let $C' = \partial F^{\times}$, so C' is a cycle in $G' = (G-e)^{\times}$. Since G' is planar, C' has BOD in G' and $C' \cup B'$ is planar for each C'-bridge B' in G'. If $C' \cup B$ were planar, then $G' + e$ would be planar, in which case $\mathrm{cr}(G) \leq 1$, a contradiction. Therefore, $C' \cup B$ is not planar.

We now introduce a convenient notion.

DEFINITION 16.5. Let G be a graph. The graph G^t is the graph whose vertices are the G-nodes and whose edges are the G-branches.

CLAIM 1. $(C' \cup B)^t$ is 3-connected.

PROOF. Let $L = (C' \cup B)^t$. If $|V(\mathrm{Nuc}(B))| = 1$, then L is a wheel and the claim follows. So assume $|V(\mathrm{Nuc}(B))| \geq 2$. We show that any two vertices of L are joined by three internally disjoint paths. For $u, w \in \mathrm{Nuc}(B)$, this is true in G, so let P_1, P_2, P_3 be such paths in G. If at least one P_i is contained in $B - C'$, then we can easily modify the others to use C' rather than $G - B$ to get three paths in L. If all three intersect C_e, then $B \cap (P_1 \cup P_2 \cup P_3)$ is two claws Y_u and Y_w. There is a Y_uY_w-path in $\mathrm{Nuc}(B)$, which returns us to the previous case.

If $u \in \mathrm{Nuc}(B)$ and $w \in C'$, then w is an attachment of B. Let Y be a claw in B with centre u and talons on C'. Using a C'-avoiding wY-path in B, if necessary, we can assume w is a talon of Y. It is then easy to use C' to extend the other two paths in Y to w.

Finally, if $u, w \in C'$, then both u and w are attachments of B, so there is a C'-avoiding path joining them. This path and the two uw-paths in C' yield the required three paths. □

DEFINITION 16.6. Let C be a cycle in a graph G and let P_1 and P_2 be disjoint C-avoiding paths in G. Then P_1 *and* P_2 *are* C*-skew paths* if the two C-bridges in $C \cup P_1 \cup P_2$ overlap.

As $C' \cup B$ has no planar embedding, [**25**] implies B has either a tripod whose attachments are in C' or two C'-skew paths.

Claim 2. If B has a tripod T, then $n = 3$, $G = H \cup T$ and $|V(G)| \le 14$.

Proof. Let S be the attachments of T. As $H \cup T$ is 2-connected and, relative to the cut S, both H'^{+} (taking H' to be any V_6 containing S) and T^{+} are non-planar. By Theorem 15.6, $\mathrm{cr}(H' \cup T) \ge 2$. Thus, $G = H' \cup T$, so $n = 3$ and, again by Theorem 15.6, $|V(G)| \le 10$. □

Thus, we can assume B has no tripod. Then B has C'-skew paths, say P_1 and P_2. Since these do not exist in $B - e$, e is in one of them. If $C \cap \mathrm{att}(B) = \emptyset$, choose e' any edge of C not in $P_1 \cup P_2$. If $C \cap \mathrm{att}(B) \ne \emptyset$, choose e' to be the other edge of C incident with the same attachment as e.

Repeat with $G - e'$. This yields C'' so that B has C''-skew paths $u_1'u_2'$ and $w_1'w_2'$ (e' incident with u_1'). Since $u_1u_2 \cup w_1w_2 \subseteq B - e'$, they are not C''-skew. In C', we have the cyclic order u_1, w_1, u_2, w_2, say. In C'' we have $u_1u_2w_1w_2$. Likewise in C' we have $u_1'u_2'w_1'w_2'$, while in C'' we have $u_1'w_1'u_2'w_2'$.

Let D and D' be 1-drawings of H having all attachments of B on faces F, F', respectively, so that the cyclic orders of $\mathrm{att}(B)$ are different in ∂F and $\partial F'$.

Claim 3. $n \ge 4$.

Proof. Let H be a subdivision of V_6 in G. We remark that if f and f' are any disjoint H-branches having internal vertices that are ends of an H-avoiding path P in G, then $H \cup P$ is a subdivision of V_8 in G.

We consider first the case that $\mathrm{att}(B)$ is not contained in any 4-cycle of H. Because we know the 1-drawings of H and $\mathrm{att}(B)$ is contained in the boundary ∂F of a face F of such a 1-drawing, ∂F is $\times v_1v_2v_3\times$. If B has attachments in both $\langle \times v_1\rangle$ and $\langle v_3\times\rangle$, then G has a subdivision of V_8, as required. Thus, we may assume that $\mathrm{att}(B)$ is contained in a 4-cycle Q of H, which we may take to be $[v_1v_2v_3v_4v_1]$.

In at least one of D and D', Q is self-crossed (otherwise the cyclic orders of $\mathrm{att}(B)$ are the same) and B is drawn in the face $\times v_1v_6v_3\times$. However, in this case $\mathrm{att}(B) \subseteq \langle \times, v_1] \cup [v_3, \times\rangle$ and at least two attachments of B are in each. In this case, we again have a subdivision of V_8 in G, as required. □

Claim 4. B has no (interior) spoke attachment.

Proof. From Claim 3, we know that $n \ge 4$. By way of contradiction, we assume B has an attachment in $\langle s_0\rangle$. From the listing of the faces of 1-drawings of V_{2n}, the only possibilities for each of ∂F and $\partial F'$ are:

: (1) $[v_0, r_0, v_1, s_1, v_{n+1}, r_n, v_n, s_0, v_0]$;
: (1') $[v_0, r_{-1}, v_{-1}, s_{-1}, v_{n-1}, r_{n-1}, v_n, s_0, v_0]$;
: (2) $\langle v_1, r_0, v_0, s_0, v_n, r_n, v_{n+1}, r_{n+1}, v_{n+2}\rangle$;
: (2') $\langle v_{-1}, r_{-1}, v_0, s_0, v_n, r_{n-1}, v_{n-1}, r_{n-2}, v_{n-2}\rangle$;
: (3) $\langle v_{n-1}, r_{n-1}, v_n, s_0, v_0, r_{-1}, v_{-1}, r_{-2}, v_{-2}\rangle$;
: (3') $\langle v_{n+1}, r_n, v_n, s_0, v_0, r_0, v_1, r_1, v_2]$;
: (4) $\langle v_{-1}, r_{-1}, v_0, s_0, v_n, r_n, v_{n+1}\rangle$;
: (4') $\langle v_{n-1}, r_{n-1}, v_n, s_0, v_0, r_0, v_1\rangle$;
: (5) $[v_0, v_1, v_2, \dots, v_n, s_0, v_0]$;
: (5') $[v_0, s_0, v_n, v_{n+1}, v_{n+2}, \dots, v_{-1}, v_0]$.

We now consider these possibilities in pairs. In every case, the ends of the skew paths will occur in the same cyclic order on the boundaries of the two faces, which is impossible.

: (1,1') $\mathrm{att}(B) \subseteq s_0$; same cyclic order, a contradiction.
: (2,2') $\mathrm{att}(B) \subseteq s_0$; same cyclic order, a contradiction.
: (3,3') $\mathrm{att}(B) \subseteq s_0$; same cyclic order, a contradiction.
: (4,4') $\mathrm{att}(B) \subseteq s_0$; same cyclic order, a contradiction.
: (5,5') $\mathrm{att}(B) \subseteq s_0$; same cyclic order, a contradiction.
: (1,2) $\mathrm{att}(B) \subseteq \langle v_1, r_0, v_0, s_0, v_n, r_n, v_n + 1]$; same cyclic order, a contradiction.
: (1,2') $\mathrm{att}(B) \subseteq [v_0, s_0, v_n]$; same cyclic order, a contradiction.
: (1,3) $\mathrm{att}(B) \subseteq s_0$; same cyclic order, a contradiction.
: (1,3') $\mathrm{att}(B) \subseteq \langle v_{n+1}, r_n, v_n, s_0, v_0, r_0, v_1]$; same cyclic order, a contradiction.
: (1,4) $\mathrm{att}(B) \subseteq \langle v_1, r_0, v_0, s_0, v_n]$; same cyclic order, a contradiction.
: (1,4') $\mathrm{att}(B) \subseteq [v_1, r_0, v_0, s_0, v_n]$; same cyclic order, a contradiction.
: (1,5) $\mathrm{att}(B) \subseteq [v_1, r_0, v_0, s_0, v_n]$; same cyclic order, a contradiction.
: (1,5') $\mathrm{att}(B) \subseteq [v_{n+1}, r_n, v_n, s_0, v_0]$; same cyclic order, a contradiction.
: (2,3) $\mathrm{att}(B) \subseteq s_0$; same cyclic order, a contradiction.
: (2,3') $\mathrm{att}(B) \subseteq \langle v_{n+1}, r_n, v_n, s_0, v_0, r_0, v_1]$; same cyclic order, a contradiction.
: (2,4) $\mathrm{att}(B) \subseteq \langle v_{n+1}, r_n, v_n, s_0, v_0]$; same cyclic order, a contradiction.
: (2,4') $\mathrm{att}(B) \subseteq \langle v_1, r_0, v_0, s_0, r_n]$; same cyclic order, a contradiction.
: (2,5) $\mathrm{att}(B) \subseteq \langle v_1, r_0, v_0, s_0, v_n]$; same cyclic order, a contradiction.
: (2,5') $\mathrm{att}(B) \subseteq [v_0, s_0, v_n, r_n, v_{n+1}, r_{n+1}, v_{n+2}\rangle$; same cyclic order, a contradiction.
: (3,4) $\mathrm{att}(B) \subseteq \langle v_{-1}, r_{-1}, v_0, s_0, v_n]$; same cyclic order, a contradiction.
: (3,4') $\mathrm{att}(B) \subseteq \langle v_{n-1}, r_{n-1}, v_n, s_0, v_0]$; same cyclic order, a contradiction.
: (3,5) $\mathrm{att}(B) \subseteq \langle v_{n-1}, r_{n-1}, v_n, s_0, v_0]$; same cyclic order, a contradiction.
: (3,5') $\mathrm{att}(B) \subseteq \langle v_{-2}, r_{-2}, v_{-1}, r_{-1}, v_0, s_0, v_n]$; same cyclic order, a contradiction.
: (4,5) $\mathrm{att}(B) \subseteq [v_0, s_0, v_n]$; same cyclic order, a contradiction.
: (4,5') $\mathrm{att}(B) \subseteq \langle v_{-1}, r_{-1}, v_0, s_0, v_n, r_n, v_{n+1}\rangle$; same cyclic order, a contradiction.

As any pair gives the same cyclic order, we always get a contradiction. □

CLAIM 5. B is not a local H-bridge.

PROOF. Suppose B is local, with $\mathrm{att}(B) \subseteq Q_0$. From Claims 3 and 4, we may assume $n \geq 4$ and B has no spoke attachment. Thus, $\mathrm{att}(B) \subseteq r_0 \cup r_n$. Moreover, B cannot have attachments in both $\langle r_0 \rangle$ and $\langle r_n \rangle$ because G has no subdivision of $V_{2(n+1)}$. On the other hand, B has at least two attachments in both r_0 and r_n or else the cyclic order of the ends of the skew paths is always the same. So we may assume $\mathrm{att}(B) \cap r_0 = \{v_0, v_1\}$. We need two attachments in r_n. From the listing of faces in 1-drawings of V_{2n}, the only possibilities for ∂F and $\partial F'$ occur when Q_0 is not self-crossed and so the cyclic orders of the attachments of B are the same in both cases, a contradiction. □

CLAIM 6. For some i, $\mathrm{att}(B) \subseteq r_i \cup r_{i+n+1}$.

PROOF. By Claims 3, 4, and 5, $n \geq 4$, B has no spoke attachments, and B is not local.

We consider in turn the possibilities for the face of $D_e[H]$ that contains $B - e$. We know B is not local, so it can only be contained in a face whose boundary has one of the following forms:

(1) $[\times, r_i, v_i, s_i, v_{i+n}, r_{i+n-1}, \times]$;
(2) $[\times, r_i, v_i, r_{i-1}, v_{i-1}, s_{i-1}, v_{n+i-1}, r_{n+i-1}, \times]$;
(3) $[\times, r_i, v_{i+1}, r_{i+2}, \ldots, v_{i+n-1}, r_{i+n-1}, \times]$;

(4) $[v_i, s_i, v_{n+i}, r_{n+i}, v_{n+i+1}, \dots, r_{i-1}, v_i]$; or
(5) $[\times, r_i, v_{i+1}, r_{i+1}, \dots, r_{n+i-1}, v_{n+i}, r_{n+i}, \times]$.

As in the proof of Claim 4, the faces of $D_e[H]$ and $D_{e'}[H]$ containing $B - e$ and $B - e'$, respectively, cannot both be of one of the types (3, 4, 5): the vertices of $\mathrm{att}(B)$ will occur in the same order in both cases.

If one of the drawings has $B - e$ or $B - e'$ in a face of type (1), then we are done: $\mathrm{att}(B) \subseteq r_i \cup r_{i+n-1}$. The remaining case is that one of the drawings has $B - e$ or $B - e'$ drawn in a face of type (2).

All other possibilities having been eliminated, we may assume (taking $i = n+1$)

$$\mathrm{att}(B) \subseteq [\times, r_1, v_1, r_0, v_0, s_0, v_n, r_n, \times] \ .$$

Because B is not local, $\mathrm{att}(B) \cap \langle r_1 \rangle \neq \emptyset$. Because $\mathrm{att}(B)$ occurs in different orders in ∂F and $\partial F'$, $\mathrm{att}(B) \cap r_n \neq \emptyset$. By way of contradiction, we suppose B also has an attachment in $[v_0, r_0, v_1\rangle$. The only other face which could allow these three attachments is $[\times, r_0, v_1, r_1, \dots, v_{i-1}, r_{i-1}, v_n, r_n, \times]$. Notice v_0 is not in this second boundary, so one attachment is in $\langle r_0 \rangle$. Because $V_{2(n+1)} \not\subseteq G$, no attachment is in $\langle r_n \rangle$. Thus $\mathrm{att}(B) \cap r_n = \{v_n\}$. But then, once again, the attachments of B occur in the same cyclic orders in ∂F and $\partial F'$, a contradiction. □

As we have seen above, the alternative to "B is neither a tree nor contains a tripod" is that B has the C'-skew paths P_1 and P_2, as well as the C''-skew paths P_1' and P_2'. Claim 6 shows the four ends of P_1 and P_2 are in $r_0 \cup r_{n+1}$. If three of them are in r_0, say, then they occur in the same cyclic order in ∂F and $\partial F'$, a contradiction. So two are in r_0 and two in r_{n+1}. If P_1 has both ends in r_0, say, then the ends of P_1 and P_2 can never interlace, a contradiction as they interlace in ∂F. So each has one end in each of r_0 and r_{n+1}. Likewise for P_1', P_2'.

Adding at most 3 paths in $B - \mathrm{att}(B)$ to $P_1 \cup P_2 \cup P_1' \cup P_2'$, we obtain $B' \subseteq B$ containing $P_1 \cup P_2 \cup P_1' \cup P_2'$ so that B' is an H-bridge in $H \cup B'$.

Recall that $n \geq 4$ by Claim 3. All the attachments of B' are in $H - \langle s_3 \rangle$. Suppose D'' is a 1-drawing of $(H \cup B') - \langle s_3 \rangle$. Then $D''[B']$ is in a face F'' of $D''[H - \langle s_3 \rangle]$. Since r_0 and r_{n+1} both have at least two attachments of B', they are both incident with F''. Thus one of the pairs P_1, P_2 and P_1', P_2' is a $\partial F''$-skew pair. Therefore, $\mathrm{cr}((H \cup B') - \langle s_3 \rangle) \geq 2$, contradicting the fact that G is 2-crossing-critical. ■

Combining Corollary 16.2 and Lemma 16.4, we immediately have the main result of this section.

THEOREM 16.7. *Let $G \in \mathcal{M}_2^3$, $V_{2n} \cong H \subseteq G$, $n \geq 3$, and suppose G has no subdivision of $V_{2(n+1)}$. If B is an H-bridge, then $|V(B)| \leq 88$.*

This completes the first main step of our effort to show that 3-connected, 2-crossing-critical graphs with no subdivision of V_{2n} have bounded size.

16.2. The number of bridges is bounded

This subsection, the final leg of this work, is devoted to showing that there is a particular subdivision H of V_{2n} in G so that there are at most $O(n^3)$ H-bridges in G that have a vertex that is not an H-node. Theorem 16.7 shows that, for any $V_{2n} \cong H \subseteq G$, all H-bridges have at most 88 vertices (when there is no subdivision of $V_{2(n+1)}$). The combination easily implies G has at most $O(n^3)$ vertices.

DEFINITION 16.8. Let G be a graph and let n be an integer, $n \geq 3$. A subdivision H of V_{2n} in G is *smooth* if, whenever B is an H-bridge with all its attachments in the same H-branch, B is just an edge that is in a digon with an edge of H.

We begin by showing that every $G \in \mathcal{M}_2^3$ with a subdivision V_{2n} has a smooth subdivision H of V_{2n}. For such an H, every vertex of G either is an H-node or is in an H-bridge that does not have all its attachments in the same H-branch. So it will be enough to show that the number of these H-bridges is $O(n^3)$.

This analysis is completed in three parts. We start with the result that there are not many H-bridges having an attachment in a particular vertex of H and an attachment in the interior of some H-branch. This is useful for H-bridges having both node and branch attachments, but is also used in the second part, which is to bound the number of H-bridges having attachments in the interiors of the same two H-branches. The final part puts these together with those H-bridges having attachments in three or more H-nodes.

We start by showing that every $G \in \mathcal{M}_2^3$ with a subdivision of V_{2n} has a smooth subdivision of V_{2n}.

LEMMA 16.9. *Let $G \in \mathcal{M}_2^3$ and suppose G contains a subdivision of V_{2n}, with $n \geq 3$. Then G has a smooth subdivision of V_{2n}.*

Proof. Choose H to be a subdivision of V_{2n} in G that minimizes the number of edges of G that are in H. We claim H is smooth.

To this end, let B be an H-bridge with all attachments in the same H-branch b and let P be a minimal subpath of b containing $\operatorname{att}(B)$. Set $K = B \cup P$ and notice that K is both H-close and 2-connected. By Lemma 5.13, K is a cycle, so B is just a path and, since G is 3-connected, just an edge. It remains to prove that P is just an edge as well.

Let $H' = (H \cup B) - \langle P \rangle$. Evidently H' is a subdivision of V_{2n} in G and $|E(H')| = |E(H)| - |E(P)| + 1$. Since $|E(H)| \leq |E(H')|$ by the choice of H, we see that $|E(P)| \leq 1$, and, therefore, P is just an edge, as required. ∎

We now turn our attention to the H-bridges of a smooth subdivision H of V_{2n}. There are three main steps.

Step 1: *Bridges attaching to a particular vertex and branch.*

The first step in bounding the number of H-bridges is to bound the number of them that can have an attachment at a particular vertex of H and in the interior of a particular H-branch. This is the content of this step.

LEMMA 16.10. *Let $G \in \mathcal{M}_2^3$, $V_{2n} \cong H \subseteq G$, $n \geq 3$ and suppose H is smooth. For a vertex (not necesssarily a node) u of H and an H-branch b, there are at most 41 H-bridges with an attachment at u and an attachment in $\langle b \rangle - u$.*

Proof. Suppose there are 42 such H-bridges. Let B_0 be one of them, let $e \in E(B_0)$ and let D be a 1-drawing of $G - e$. If $u \notin \langle b \rangle$, then at most 4 faces of $D[H]$ are incident with $\langle b \rangle$, and therefore at least 11 of these H-bridges (other than B_0) are in the same face F of $D[H]$. If $u \in \langle b \rangle$, then precisely two faces of $D[H]$ are incident with u, so at least 21 of these bridges are in the same face F of $D[H]$ and of these at least 11 have an attachment in the same component of $D[b-u] \cap (\partial F)^{\times}$. In both cases, let $\mathcal{B}$ be the set of 11 bridges, contained in F, having u as an attachment and an attachment in the same component b' of $D[b-u]\cap(\partial F)^\times$. As $D[(\partial F)^\times\cup(\cup_{B\in\mathcal{B}}\mathcal{B})]$ is planar with $(\partial F)^\times$ bounding a face, no two $(\partial F)^\times$-bridges in $\mathcal{B}$ overlap.

Let $P=b'$ and $Q=(\partial F)^\times-\langle P\rangle$. Lemma 4.8 applies to $(\partial F)^\times$, P, Q, $\mathcal{B}$. As there are no digons disjoint from H, there is a unique (up to inversion) ordering $B_1,\dots,B_{11}$ of $\mathcal{B}$ so that $P=P_{B_1}\dots P_{B_{11}}$ and $Q=Q_{B_1}\dots Q_{B_{11}}$.

Because $u\in Q_{B_1}\cap Q_{B_2}\cap\dots\cap Q_{B_{11}}$ and the Q_{B_i} are internally disjoint subpaths of Q, all of $Q_{B_2},\dots,Q_{B_{10}}$ are just u. For $i=1,\dots,11$, let a_i and a_i' be the ends of P_i, so that $P=(\dots,a_2,\dots,a_2',\dots,a_3,\dots,a_3',\dots,a_{10},\dots,a_{10}',\dots)$.

Claim 1. For $i\in\{2,\dots,9\}$, $a_i\neq a_{i+1}'$.

Proof. Otherwise, $a_i=a_i'=a_{i+1}=a_{i+1}'$, implying that B_i and B_{i+1} constitute a digon disjoint from H, which is impossible. □

For $i,j\in\{2,3,\dots,10\}$ with $i<j$, set $K_{ij}=(\cup_{k=i}^{j}B_k)\cup a_iPa_j'$.

Claim 2. For $i,j\in\{2,\dots,10\}$ with $i<j$, K_{ij} is 2-connected.

Proof. Let R_i be an H-avoiding ua_i-path in B_i, and R_j an H-avoiding ua_j'-path in B_j. Then $C_{ij}:=R_i\cup R_j\cup a_iPa_j'\subseteq K_{ij}$ is a cycle containing u and a_iPa_j'.

For $x\in B_k$, $i\le k\le j$, $x\notin H$, for any H-node $w\neq u$, G has 3 internally disjoint xw-paths; at least two of these leave B_k in a_kPa_k', and so no cut vertex of K_{ij} separates x from C_{ij}. □

Since b' is not crossed in D, $D[K_{i,i+2}]$ is clean and is contained in $F\cup\partial F$. There is a unique face F_i of $D[K_{i,i+2}]$ so that $F_i\not\subseteq F$; since $K_{i,i+2}$ is 2-connected, F_i is bounded by a cycle C_i. As $D[K_{i,i+2}]\subseteq F\cup\partial F$, $\partial F\subseteq F_i\cup\partial F_i$. As $D[u]\in\partial F\cap D[K_{i,i+2}]$, $D[u]\in\partial F_i$. Likewise $D[a_iPa_{i+2}']\subseteq\partial F_i$.

Thus, $u\in C_i$ and $a_iPa_{i+2}'\subseteq C_i$. Therefore, $C_i\cap H$ is u and a_iPa_{i+2}', from which we deduce that there is a C_i-bridge M_i so that $H\subseteq C_i\cup M_i$. Observe that B_{i+1} is a C_i-bridge different from M_i.

For $i=2,5,8$, let e_i be an edge of B_{i+1} incident with u, and let D_i be a 1-drawing of $G-e_i$.

Claim 3. For $i\in\{2,5,8\}$, C_i has BOD in G and $D_i[C_i]$ is not clean.

Proof. At most one of $D_2[C_i]$, $i\in\{2,5,8\}$ is crossed, so for at least one $i\in\{5,8\}$, $D_e[C_i]$ is clean. It follows that C_i has BOD in $G-e$.

By Claim 1, $a_3\neq a_i$, whence $B_3\subseteq M_i$, and $B_3-e\subseteq M_i-e$. Furthermore, $u\in H$, so $u\in\operatorname{att}(M_i-e)$. Thus $\operatorname{att}_{G-e}(M_i-e)=\operatorname{att}_G(M_i)$ and M_i-e is a C_i-bridge in $G-e$. We conclude that the overlap diagrams for C_i in $G-e$ and G are isomorphic and, therefore, C_i has BOD in G.

We now show that all three C_j, $j\in\{2,5,8\}$, have BOD in G. If $D_i[C_i]$ is clean, then $D_i[C_i\cup M_i]$ is a 1-drawing of $C_i\cup M_i$, implying via Corollary 4.7 that $\operatorname{cr}(G)\le 1$, a contradiction. So $D_i[C_i]$ is not clean, and, therefore, for $j\in\{2,5,8\}\setminus\{i\}$, $D_i[C_j]$ is clean. Thus, C_j has BOD in $G-e_i$, and, following the argument above for C_i, we deduce that C_j has BOD in G. □

Claim 4. For $i\in\{2,5,8\}$, one face of $D_i[C_i]$ contains all H-nodes, other than (possibly) u.

PROOF. Let e'_i be the edge of H so that $D_i[e'_i]$ crosses $D_i[a_iba'_{i+2}]$ and let b'_i be the H-branch containing e'_i. If $n=3$, let R be a hexagon in H containing b and b'_i. For $n\geq 4$, both b and b'_i are in the rim R of H.

Since b and b'_i are disjoint, for $n\geq 3$, $R-(\langle b\rangle\cup\langle b'\rangle)$ has two components, each with at least two nodes of H. Either of these with $\leq n$ nodes has all its nodes adjacent by spokes to the other component. Obviously, there is at least one such.

Observe that if A is any path in $R-(\langle b\rangle\cup\langle b'_i\rangle)$ such that $D_i[A]$ has a vertex in each face of $D_i[C_i]$, then $u\in V(A)$ and the two paths P,P' in A having u as an end are such that $D_i[P]$ and $D_i[P']$ are in different faces of $D_i[C_i]$.

Let K be a component of $R-(\langle b\rangle\cup\langle b'_i\rangle)$ not containing u and let L be the other. Then $D_i[K]$ is in the closure of a face F_i of $D_i[C_i]$. We claim that $D_i[L]\subseteq F_i\cup\{u\}$.

Any H-node w in L that is joined by a spoke to an H-node w' in K has $D_i[w]\subseteq F_i\cup D_i[u]$, since otherwise $D_i[ww']$ crosses $D_i[C_i]$.

If there is an H-node w in L that is not adjacent by a spoke to any vertex in K, then w is adjacent by a spoke to another H-node w' in L and, moreover, w and w' are the first and last nodes of L. As $D_i[ww']$ is disjoint from $D_i[C_i]$, we deduce that there is a face F of $D_i[C_i]$ so that $D_i[w]$ and $D_i[w']$ are both in $F\cup D_i[u]$. Therefore, $D_i[L]$ is contained in that face. As at least one H-node in L is adjacent by a spoke to an H-node in K, we conclude that $D_i[L]\subseteq F_i\cup D_i[u]$. □

Let F_i be the face of $D_i[C_i]$ containing all the H-nodes and let F'_i be the other face of $D_i[C_i]$.

CLAIM 5. For $i\in\{2,5,8\}$, the crossing in D_i is not in $\langle a_{i+1},b,a'_{i+1}\rangle$.

PROOF. Suppose by way of contradiction that e'_i is an edge of $G-e_i$ so that $D_i[e'_i]$ crosses $\langle a_{i+1},b,a'_{i+1}\rangle$. Clearly, $a_{i+1}\neq a'_{i+1}$. Since $H-\langle b\rangle$ is 2-connected, there is a cycle $C'\subseteq H$ containing e'_i. Let P be an H-avoiding $a_{i+1}a'_{i+1}$-path in B_{i+1} and let C be the cycle $P\cup[a_{i+1},b,a'_{i+1}]$. Then C and C' are graph-theoretically disjoint and $D_i[C]\cap D_i[C']$ contains the crossing of D_i. But then $D_i[C]$ and $D_i[C']$ must cross a second time, a contradiction. □

CLAIM 6. The only C_i-bridge that overlaps B_{i+1} is M_i.

PROOF. Let B be a C_i-bridge different from M_i overlapping B_{i+1}. Then $\mathrm{att}(B)\subseteq[a_iba'_{i+2}]\cup\{u\}$. As H is smooth, $u\in\mathrm{att}(B)$. We claim both B_{i+1} and B overlap M_i.

By Claim 1, $a_i\neq a'_{i+1}$, so B_{i+1} either has an attachment in $\langle a_i,a'_{i+2}\rangle$ or it has both a_i and a'_{i+2} as attachments. In either case, B_{i+1} overlaps M_i (which has attachments at u,a_i,a'_{i+2}).

Likewise B either has two attachments in $[a_i,a'_{i+2}]$ or at least one attachment in $\langle a_{i+1},a'_{i+1}\rangle\subseteq\langle a_i,a'_{i+2}\rangle$, so B overlaps M_i. But now B_{i+1}, B_i, and M_i make a triangle in $OD(C_i)$, contradicting Claim 3. □

Let b' be the H-branch that crosses C_i in D_i and let x be the H-node so that the crossing is in $[x,b',u]$.

CLAIM 7. Let L be the graph $[D_i[G-e_i]\cap(\mathrm{cl}(F'_i))]^\times\cup B_{i+1}$. Then the C_i-bridge containing $[\times,b',u]$ overlaps B_{i+1} in L.

PROOF. If L embeds in the plane with C_i bounding a face, then this embedding combines with D_i restricted to the closure of F to yield a 1-drawing of G, which is impossible. As each individual C_i-bridge B in L has $C_i \cup B$ planar, there are overlapping C_i-bridges in L.

By definition, L is planar with all C_i-bridges other than B_{i+1} on the same side of C_i. Therefore B_{i+1} overlaps some other C_i-bridge in L. By Claim 6, this is not any C_i-bridge other than $D_i[M_i]^\times \cap D_i[L]$, that is, the one containing $[\times, b', u]$. □

By Claim 4, $[a_i, b, a'_{i+2}] - \times$ has a component A containing $\mathrm{att}(B_{i+1}) - u$. Let z be the one of a_i and a_{i+2} that is an end of A and let Q be the minimal subpath of A containing all of z, a_{i+1}, a'_{i+1}. By Claim 7, M_i has an attachment $w_i \in [zQ\rangle$ and an H-avoiding path Q_i from w_i to a vertex $x_i \in \langle \times, b', u\rangle$. Notice that, if $j \in \{2,5,8\} \setminus \{i\}$, then $Q_i \cap C_j = \emptyset$.

There are at most two H-branches (or subpaths thereof) incident with u that can cross b. Thus for some $i, j \in \{2,5,8\}$, $b'_i = b'_j$. Choose the labelling so that x_i is no further in b'_i from u than x_j is. Since xb'_ju contains x_i, $D_j[x_i] \subseteq F'_j$ but $D_j[w_i] \subseteq F_j$. Since $Q_i \cap C_j = \emptyset$, $D_j[Q_i]$ crosses C_j, the final contradiction. ■

The other steps in the argument are to show that a smooth subdivision H of V_{2n} in G has few bridges with attachments in the interiors of distinct H-branches. There are two parts to this: either the branches do or do not have a node in common. We first deal with the latter case.

Step 2: *H-bridges joining interiors of disjoint H-branches.*

LEMMA 16.11. *Let $G \in \mathcal{M}_2^3$, $V_{2n} \cong H \subseteq G$, $n \geq 3$, H smooth and suppose G has no subdivision of $V_{2(n+1)}$. If b_1, b_2 are disjoint H-branches, then there are at most $164n + 9$ H-bridges having attachments in both $\langle b_1\rangle$ and $\langle b_2\rangle$.*

Proof. Suppose there is a set $\mathcal{B}$ of $164n + 10$ H-bridges having attachments in both $\langle b_1\rangle$ and $\langle b_2\rangle$. Let $B_0 \in \mathcal{B}$ and let $e \in B_0$. In D_e, at most 4 faces are incident with $\langle b_1\rangle$, so there is a set $\mathcal{B}'$ consisting of $41n+3$ elements of $\mathcal{B} \setminus \{B_0\}$ in the same face of $D_e[H]$. By Lemma 4.8, there is a unique ordering $(B_1, \dots, B_{41n+3})$ of the elements of $\mathcal{B}'$ so they appear in this order in both $\langle b_1\rangle$ and $\langle b_2\rangle$. It follows that $B_2, \dots, B_{41n+2}$ have all attachments in $\langle b_1\rangle \cup \langle b_2\rangle$. By Lemmas 4.8 and 16.10, B_i and B_{i+41} are totally disjoint. So there are $n+1$ totally disjoint $\langle b_1\rangle\,\langle b_2\rangle$-paths with their ends having the same relative orders on both.

We aim to use these disjoint paths to find a subdivision of $V_{2(n+1)}$ in G. We need the following new notion.

DEFINITION 16.12. Let $e = uw$ and $f = xy$ be edges in a graph G. Two cycles C and C' in G are *ef-twisting* if $C = (u, e, w, \dots, x, f, y, \dots)$ and $C' = (u, e, w, \dots, y, f, x, \dots)$, i.e., C and C' traverse the edges e and f in opposite ways.

We note that V_6 has edge-twisting cycles: if $e = uw$ and $f = xy$ are disjoint edges in V_6, with u, x not adjacent, then the 4-cycle (u, w, x, y, u) and the 6-cycle (u, w, z, y, x, z', u) are ef-twisting.

Next suppose $n \geq 4$. There are three possibilities for b_1 and b_2.

: Case 1: *Both b_1 and b_2 are in R.* We may assume without loss of generality (recall that b_1 and b_2 are not adjacent) that $b_1 = r_0$, $b_2 = r_i$, $2 \leq i \leq n$. Set $H' = R \cup s_0 \cup s_1 \cup s_2$, so $H' \cong V_6$. Then b_1 and b_2 are in disjoint H'-branches and so H', and therefore H, contains b_1b_2-twisting cycles.

: Case 2: *One is in R, the other is a spoke.* We may assume without loss of generality that $b_1 = r_0$, $b_2 = s_i, i \notin \{0,1\}$. Set $H' = R \cup s_0 \cup s_1 \cup s_i$. Then b_1 and b_2 are in disjoint H'-branches, so H', and therefore H, contains $b_1 b_2$-twisting cycles.

: Case 3: *Both b_1 and b_2 are spokes.* We may assume without any loss of generality that $b_1 = s_0, b_2 = s_i$. Then there exists $j \in \{0, \dots, n-1\} \setminus \{0, i\}$. Set $H' = R \cup s_0 \cup s_i \cup s_j$. Then b_1 and b_2 are in disjoint H'-branches and so H', and therefore H, contains $b_1 b_2$-twisting cycles.

Choose the cycle C in the twisting pair in H for b_1 and b_2 so that C traverses b_1 and b_2 in order so that the ends u_i, w_i of the $n+1$ disjoint paths occur in C as $u_1, u_2, \dots, u_{n+1}, \dots, w_1, \dots, w_{n+1}$. Then C and these paths are a subdivision of $V_{2(n+1)}$ in G, contradicting the assumption that G has no subdivision of $V_{2(n+1)}$. ∎

Next is the third and final consideration.

Step 3: *H-bridges joining interiors of H-branches having a common node.*

LEMMA 16.13. *Let $G \in \mathcal{M}_2^3$, $V_{2n} \cong H \subseteq G$, $n \geq 3$, and let b_1, b_2 be adjacent H-branches. Then at most 2 H-bridges have attachments in both $\langle b_1 \rangle$ and $\langle b_2 \rangle$.*

Proof. By way of contradiction, suppose there is a set $\{B_1, B_2, B_3\}$ of 3 such H-bridges. For each $i \in \{1,2,3\}$, let $e_i \in B_i$. There is precisely one face F_i, of a 1-drawing D_i of $G - e_i$, that is incident with both $\langle b_1 \rangle$ and $\langle b_2 \rangle$. Thus, for each B_j, $j \neq i$, $D_i[B_j] \subseteq F_i$. Clearly for $\{j,k\} = \{1,2,3\} \setminus \{i\}$, B_j and B_k do not overlap on F_i. In particular, their attachments in b_1 and b_2 are in the same order as we traverse them from their common end u. Thus we may assume B_1, B_2, B_3 appear in this order from u on both b_1 and b_2.

Notice that $\mathrm{att}(B_3) \neq \mathrm{att}(B_2)$. Therefore, there is a cycle $C \subseteq B_2 \cup b_1 \cup b_2$ consisting of a $\langle b_1 \rangle \langle b_2 \rangle$-path in B_2 and a subpath of $b_1 \cup b_2$ containing u, such that C does not contain some attachment w of B_3. Reselect $e_3 \in B_3$ to be incident with w. Let M_C be the C-bridge so that $H \subseteq C \cup M_C$.

Then $w \in \mathrm{Nuc}(M_C)$, so $B_3 \subseteq M_C$. Furthermore, if e_3 is incident with an attachment x of M_C, then x is contained in R. In particular, it is incident with another edge of M_C. Thus, $M_C - e_3$ is a C-bridge in $G - e_3$ having the same attachments as M_C has in G. Because C is H-close, $D_1[C]$ is clean; furthermore, $D_1[C \cup M_C]$ is a 1-drawing of $C \cup M_C$. Since $D_3[C]$ is also clean, C has BOD in $G - e_3$ and hence in G. Corollary 4.7 implies the contradiction that $\mathrm{cr}(G) \leq 1$. ∎

We end this section with the asserted finiteness of 3-connected 2-crossing-critical graphs with no subdivision of V_{2n+2}.

THEOREM 16.14. *Suppose $G \in \mathcal{M}_2^3$ and there is an $n \geq 3$ so that G has a subdivision of V_{2n}, but no subdivision of $V_{2(n+1)}$. Then $|V(G)| = O(n^3)$.*

Proof. By Lemma 16.9, G has a smooth subdivision H of V_{2n}. We may assume no H-bridge contains a tripod, as otherwise $|V(G)| \leq 14$ by Lemma 16.4.

We first claim that a vertex u of H that is not an H-node is an attachment of some H-bridge B not having all its attachments in the same H-branch. Since u has degree 2 in H and degree greater than 2 in G, u is an attachment of some H-bridge. Because H is smooth, an H-bridge that has all its attachments in the same H-branch is an edge in a digon. If all the H-bridges attaching at u are such edges, then u has only two neighbours and G is not 3-connected, a contradiction.

Thus, every vertex of G is either an H-node or is in some H-bridge that does not have all its attachments in the same H-branch. We bound the number of these H-bridges as follows.

We claim that, for any three H-nodes u, v, w, at most two H-bridges have all three of u, v, w as attachments. To see this, suppose three nontrivial H-bridges $B_i, i = 1, 2, 3$, all have all of u, v, w as attachments. Each B_i contains a claw Y_i having u, v, w as talons. Then $Y_1 \cup Y_2 \cup Y_3 \cup H$ contains a subdivision of $K_{3,4}$, in which case 2-criticality implies G is $K_{3,4}$. Thus, at most two H-bridges have attachments in any three nodes. So there are at most $2\binom{2n}{3}$ nontrivial H-bridges with only node attachments.

Every other H-bridge of concern has an attachment in the interior of some H-branch and at some vertex of H not in that H-branch. Lemma 16.10 implies that there are at most $(2n)(3n)41$ H-bridges with an attachment in an H-node and in an open H-branch.

Lemma 16.11 implies there are at most $(\binom{3n}{2} - 6n)(164n + 9)$ H-bridges having attachments in the interiors of disjoint H-branches.

Lemma 16.13 implies there are at most 2 H-bridges with attachments on two given adjacent H-branches and so there are at most $6n(2)$ H-bridges with attachments on two adjacent H-branches.

Every H-bridge has at most 88 vertices, and every vertex of G is either an H-node or in one of these enumerated H-bridges. Therefore,

$$|V(G)| \leq 88 \left\{ 2\binom{2n}{3} + 2n \cdot 3n \cdot 41 + 6n(2) + \left[\binom{3n}{2} - 6n\right]\left[164n + 9\right] \right\} . \qquad \blacksquare$$

CHAPTER 17

Summary

This short section provides a single theorem and some remarks summarizing the current state of knowledge about 2-crossing-critical graphs.

THEOREM 17.1 (Classification of 2-crossing-critical graphs). *Let G be a 2-crossing-critical graph.*

(1) Then G has minimum degree at least two and is a subdivision of a 2-crossing-critical graph with minimum degree at least three.

Thus, we henceforth assume G has minimum degree at least three.

(2) If G is 3-connected and contains a subdivision of V_{10}, then $G \in \mathcal{T}(\mathcal{S})$ (Definition 2.12). That is, G is a twisted circular sequence of tiles, each tile being one of the 42 elements of $\mathcal{S}$ (Definition 2.10).

(3) If G is 3-connected and does not have a subdivision of V_{10}, then G has at most three million vertices (so there are only finitely many such examples). Each of these examples either
- *has a subdivision of V_8 or*
- *is either one of the four graphs described in Theorem 15.6 or obtained from a 2-crossing-critical peripherally-4-connected graph with at most ten vertices by replacing each vertex v having precisely three neighbors with one of at most twenty patches, each patch having at most six vertices (so G has at most sixty vertices).*

(4) If G is not 3-connected, then either
- *G is one of 13 examples that are not 2-connected, or*
- *G is 2-connected, has two nonplanar cleavage units, and is one of 36 graphs, or*
- *G is 2-connected, has one nonplanar cleavage unit, and is obtained from a 3-connected 2-crossing-critical graph by replacing digons with digonal paths.*

We conclude with some remarks on what remains to be done to find all 2-crossing-critical graphs.

REMARK 17.2. In Section 15.7, we provided a method for finding all 3-connected, 2-crossing-critical graphs not containing a subdivision of V_8. It would be desirable for this program to be completed.

REMARK 17.3. The remaining unclassified 3-connected, 2-crossing-critical graphs have a subdivision of V_8 but not of V_{10}. The works of Urrutia [**36**] and Austin [**3**] have found many of these, but more work is needed to find a complete set. It may be helpful to note that we have found all such examples that do not have a representativity 2 embedding in the projective plane. The known instances are all quite small, so it is reasonable to expect that each of these has at most 60 vertices or so.

ACKNOWLEDGEMENTS

Initial impetus to this project came through Shengjun Pan, who described mechanisms for proving a version of Theorem 2.14 (for G containing a subdivision of V_{2n}, with $2n$ likely somewhat larger than 10).

We are grateful to CIMAT for hosting us on multiple occasions for work on this project. In particular, we appreciate the support of Jose Carlos Gómez Larrañaga, then director of CIMAT.

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